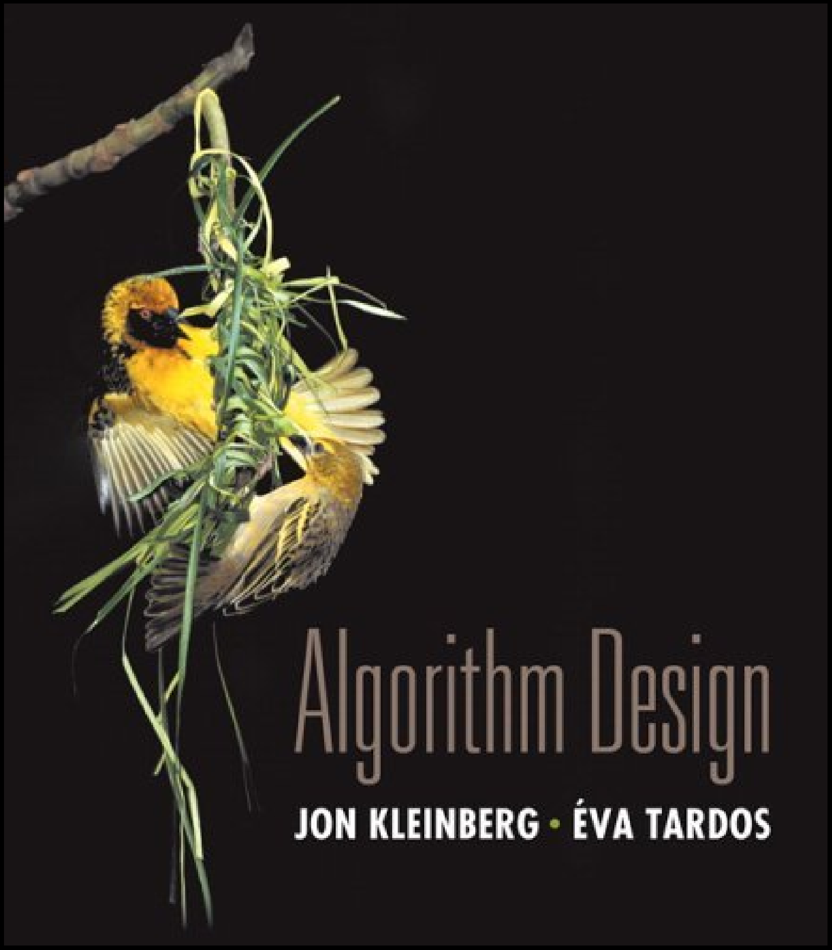


Chapter 6

Dynamic Programming

Algorithmic Paradigms

Greedy. Build up a solution incrementally, myopically optimizing some local criterion.

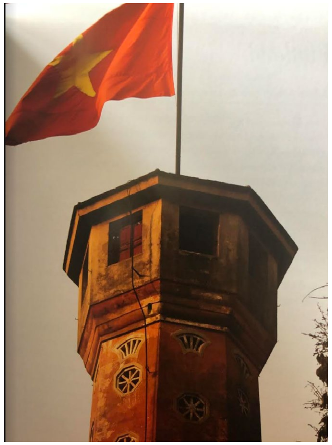
Divide-and-conquer. Break up a problem into sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

Dynamic programming. Break up a problem into a series of overlapping and/or multiple sub-problems (in sequence) and build up solutions to larger sub-problems until the original problem.

Algorithmic Paradigms

Keys

- Identify a recurrence
  - Follow a natural linear sequence
  - Generalize the problem (adding a new variable)
- Avoid redundancy
  - Memorization
  - Removing recursion
- Tower of Hanoi (n disks on three pegs, 2^n-1 moves)
  - Multiple subproblems in sequence
  - Tower of Brahma (64 disks, end of the world)
  - Optimization for 4 or more pegs is still open



Dynamic Programming History

Bellman. [1950s] Pioneered the systematic study of dynamic programming.

Etymology.

Dynamic programming = planning over time.

Secretary of Defense was hostile to mathematical research.

Bellman sought an impressive name to avoid confrontation.

> "it's impossible to use dynamic in a pejorative sense"
> "something not even a Congressman could object to"

Reference: Bellman, R. E. *Eye of the Hurricane, An Autobiography.*

Dynamic Programming Applications

Areas.

- Bioinformatics.
- Control theory.
- Information theory.
- Operations research.
- Computer science: theory, graphics, AI, compilers, systems, ….

Some famous dynamic programming algorithms.

- Unix diff for comparing two files.
- Viterbi for hidden Markov models.
- Smith-Waterman for genetic sequence alignment.
- Bellman-Ford for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.

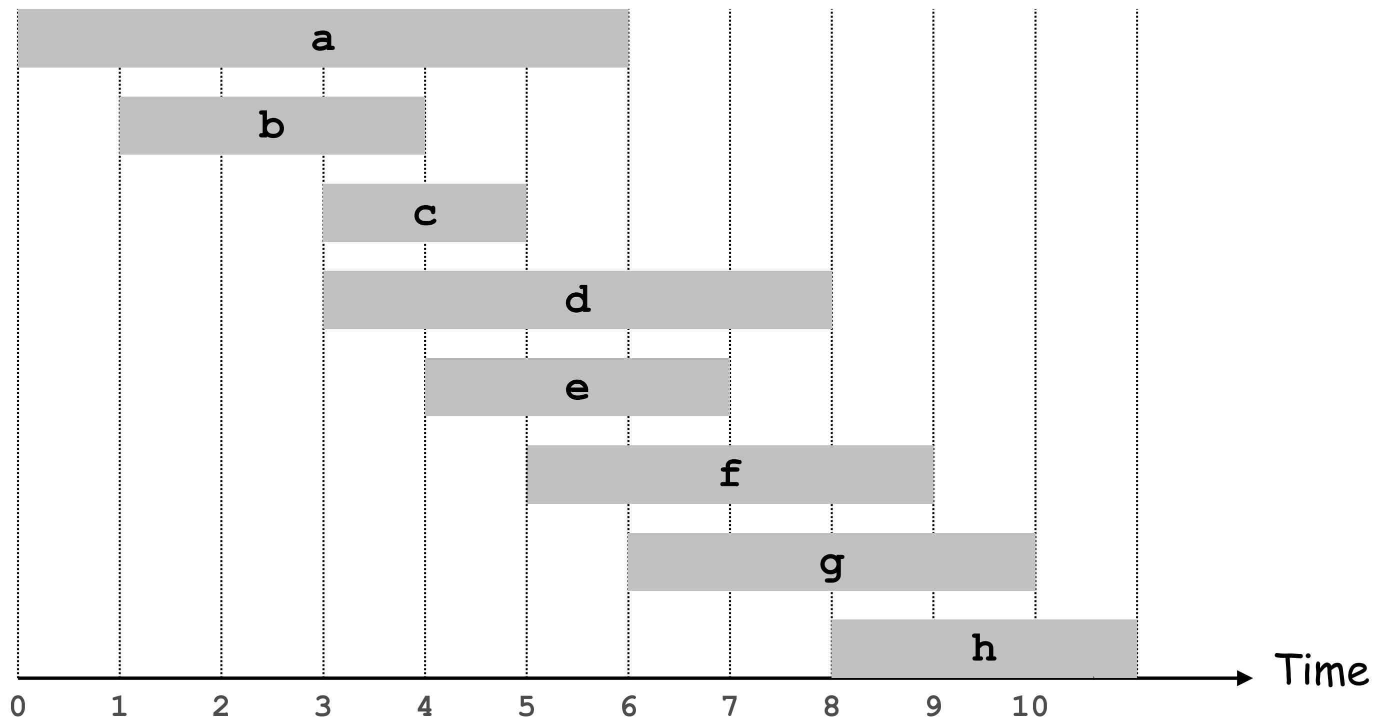
6.1 Weighted Interval Scheduling

Follow a natural linear sequence, but binary choice

Weighted Interval Scheduling

Weighted interval scheduling problem.

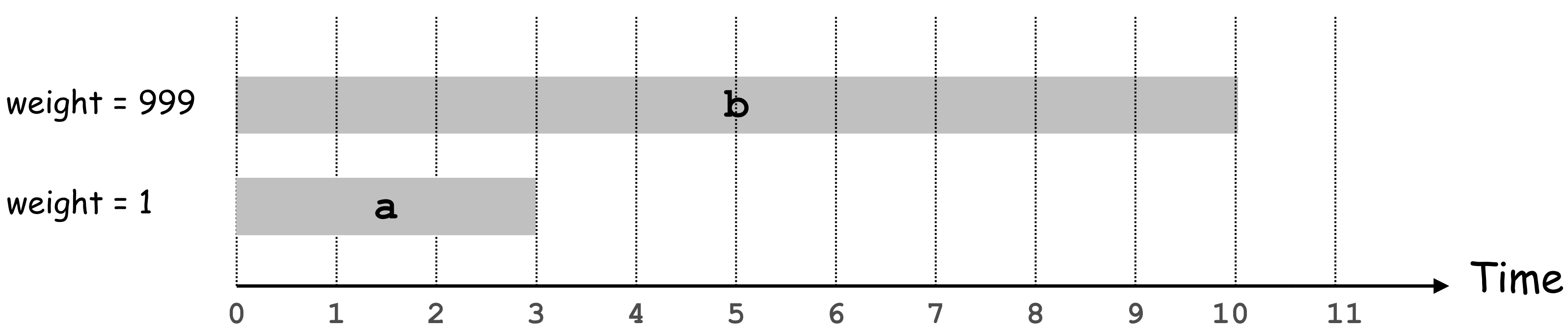
- Job j starts at s_j, finishes at f_j, and has weight or value v_j .
- Two jobs compatible if they don't overlap.
- Goal: find maximum weight subset of mutually compatible jobs.



