Note: For answer to each question, please explain your answer in plain English first. There are a total of 100 pts plus 10 bonus points. It is your responsibility to make sure that you have all the pages!
Problem 1. 20 points, 4 points each) True or False?
For each of the statements below, determine whether it is True or False. Briefly explain your answer or provide a counterexample for each of them.

(a) Given $T(n) = 3n^3 + 2n^2 \lg n + n + 2$
   (a) $T(n) = O(n^4)$
   (b) $T(n) = \Omega(n^2 \lg n)$
   (c) $T(n) = \Theta(n^3)$
   (d) $T(n) = O(n^3)$

(b) If $T(n) = 8T(n/2) + \Theta(n^2)$ then $T(n) = \Theta(n^2 \lg n)$.

(c) Let $G$ be an undirected graph on $n$ nodes. $G$ has $n - 1$ edges if (1) $G$ is connected and (2) $G$ does not contain a cycle.

(d) In stable marriage, if $w$ is ranked last in $m$’s preference list and $m$ is ranked last in $w$’s list, then $(m, w)$ will not appear in any stable matching.

(e) In HW2 assignment, the golden ratio search can always beat Peter’s proposed approach for a given quadratic quality function and a given search range.
(Problem 2. 20 points) Given \( n \) girls and \( 2n \) boys, each with her/his private preference orders,

1. (5 points) Design a girl-initiated extended G-S algorithm such that it is female-optimal.
2. (5 points) Briefly show that your algorithm is still stable.
3. (5 points) Suppose each girl can marry exactly two boys, define a new concept of stability.
4. (5 points) Design an extended G-S algorithm for 3. that terminates and all matches are stable.
(Problem 3. 20 points) Extend the Weighted Interval Scheduling as following: each schedule can only include up to \( k \) out of total \( n \) jobs. The objective is still to find a compatible subset with the maximum total value.

1. (15 points) Provide a recursive solution in plain English first, followed by pseudo code and complexity analysis.

2. (5 points) Use the example in the class notes to illustrate your algorithm with a solution (i.e., selected jobs) for \( k = 2 \). The weight distribution from job \( a \) to job \( h \) is 2, 8, 7, 10, 3, 9, 13, and 11, respectively.
(Problem 4. 20 points) Extend question 5.7 to a 3-D $n \times n \times n$ grid graph. Two nodes $(i, j, k)$ and $(i', j', k')$ are neighbors if and only if $|i - i'| + |j - j'| + |k - k'| = 1$, i.e., each node has up to six neighbors. A node is local minimum if it has a minimum value among its 1-hop neighborhood. Provide a high-level solution in plain English, followed by pseudo code and complexity analysis.

- (5 points) Find a simple $\Theta(n^3)$ algorithm that finds a local minimum.
- (15 points) Enhance the simple algorithm so that the complexity of the enhanced algorithm is less than $\Theta(n^3)$. Provide a proof of algorithm complexity.
Problem 5. 20 points) Given a 2-D grid graph (like the graph in question 5.7), the distance of two adjacent nodes is 1.

1. (5 points) Find a recursive solution that determines the number of shortest paths from (0, 0) to (m, n).

2. (5 points) Determine the number of shortest paths in terms of m and n. Use m = 3 and n = 2 as an example to illustrate.

3. (5 points) Provide an iterative solution for the same problem.

4. (5 points) Show and prove the complexity of two solutions.
(Bonus problem, 10 pts), Suppose $C$ is a given set of currency denominations $C = \{1, p, 2p^2, 3p^3, ..., np^n\}$, where $p > 1$ and $n \geq 0$ are integer.

1. (2 points) Show how does the greedy algorithm described in the class notes make changes for 80 when $p = 2$ and $n = 4$.

2. (8 points) Show that the greedy algorithm discussed in the class always finds an optimal solution in terms of minimizing the number of coins for changes.