Unweighted Interval Scheduling Review

Recall. Greedy algorithm works if all weights are 1.
- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

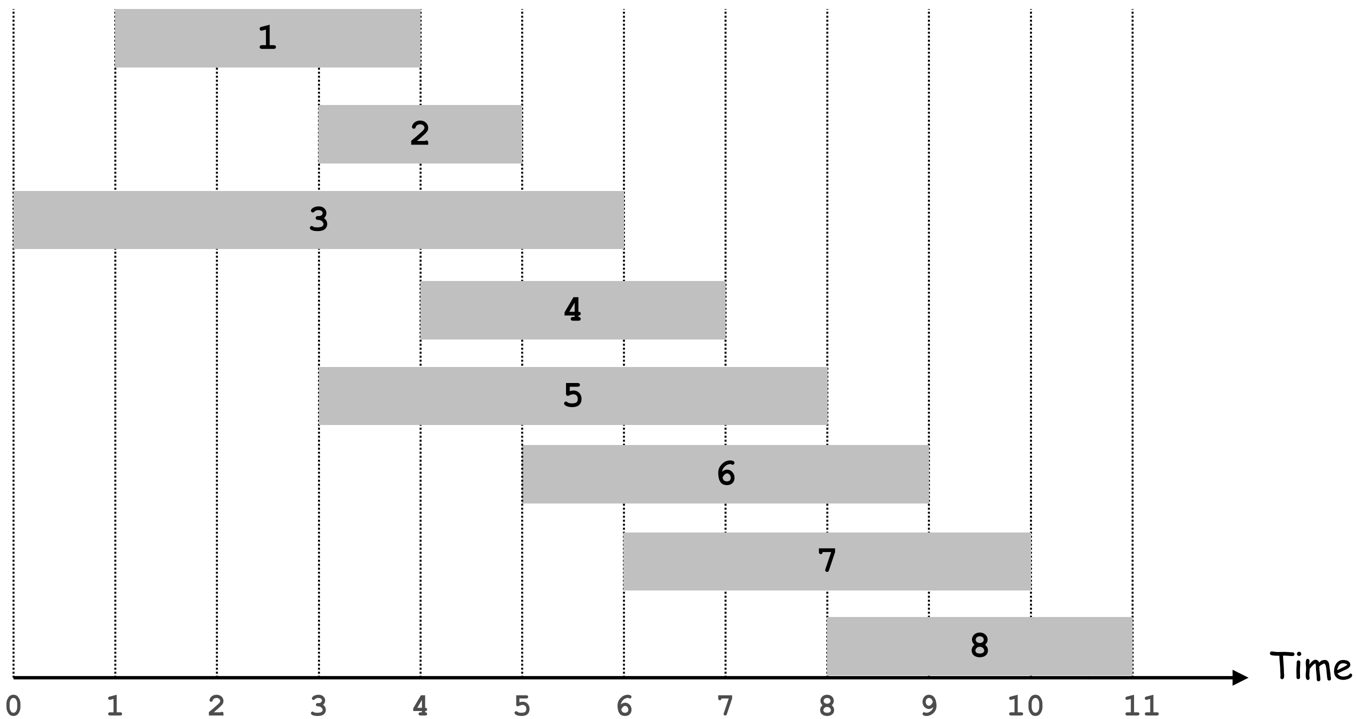
Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.



Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \le f_2 \le \ldots \le f_n$.
Def. $p(j)$ = largest index $i < j$ such that job i is compatible with j.

Ex: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$.



Dynamic Programming: Binary Choice

Notation. OPT(j) = value of optimal solution to the problem consisting of job requests 1, 2, ..., j.

Case 1: OPT selects job j.
- collect profit v_j
- can't use incompatible jobs { p(j) + 1, p(j) + 2, ..., j - 1 }
- must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., p(j)

optimal substructure

Case 2: OPT does not select job j.
- must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., j-1

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ v_j + OPT(p(j)),\ OPT(j-1) \} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Brute Force

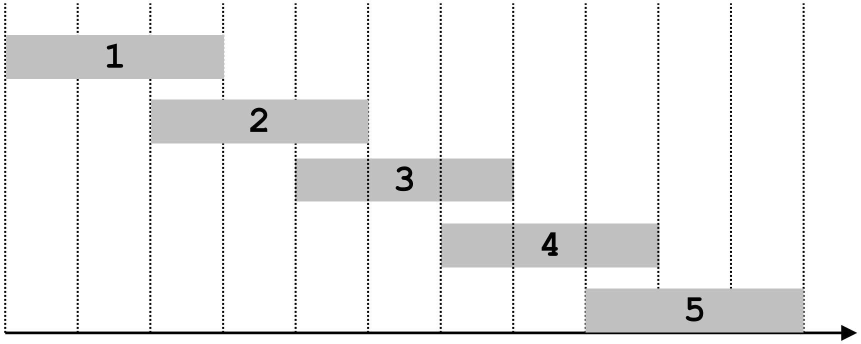
Brute force algorithm.

```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

Compute p(1), p(2), …, p(n)

Compute-Opt(j) {
   if (j = 0)
      return 0
   else
      return max(vj + Compute-Opt(p(j)), Compute-Opt(j-1))
}
```

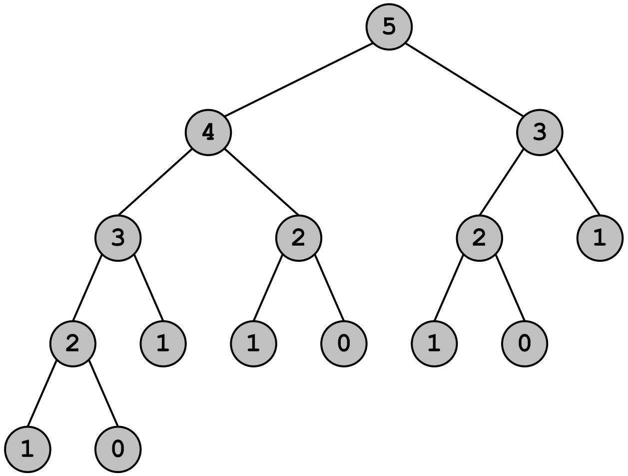
Weighted Interval Scheduling: Brute Force

Observation. Recursive algorithm fails spectacularly because of redundant sub-problems ⇒ exponential algorithms.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



p(1) = 0, p(j) = j-2



Weighted Interval Scheduling: Memoization

Memoization. Store results of each sub-problem in a cache; lookup as needed.

```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.
Compute p(1), p(2), …, p(n)

for j = 1 to n
   M[j] = empty        ← global array
M[0] = 0

M-Compute-Opt(j) {
   if (M[j] is empty)
      M[j] = max(vj + M-Compute-Opt(p(j)), M-Compute-Opt(j-1))
   return M[j]
}
```

Weighted Interval Scheduling: Running Time

Claim. Memoized version of algorithm takes $O(n \log n)$ time.
- Sort by finish time: $O(n \log n)$.
- Computing $p(\cdot)$: $O(n \log n)$ via sorting by start time.

- `M-Compute-Opt(j)`: each invocation takes $O(1)$ time and either
  - (i) returns an existing value `M[j]`
  - (ii) fills in one new entry `M[j]` and makes two recursive calls

- Progress measure Φ = # nonempty entries of `M[]`.
  - initially $\Phi = 0$, throughout $\Phi \leq n$.
  - (ii) increases Φ by 1 $\Rightarrow$ at most 2n recursive calls.

- Overall running time of `M-Compute-Opt(n)` is $O(n)$. ▪

Remark. $O(n)$ if jobs are pre-sorted by start and finish times.

Weighted Interval Scheduling: Finding a Solution

Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?

A. Do some post-processing.

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
   if (j = 0)
      output nothing
   else if (v_j + M[p(j)] > M[j-1])
      print j
      Find-Solution(p(j))
   else
      Find-Solution(j-1)
}
```

# of recursive calls $\leq n \Rightarrow O(n)$.

Weighted Interval Scheduling: Bottom-Up

Bottom-up dynamic programming. Unwind recursion for tail-recursion.

```
Input: n, s_1,…,s_n , f_1,…,f_n , v_1,…,v_n

Sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n.

Compute p(1), p(2), …, p(n)

Iterative-Compute-Opt {
   M[0] = 0
   for j = 1 to n
      M[j] = max(v_j + M[p(j)], M[j-1])
}
```

Dijkstra's 1968 letter: Go To Statement Considered Harmful.

6.3 Segmented Least Squares

Follow a natural linear sequence, but multiway choice

Segmented Least Squares

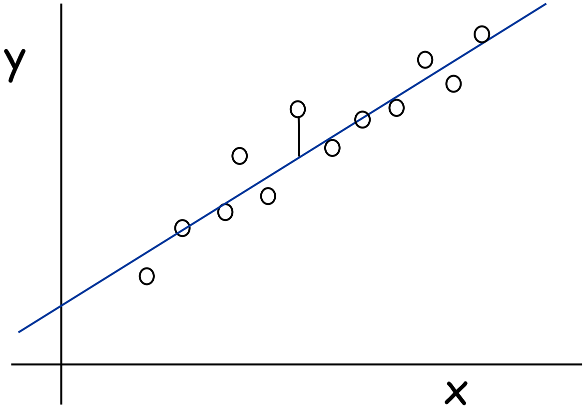
Least squares.

Foundational problem in statistic and numerical analysis.

Given n points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

Find a line $y = ax + b$ that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$



Solution. Calculus $\Rightarrow$ min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented least squares.

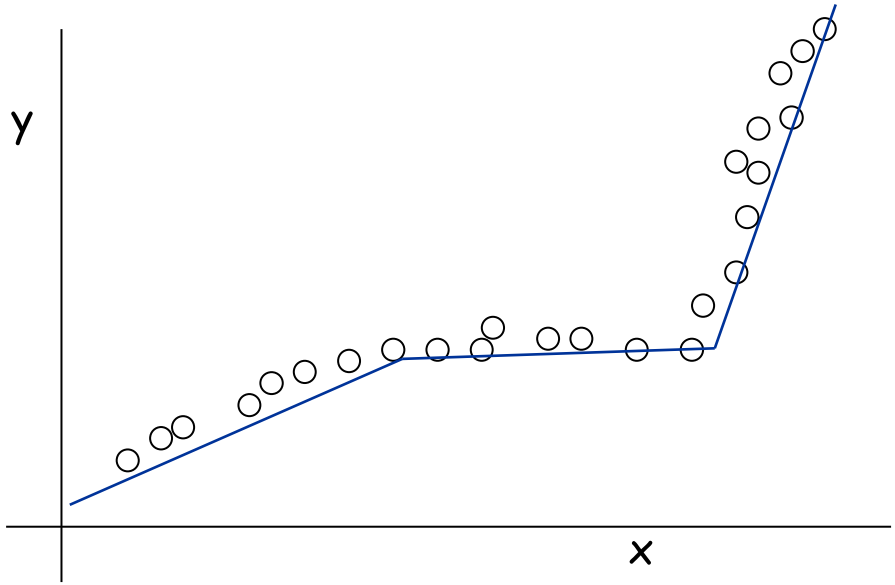
Points lie roughly on a sequence of several line segments.

Given n points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes $f(x)$.

Q. What's a reasonable choice for $f(x)$ to balance accuracy and parsimony?

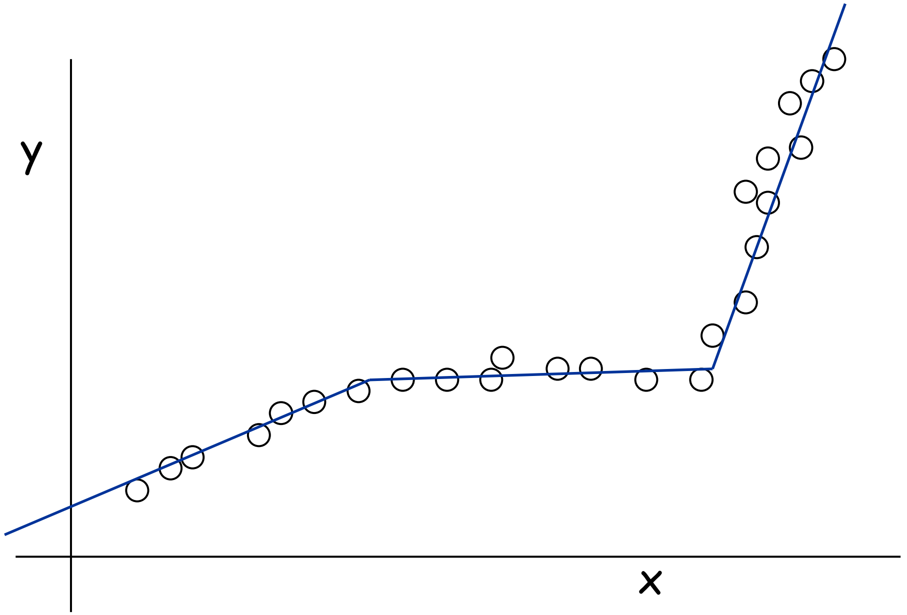
goodness of fit

number of lines



Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes:
  - the sum of the sums of the squared errors E in each segment
  - the number of lines L
- Tradeoff function: $E + c L$, for some constant $c > 0$.



Dynamic Programming: Multiway Choice

Notation.

$OPT(j)$ = minimum cost for points $p_1, p_{i+1}, \ldots, p_j$.

$e(i, j)$ = minimum sum of squares for points $p_i, p_{i+1}, \ldots, p_j$.

To compute OPT(j):

Last segment uses points $p_i, p_{i+1}, \ldots, p_j$ for some i.

Cost = $e(i, j) + c + OPT(i-1)$.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min\limits_{1 \le i \le j} \{ \; e(i,j) + c + OPT(i-1) \} & \text{otherwise} \end{cases}$$

Segmented Least Squares: Algorithm

```
INPUT: n, p1,…,pN , c

Segmented-Least-Squares() {
   M[0] = 0
   for j = 1 to n
      for i = j down to 1
         compute the least square error eij for
         the segment pi,…, pj

   for j = 1 to n
      M[j] = min 1 ≤ i ≤ j (eij + c + M[i-1])

   return M[n]
}
```

Running time. $O(n^3)$. ← can be improved to $O(n^2)$ by pre-computing various statistics

Bottleneck = computing $e(i, j)$ for $O(n^2)$ pairs, $O(n)$ per pair using previous formula.

6.4 Knapsack Problem

Generalize the problem (by adding a new variable)

Knapsack Problem

Knapsack problem.

- Given n objects and a "knapsack."
- Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.
- Knapsack has capacity of W kilograms.
- Goal: fill knapsack to maximize total value.

Ex: { 3, 4 } has value 40.

W = 11

#	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Greedy: repeatedly add item with maximum ratio v_i / w_i.
Ex: { 5, 2, 1 } achieves only value = 35 $\Rightarrow$ greedy not optimal.

Dynamic Programming: False Start

Def. OPT(i) = max profit subset of items 1, …, i.

Case 1: OPT does not select item i.
- OPT selects best of { 1, 2, …, i-1 }

Case 2: OPT selects item i.
- accepting item i does not immediately imply that we will have to reject other items
- without knowing what other items were selected before i, we don't even know if we have enough room for i

Conclusion. Need more sub-problems!

Dynamic Programming: Adding a New Variable

Def. OPT(i, w) = max profit subset of items 1, …, i with weight limit w.

Case 1: OPT does not select item i.
- OPT selects best of { 1, 2, …, i-1 } using weight limit w

Case 2: OPT selects item i.
- new weight limit = w – w_i
- OPT selects best of { 1, 2, …, i-1 } using this new weight limit

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max\{\, OPT(i-1, w),\quad v_i + OPT(i-1, w-w_i)\,\} & \text{otherwise} \end{cases}$$

Knapsack Problem: Bottom-Up

Knapsack. Fill up an n-by-W array.

```
Input: n, W, w1,…,wN, v1,…,vN

for w = 0 to W
   M[0, w] = 0

for i = 1 to n
   for w = 1 to W
      if (wi > w)
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max {M[i-1, w], vi + M[i-1, w-wi ]}

return M[n, W]
```

Knapsack Algorithm

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Problem: Running Time

Running time. $\Theta(n\ W)$.

- Not polynomial in input size!
- "Pseudo-polynomial."
- Decision version of Knapsack is NP-complete. [Chapter 8]

Knapsack approximation algorithm. There exists a poly-time algorithm that produces a feasible solution that has value within 0.01% of optimum. [Section 11.8]

Matrix-chain Multiplication

A sequence of matrix multiplication: $A_1A_2 \ldots A_n$, where A_i: $p_{i-1} \times p_i$

# of different parameterizations: Catalan number $\Omega(n^4/n^{3/2})$

Example: A1: 10 x 100, A2: 100 x 5, A3: 5 x 50

((A1 A2) A3): 10 x 100 x 5 + 10 x 5 x 50 = 7,500

(A1 (A2 A3)): 100 x 5 x 50 + 10 x 100 x 50 = 25,000

M[i, j]: minimum cost from A_i to A_j, then M[1, n]

s table gives index (location)

$$
\begin{aligned}
&(A_1(A_2(A_3A_4)))\ ,\\
&(A_1((A_2A_3)A_4))\ ,\\
&((A_1A_2)(A_3A_4))\ ,\\
&((A_1(A_2A_3))A_4)\ ,\\
&(((A_1A_2)A_3)A_4)\ .
\end{aligned}
$$

$$
m[i,j] = \begin{cases} 0 & \text{if } i = j\ , \\ \min\limits_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\} & \text{if } i < j\ . \end{cases}
$$

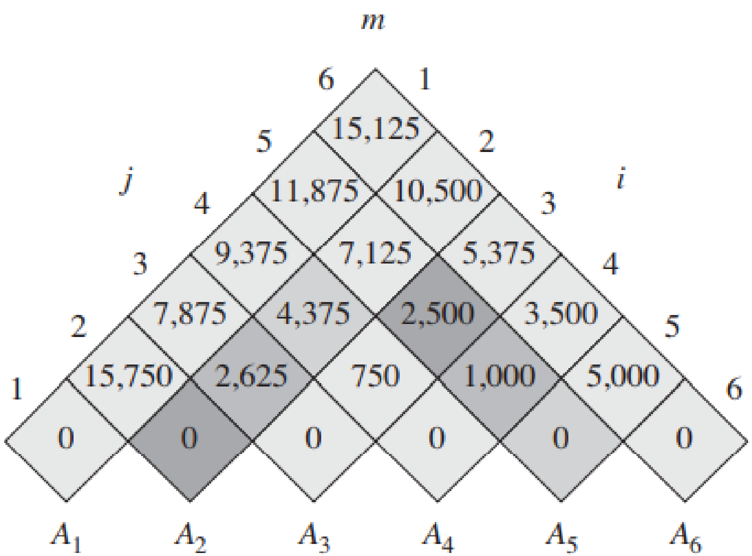
Solution: Build a pyramid bottom-up

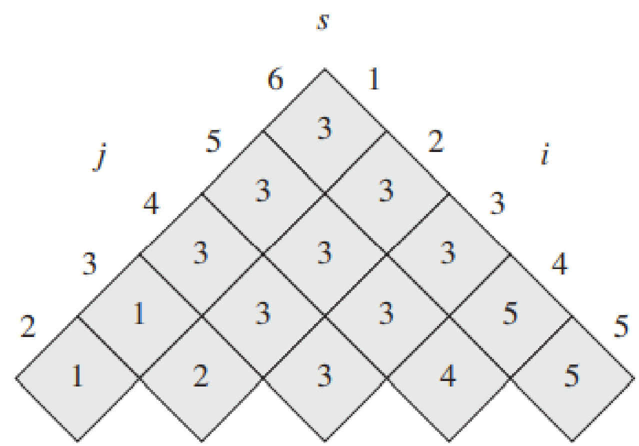
Generalize the problem by solving all subproblems:

m[i, j] with j-i = k: 1, 2, 3, … n-1

Solution: bottom-up from small ranges to the final range [1, n]

Complexity: $O(n^3)$





matrix	A_1	A_2	A_3	A_4	A_5	A_6
dimension	30×35	35×15	15×5	5×10	10×20	20×25

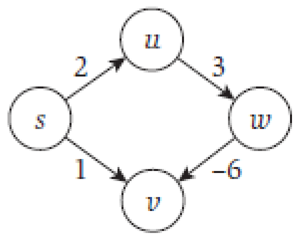
Shortest path: using a new variable i

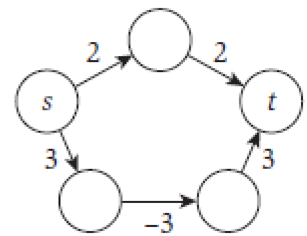
Shortest path: greedy (Dijkstra) and dynamic programming solutions

Both are based on optimal-substructure property

However, Dijkstra’s solution fails when there is a negative edge

Add a positive number to all edges does not work (see right figure)





Bellman-Ford OPT(i, v): shortest path from v to the dest. with i edge

Original problem: OPT(n-1, s)

If $i > 0$ then

$$\mathrm{OPT}(i, v) = \min(\mathrm{OPT}(i-1, v), \min_{w \in V}(\mathrm{OPT}(i-1, w) + c_{vw})).$$

Pull or Push implementation

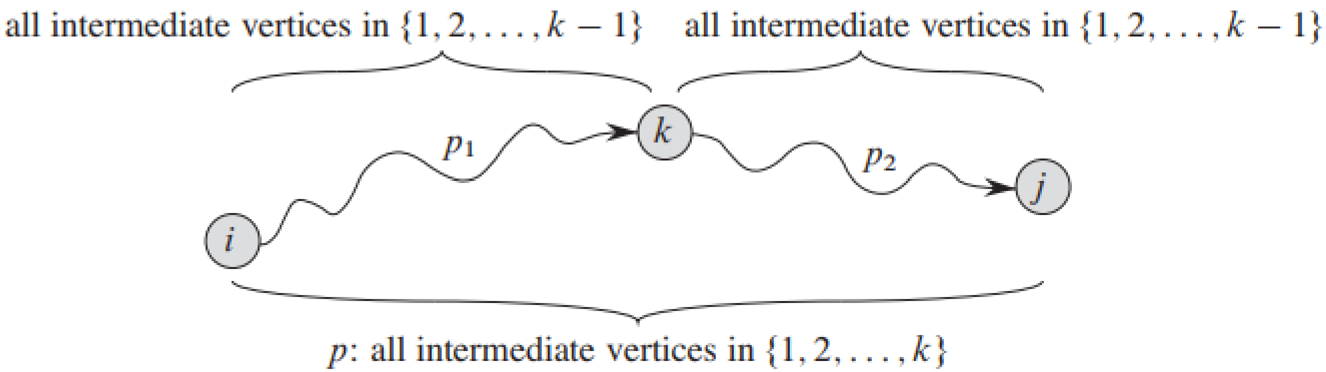
All-Pair Shortest Path: using a new set

Floyd-Warshall algorithm with complexity $O(n^3)$

Key: increase the size k of intermediate node set {1, 2,…, k}

step by step (using a special matrix multiplication)

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 , \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 . \end{cases}$$



6.5 RNA Secondary Structure

Multiple subproblems

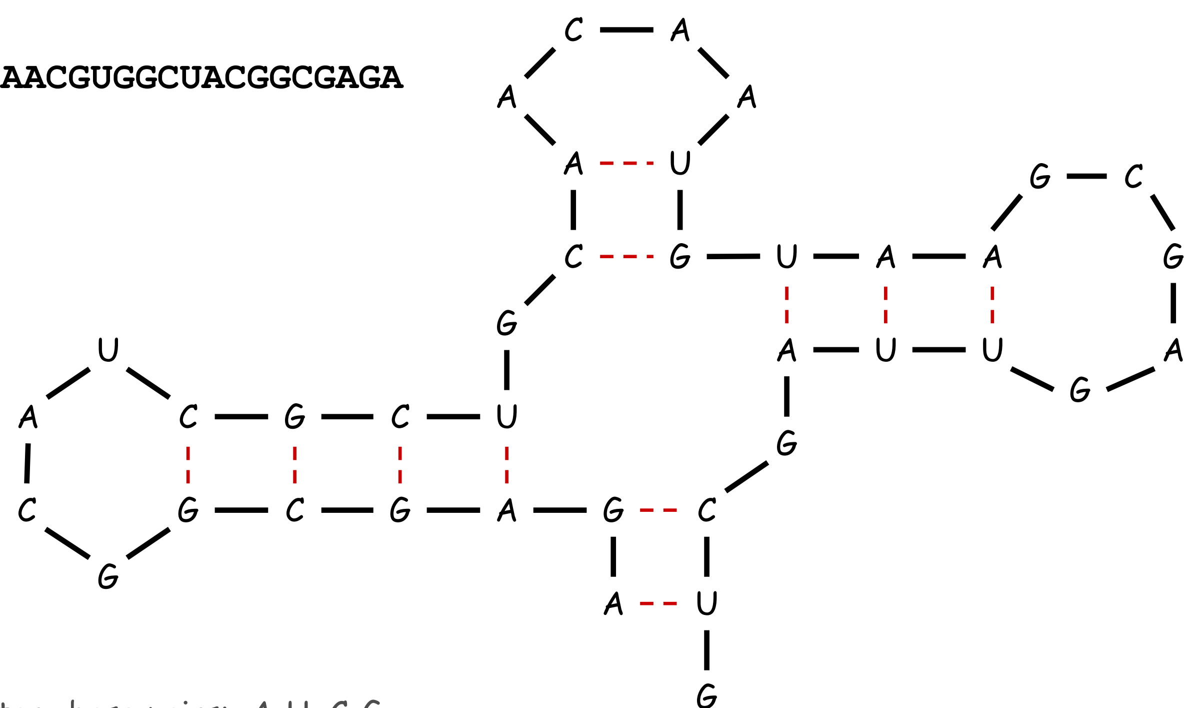
Bioinformatics: methods and software tools for understanding biological data, when the data sets are large and complex.

RNA Secondary Structure

RNA. String B = $b_1 b_2 \ldots b_n$ over alphabet { A, C, G, U }.

Secondary structure. RNA is single-stranded so it tends to loop back and form base pairs with itself. This structure is essential for understanding behavior of molecule.

Ex: **GUCGAUUGAGCGAAUGUAACAACGUGGCUACGGCGAGA**



complementary base pairs: A-U, C-G

RNA Secondary Structure

Secondary structure. A set of pairs $S = \{ (b_i, b_j) \}$ that satisfy:

[Watson-Crick.] S is a matching and each pair in S is a Watson-Crick complement: A-U, U-A, C-G, or G-C.

[No sharp turns.] The ends of each pair are separated by at least 4 intervening bases. If $(b_i, b_j) \in S$, then $i < j - 4$.

[Non-crossing.] If (b_i, b_j) and (b_k, b_l) are two pairs in S, then we cannot have $i < k < j < l$.

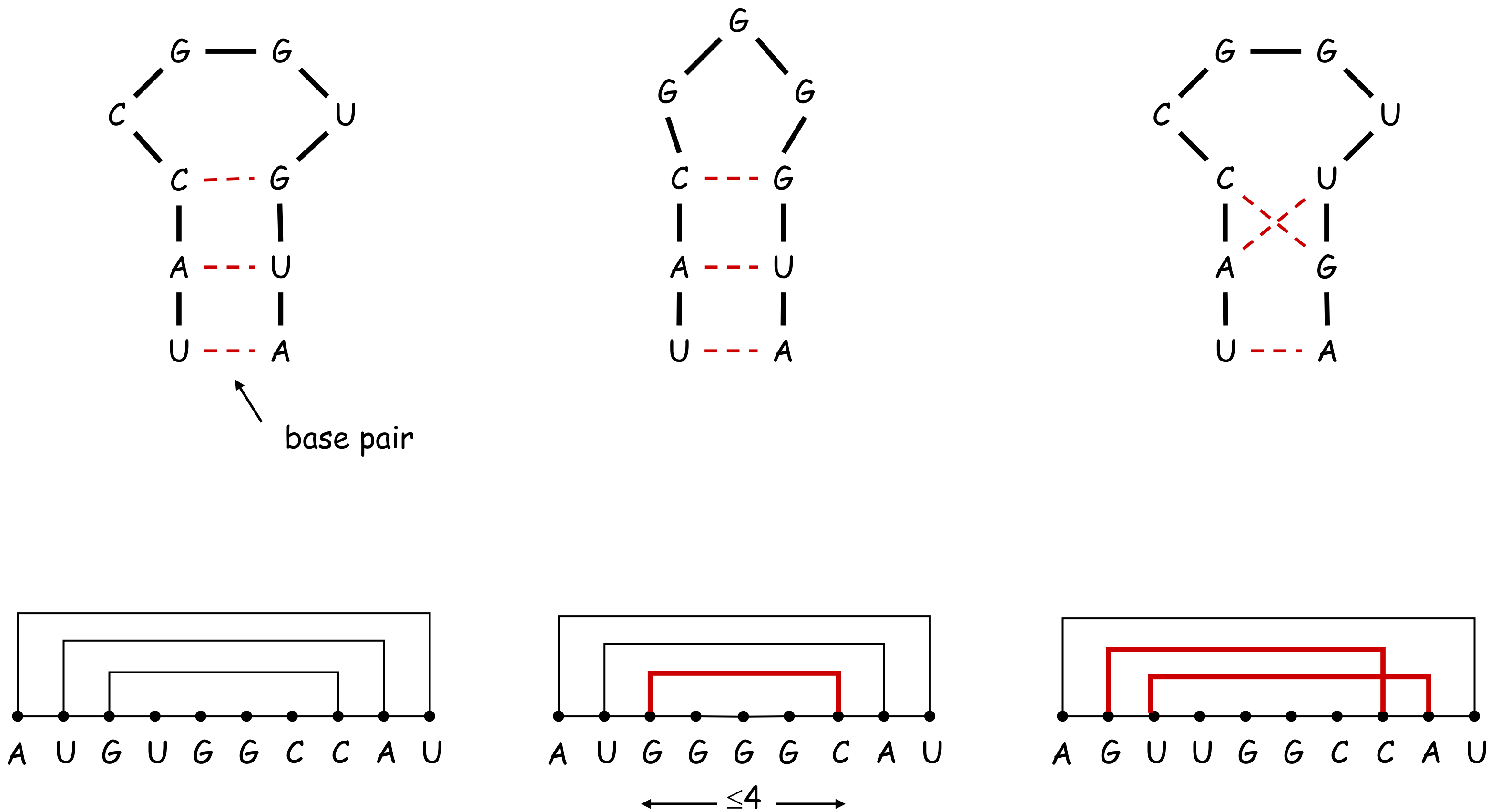
Free energy. Usual hypothesis is that an RNA molecule will form the secondary structure with the optimum total free energy.

approximate by number of base pairs

Goal. Given an RNA molecule $B = b_1 b_2 \ldots b_n$, find a secondary structure S that maximizes the number of base pairs.

RNA Secondary Structure: Examples

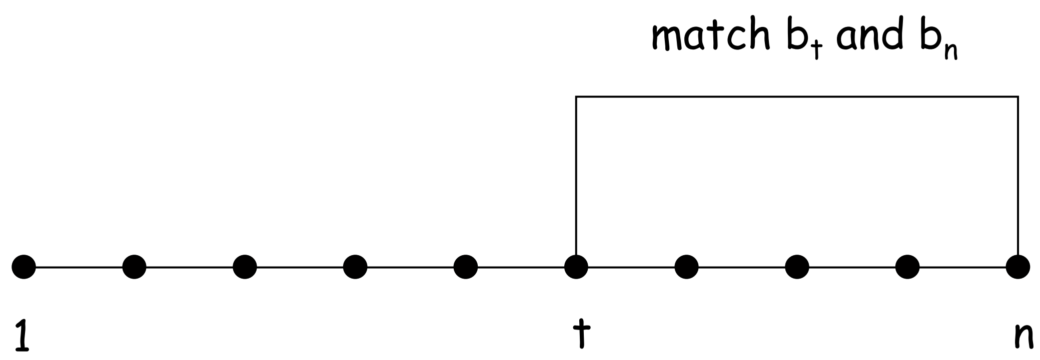
Examples.



ok | sharp turn | crossing

RNA Secondary Structure: Subproblems

First attempt. $OPT(j)$ = maximum number of base pairs in a secondary structure of the substring $b_1b_2 \ldots b_j$.



Difficulty. Results in two sub-problems.

- Finding secondary structure in: $b_1b_2 \ldots b_{t-1}$. ← $OPT(t-1)$
- Finding secondary structure in: $b_{t+1}b_{t+2} \ldots b_{n-1}$. ← need more sub-problems

Dynamic Programming Over Intervals

Notation. $OPT(i, j)$ = maximum number of base pairs in a secondary structure of the substring $b_i b_{i+1} \ldots b_j$.

Case 1. If $i \geq j - 4$.
- $OPT(i, j) = 0$ by no-sharp turns condition.

Case 2. Base b_j is not involved in a pair.
- $OPT(i, j) = OPT(i, j-1)$

Case 3. Base b_j pairs with b_t for some $i \leq t < j - 4$.
- non-crossing constraint decouples resulting sub-problems
- $OPT(i, j) = 1 + \max_t \{ OPT(i, t-1) + OPT(t+1, j-1) \}$

take max over t such that $i \leq t < j-4$ and b_t and b_j are Watson-Crick complements

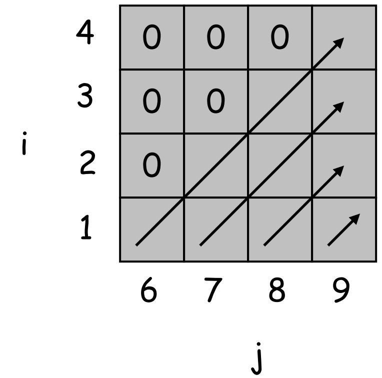
Remark. Same core idea in CKY algorithm to parse context-free grammars.

Bottom Up Dynamic Programming Over Intervals

Q. What order to solve the sub-problems?
A. Do shortest intervals first.

```
RNA(b1,…,bn) {
   for k = 5, 6, …, n-1
      for i = 1, 2, …, n-k
         j = i + k
         Compute M[i, j]

   return M[1, n]
}
```

using recurrence



Running time. $O(n^3)$.

Dynamic Programming Summary

Recipe.

- Characterize structure of problem.
- Recursively define value of optimal solution.
- Compute value of optimal solution.
- Construct optimal solution from computed information.

Dynamic programming techniques.

- Linear sequence with binary choice: weighted interval scheduling.
- Linear sequence with multi-way choice: segmented least squares.
- Adding a new variable: knapsack and shortest path
- Adding a new set: all-pair shortest paths
- All subproblems: sequence of matrix multiplications.
- Multiple subproblems : RNA secondary structure.

Top-down vs. bottom-up: recursion vs. iteration

6.6 Sequence Alignment

Multiple subproblems

Computational biology: development and application of data-analytical and theoretical methods, mathematical modelling and computational simulation techniques to the study of biological, ecological, behavioral, and social systems.

String Similarity

How similar are two strings?

ocurrance

occurrence

o	c	u	r	r	a	n	c	e	–
o	c	c	u	r	r	e	n	c	e

6 mismatches, 1 gap

o	c	–	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

1 mismatch, 1 gap

o	c	–	u	r	r	–	a	n	c	e
o	c	c	u	r	r	e	–	n	c	e

0 mismatches, 3 gaps

Edit Distance

Applications.
- Basis for Unix diff.
- Speech recognition.
- Computational biology.

Edit distance. [Levenshtein 1966, Needleman-Wunsch 1970]
- Gap penalty δ; mismatch penalty α_{pq}.
- Cost = sum of gap and mismatch penalties.

C	T	G	A	C	C	T	A	C	C	T
C	C	T	G	A	C	T	A	C	A	T

$\alpha_{TC} + \alpha_{GT} + \alpha_{AG} + 2\alpha_{CA}$

-	C	T	G	A	C	C	T	A	C	C	T
C	C	T	G	A	C	-	T	A	C	A	T

$2\delta + \alpha_{CA}$

Sequence Alignment

Goal: Given two strings $X = x_1\ x_2 \ldots x_m$ and $Y = y_1\ y_2 \ldots y_n$ find alignment of minimum cost.

Def. An alignment M is a set of ordered pairs x_i-y_j such that each item occurs in at most one pair and no crossings.

Def. The pair x_i-y_j and $x_{i'}$-$y_{j'}$ cross if $i < i'$, but $j > j'$.

$$\text{cost}(M) = \underbrace{\sum_{(x_i, y_j) \in M} \alpha_{x_i y_j}}_{\text{mismatch}} + \underbrace{\sum_{i : x_i \text{ unmatched}} \delta + \sum_{j : y_j \text{ unmatched}} \delta}_{\text{gap}}$$

Ex: CTACCG vs. TACATG.

Sol: $M = x_2$-y_1, x_3-y_2, x_4-y_3, x_5-y_4, x_6-y_6.

x_1	x_2	x_3	x_4	x_5		x_6
C	T	A	C	C	-	G
-	T	A	C	A	T	G
	y_1	y_2	y_3	y_4	y_5	y_6

Sequence Alignment: Problem Structure

Def. OPT(i, j) = min cost of aligning strings $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$.

Case 1: OPT matches x_i-y_j.

- pay mismatch for x_i-y_j + min cost of aligning two strings $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_{j-1}$

Case 2a: OPT leaves x_i unmatched.

- pay gap for x_i and min cost of aligning $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_j$

Case 2b: OPT leaves y_j unmatched.

- pay gap for y_j and min cost of aligning $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_{j-1}$

$$OPT(i,j) = \begin{cases} j\delta & \text{if } i = 0 \\ \min \begin{cases} \alpha_{x_i y_j} + OPT(i-1, j-1) \\ \delta + OPT(i-1, j) \\ \delta + OPT(i, j-1) \end{cases} & \text{otherwise} \\ i\delta & \text{if } j = 0 \end{cases}$$

Sequence Alignment: Algorithm

```
Sequence-Alignment(m, n, x1x2...xm, y1y2...yn, δ, α) {
   for i = 0 to m
      M[i, 0] = iδ
   for j = 0 to n
      M[0, j] = jδ

   for i = 1 to m
      for j = 1 to n
         M[i, j] = min(α[xi, yj] + M[i-1, j-1],
                       δ + M[i-1, j],
                       δ + M[i, j-1])
   return M[m, n]
}
```

Analysis. $\Theta(mn)$ time and space.

English words or sentences: m, n $\leq$ 10.

Computational biology: m = n = 100,000. 10 billions ops OK, but 10GB array?

6.7 Sequence Alignment in Linear Space

Dynamic programming combined with divide-and-conquer

Sequence Alignment: Linear Space

Q. Can we avoid using quadratic space?

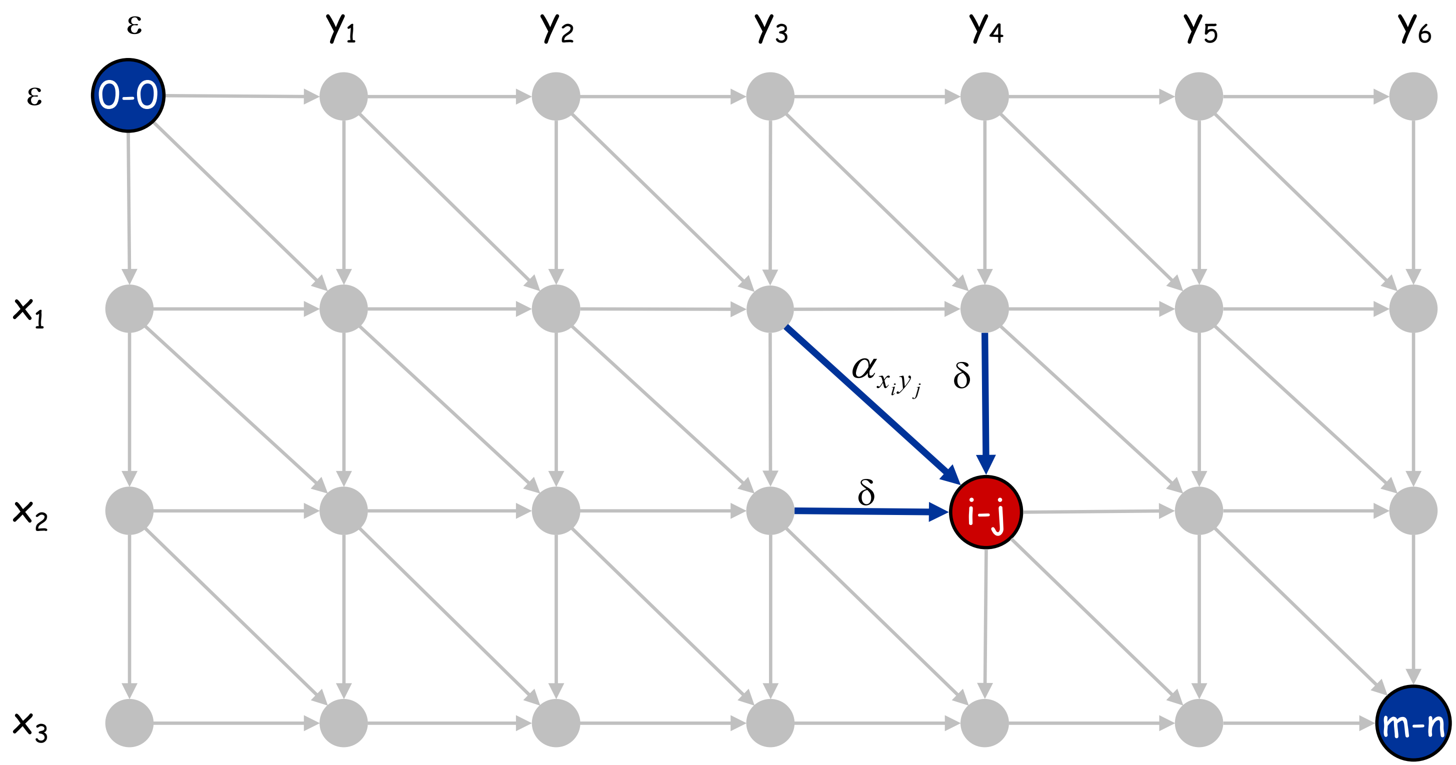
Easy. Optimal value in O(m + n) space and O(mn) time.
- Compute OPT(i, •) from OPT(i-1, •).
- No longer a simple way to recover alignment itself.

Theorem. [Hirschberg 1975] Optimal alignment in O(m + n) space and O(mn) time.
- Clever combination of divide-and-conquer and dynamic programming.
- Inspired by idea of Savitch from complexity theory.

Sequence Alignment: Linear Space

Edit distance graph.

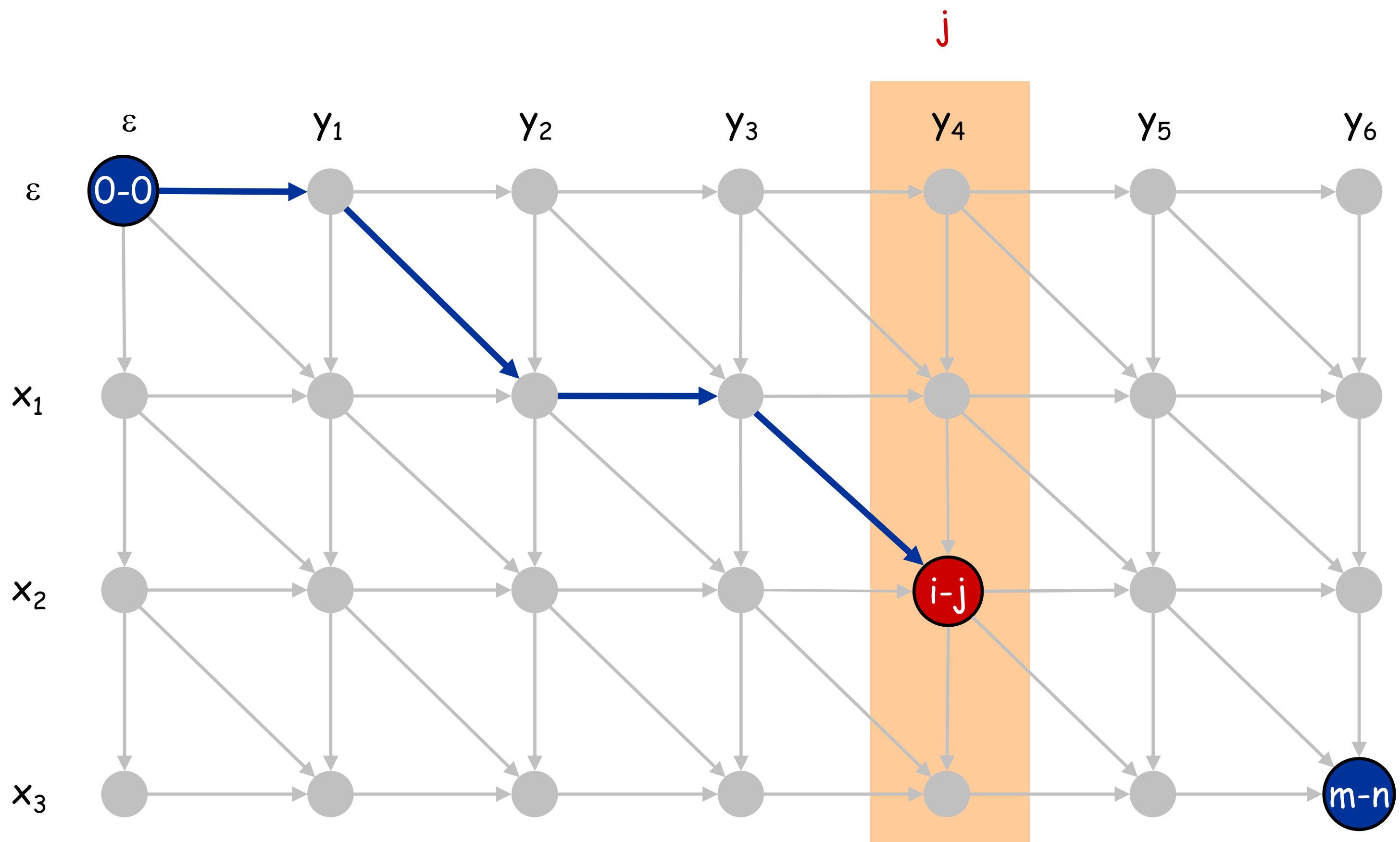
- Let f(i, j) be shortest path from (0,0) to (i, j).
- Observation: f(i, j) = OPT(i, j).



Sequence Alignment: Linear Space

Edit distance graph.

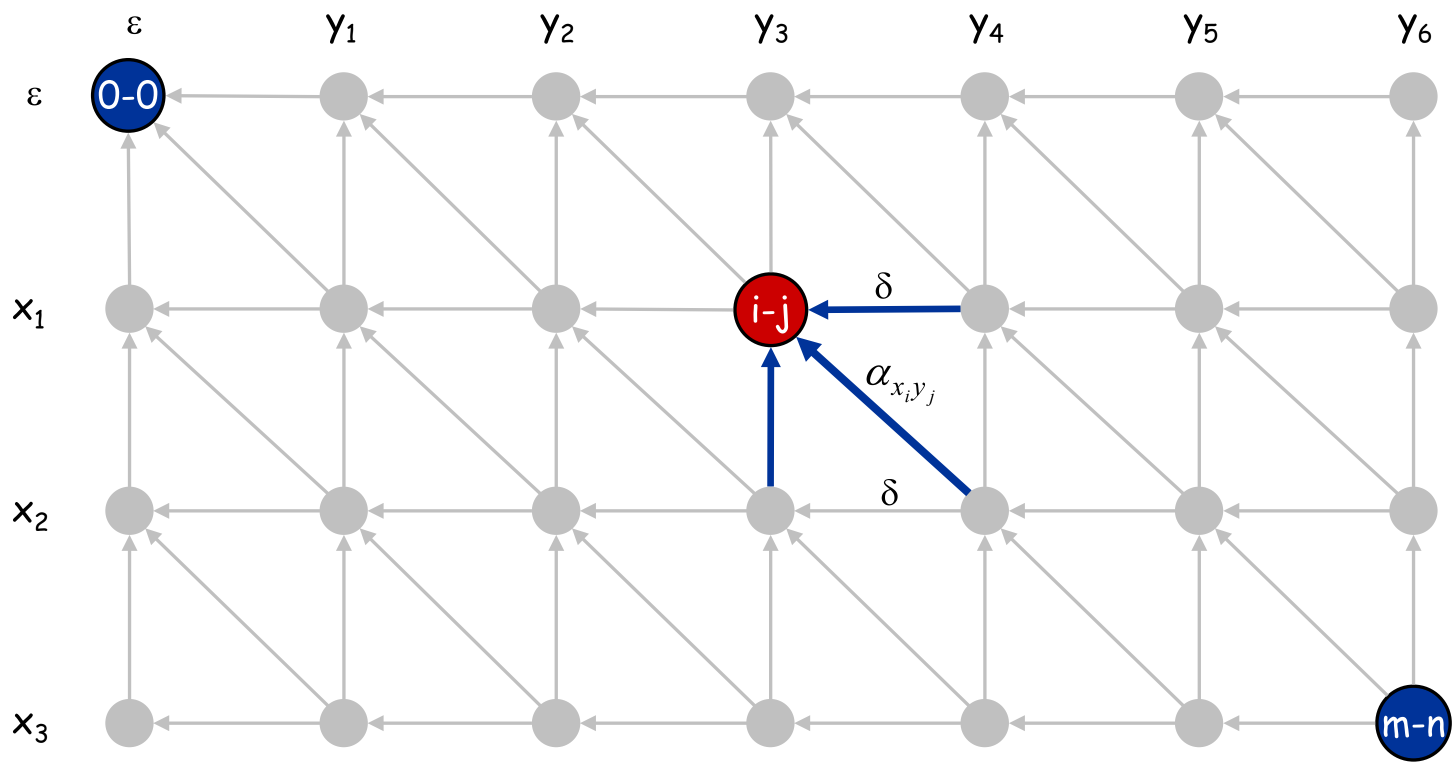
- Let f(i, j) be shortest path from (0,0) to (i, j).
- Can compute f (•, j) for any j in O(mn) time and O(m + n) space.



Sequence Alignment: Linear Space

Edit distance graph.

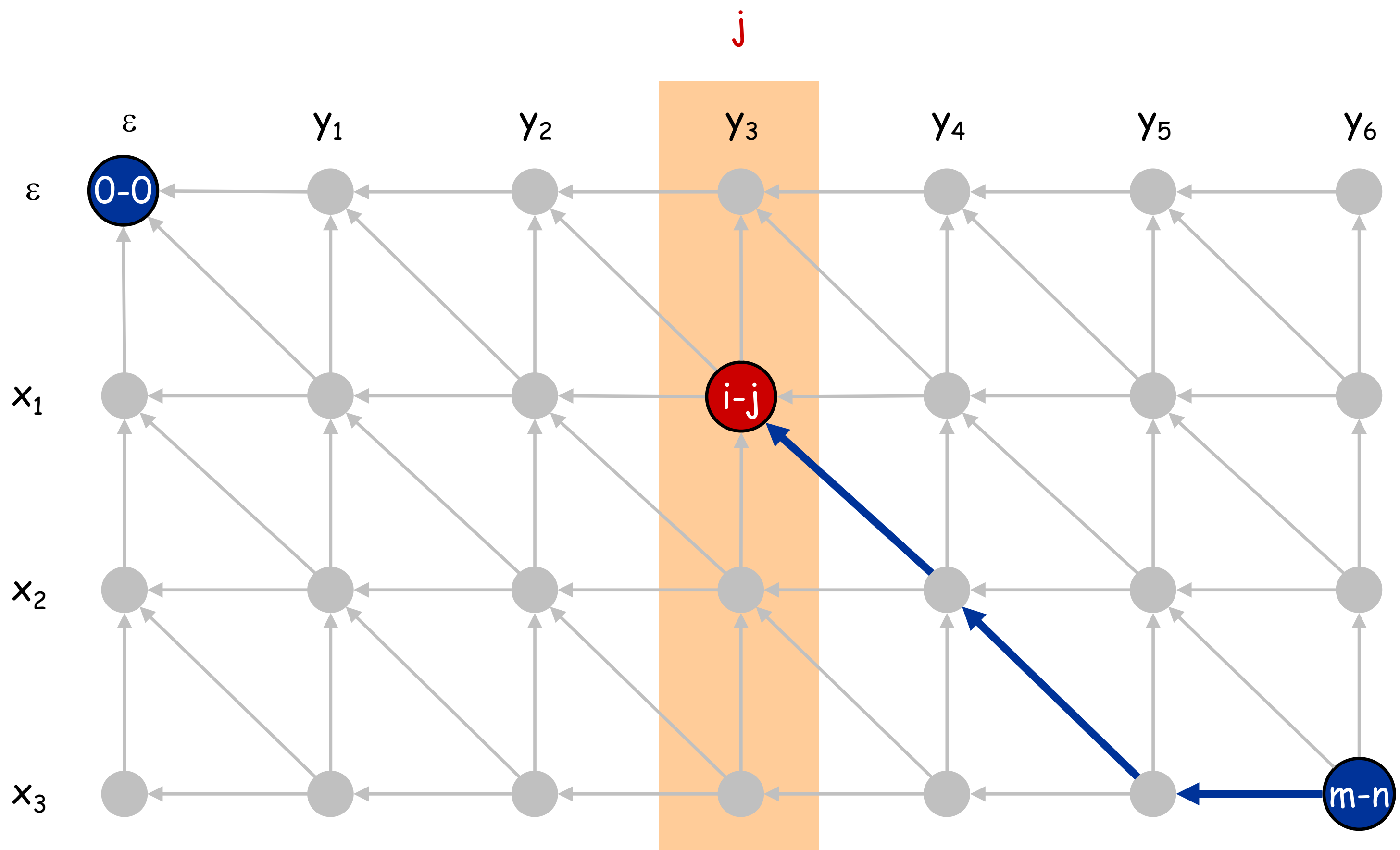
- Let $g(i, j)$ be shortest path from (i, j) to (m, n).
- Can compute by reversing the edge orientations and inverting the roles of $(0, 0)$ and (m, n)



Sequence Alignment: Linear Space

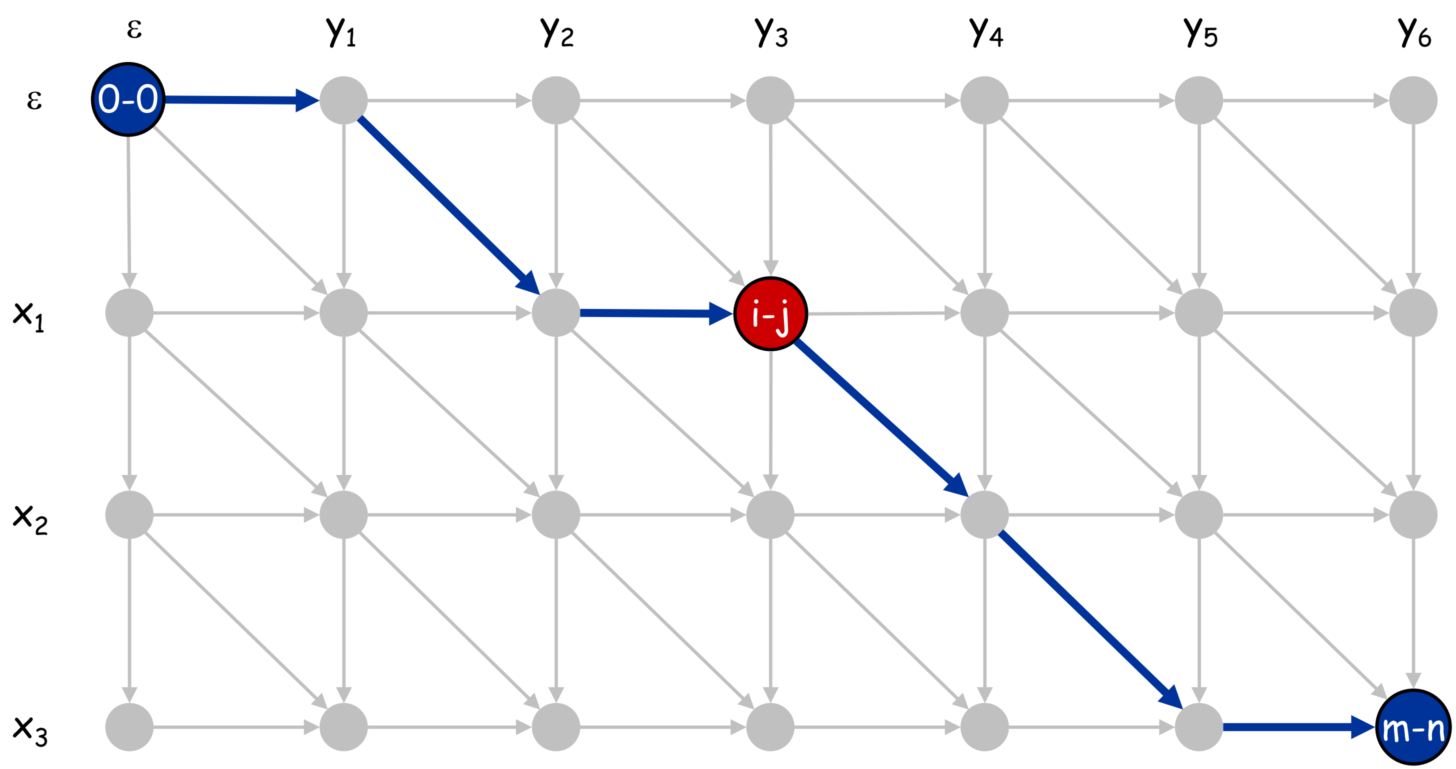
Edit distance graph.

- Let $g(i, j)$ be shortest path from (i, j) to (m, n).
- Can compute $g(\bullet, j)$ for any j in $O(mn)$ time and $O(m + n)$ space.



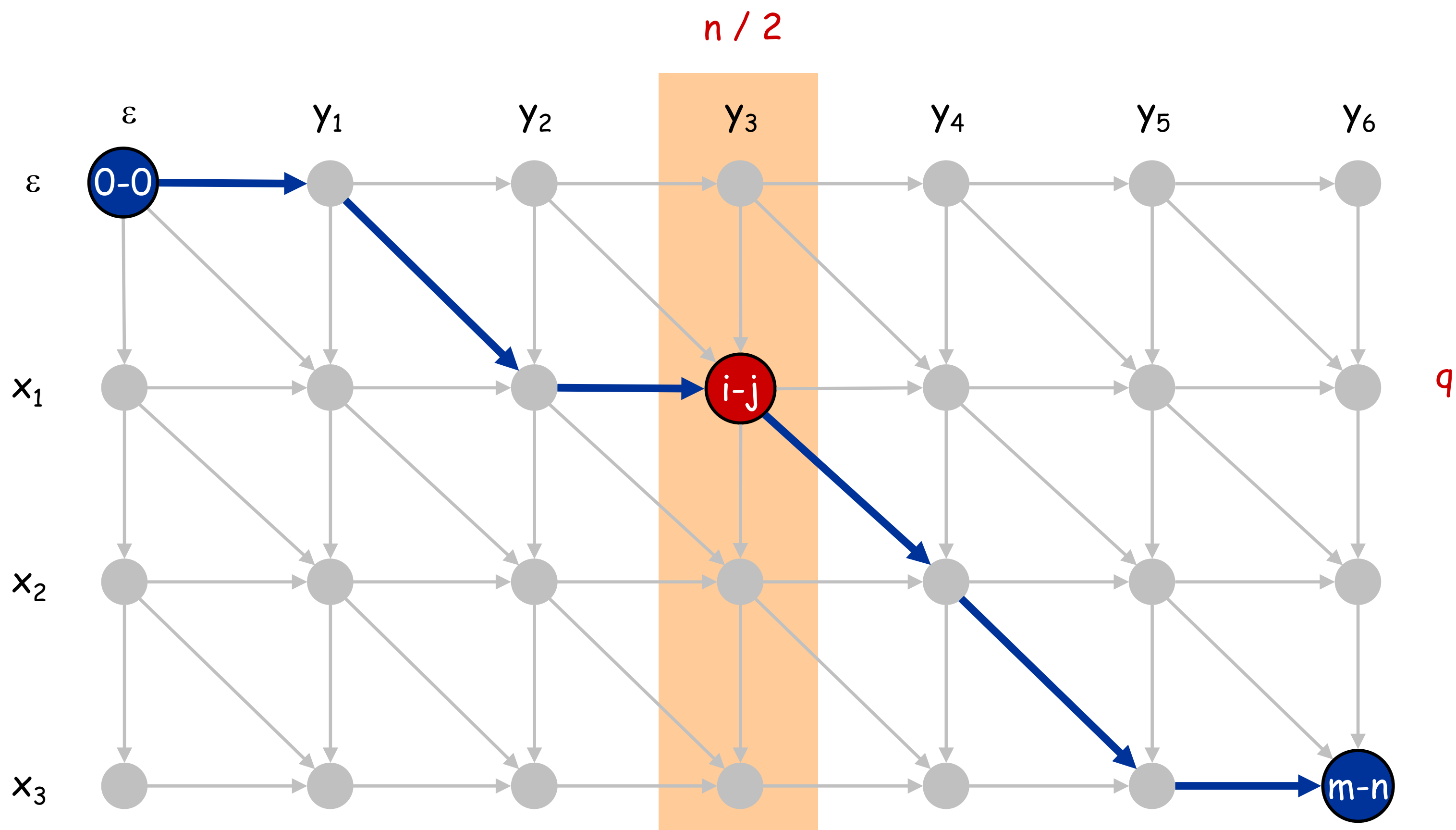
Sequence Alignment: Linear Space

Observation 1. The cost of the shortest path that uses (i, j) is $f(i, j) + g(i, j)$.



Sequence Alignment: Linear Space

Observation 2. let q be an index that minimizes f(q, n/2) + g(q, n/2). Then, the shortest path from (0, 0) to (m, n) uses (q, n/2).



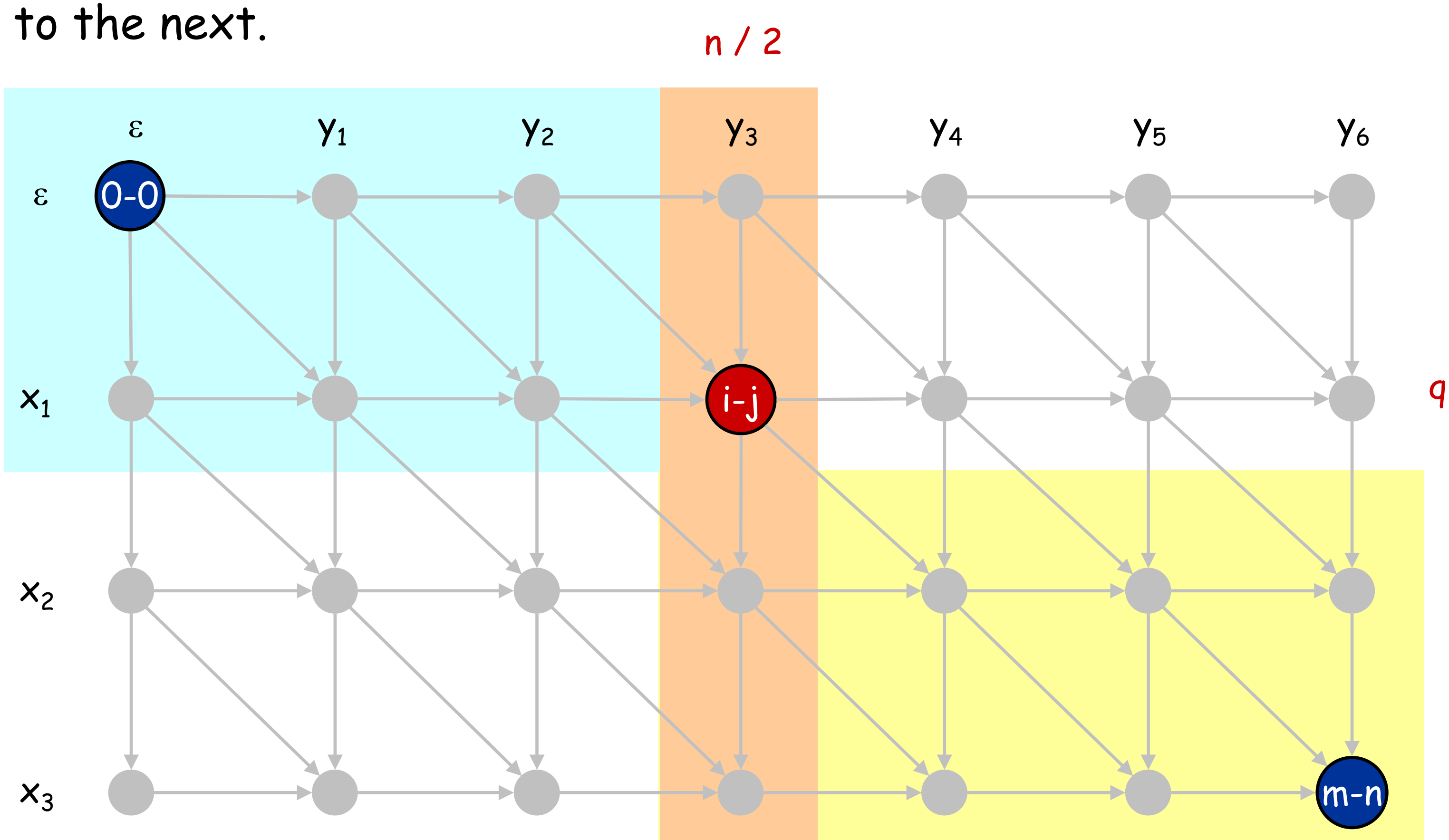
Sequence Alignment: Linear Space

Divide: find index q that minimizes $f(q, n/2) + g(q, n/2)$ using DP.

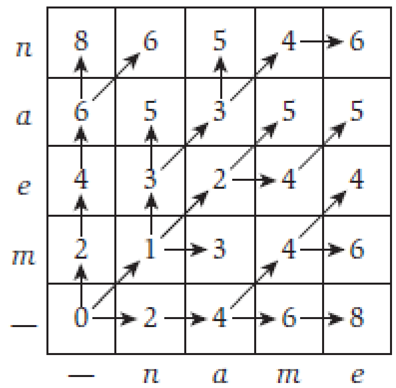
Align x_q and $y_{n/2}$.

Conquer: recursively compute optimal alignment in each piece.

Apply recursive calls sequentially and reuse the working space from one call to the next.



Example: match "mean" with "name"
gap: 2, mismatch: 1 or 3 (vowel with consonant)




```
Divide-and-Conquer-Alignment(X,Y)
  Let m be the number of symbols in X
  Let n be the number of symbols in Y
  If m ≤ 2 or n ≤ 2 then
     Compute optimal alignment using Alignment(X,Y)
  Call Space-Efficient-Alignment(X,Y[1:n/2])
  Call Backward-Space-Efficient-Alignment(X,Y[n/2+1:n])
  Let q be the index minimizing f(q,n/2)+g(q,n/2)
  Add (q,n/2) to global list P
  Divide-and-Conquer-Alignment(X[1:q],Y[1:n/2])
  Divide-and-Conquer-Alignment(X[q+1:n],Y[n/2+1:n])
  Return P
```

Theorem. Let T(m, n) = max running time of algorithm on strings of length at most m and n. T(m, n) = O(mn log n).

$$T(m,n) \;\le\; 2T(m,\, n/2) + O(mn) \;\;\Rightarrow\;\; T(m,n) \;=\; O(mn \log n)$$

Remark. Analysis is not tight because two sub-problems are of size (q, n/2) and (m - q, n/2). In next slide, we save log n factor.

Sequence Alignment: Running Time Analysis

Theorem. Let T(m, n) = max running time of algorithm on strings of length m and n. T(m, n) = O(mn).

Pf. (by induction on n)

- O(mn) time to compute f(•, n/2) and g (•, n/2) and find index q.
- T(q, n/2) + T(m - q, n/2) time for two recursive calls.
- Choose constant c so that:

$$
\begin{aligned}
T(m,\ 2) &\leq cm \\
T(2,\ n) &\leq cn \\
T(m,\ n) &\leq cmn + T(q,\ n/2) + T(m-q,\ n/2)
\end{aligned}
$$

- Base cases: m = 2 or n = 2.
- Inductive hypothesis: T(m, n) ≤ 2cmn.

$$
\begin{aligned}
T(m,n) &\leq T(q,n/2) + T(m-q,n/2) + cmn \\
&\leq 2cqn/2 + 2c(m-q)n/2 + cmn \\
&= cqn + cmn - cqn + cmn \\
&= 2cmn
\end{aligned}
$$

Parsimony theory

Principle of parsimony
- A theory should provide the simplest possible explanation for a phenomenon.

Occam's razor

The simplest of two competing theories is to be preferred.

The KISS principle

Keep in Simple, Stupid!

Good theory

Exhibits an aesthetic quality, that a good theory is beautiful or natural.

Examples

Dijkstra, "Self-stabilizing systems in spite of distributed control", Comm. of the ACM, 17 (11): 643–644, 1974.

Kleinberg, "Navigation in a small world", Nature, 406 (6798): 845. 2000.

Searching for the Simplest Solution!

House connections in a village: n houses are connected using cables and switches to form a tree, where interior nodes are switches.

1. Find two houses that are the farthest apart in the connection, assuming each cable section has a different length.

1. Suppose each household has an occupancy limit and each cable section has bandwidth limit. Links should support all possible simultaneous pairwise telephone conversations (unit bandwidth) between houses (i.e., hose model). What is the schedule of m (> n) persons to houses with the maximum elasticity for future grow (i.e., maximum uniform growth in occupancy)?

Example: n=5 and m=8