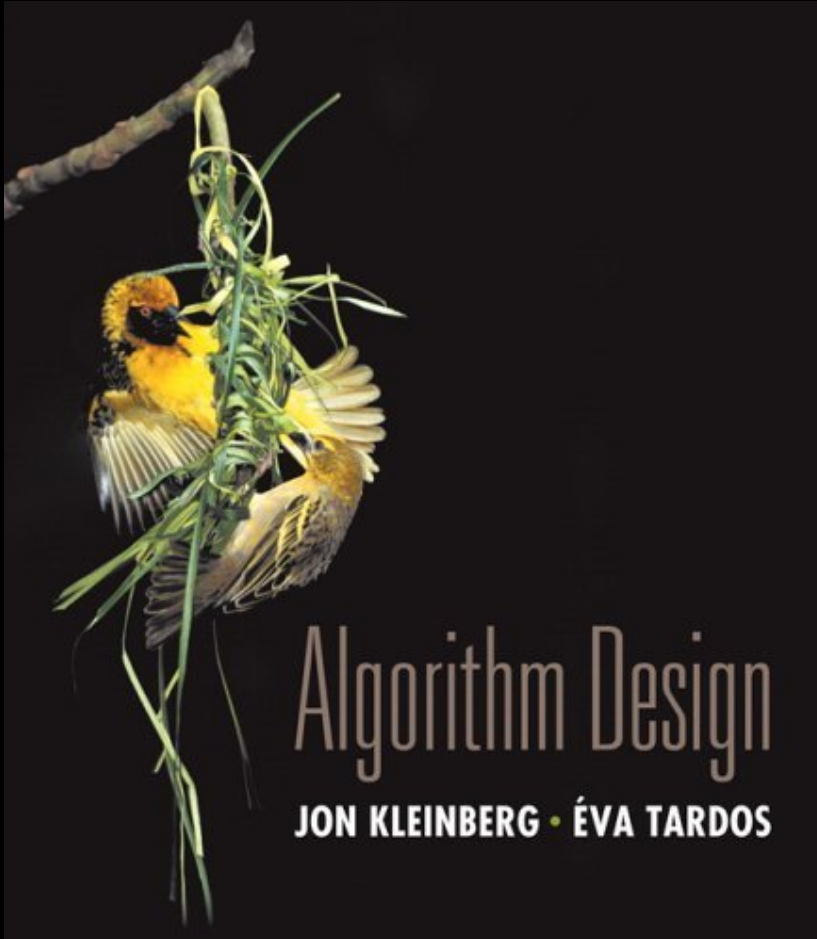
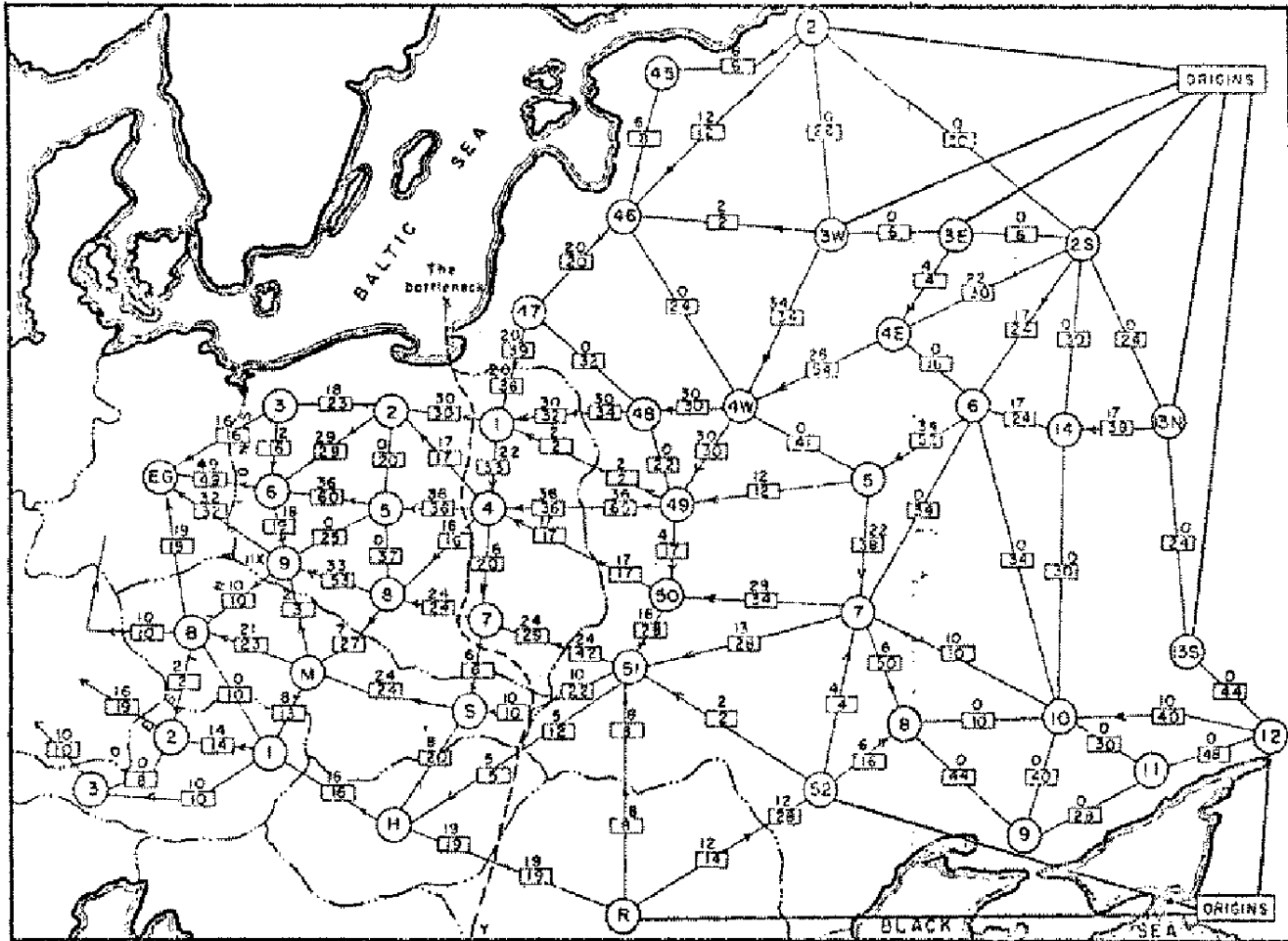


Chapter 7

Network Flow



Soviet Rail Network, 1955



Reference: *On the history of the transportation and maximum flow problems.*
Alexander Schrijver in *Math Programming*, 91: 3, 2002.

Maximum Flow and Minimum Cut

Max flow and min cut.

Two very rich algorithmic problems.

Cornerstone problems in combinatorial optimization.

Beautiful mathematical duality.

Nontrivial applications / reductions.

Data mining.

Open-pit mining.

Project selection.

Airline scheduling.

Bipartite matching.

Baseball elimination.

Image segmentation.

Network connectivity.

Network reliability.

Distributed computing.

Egalitarian stable matching.

Security of statistical data.

Network intrusion detection.

Multi-camera scene reconstruction.

Many many more ...

Minimum Cut Problem

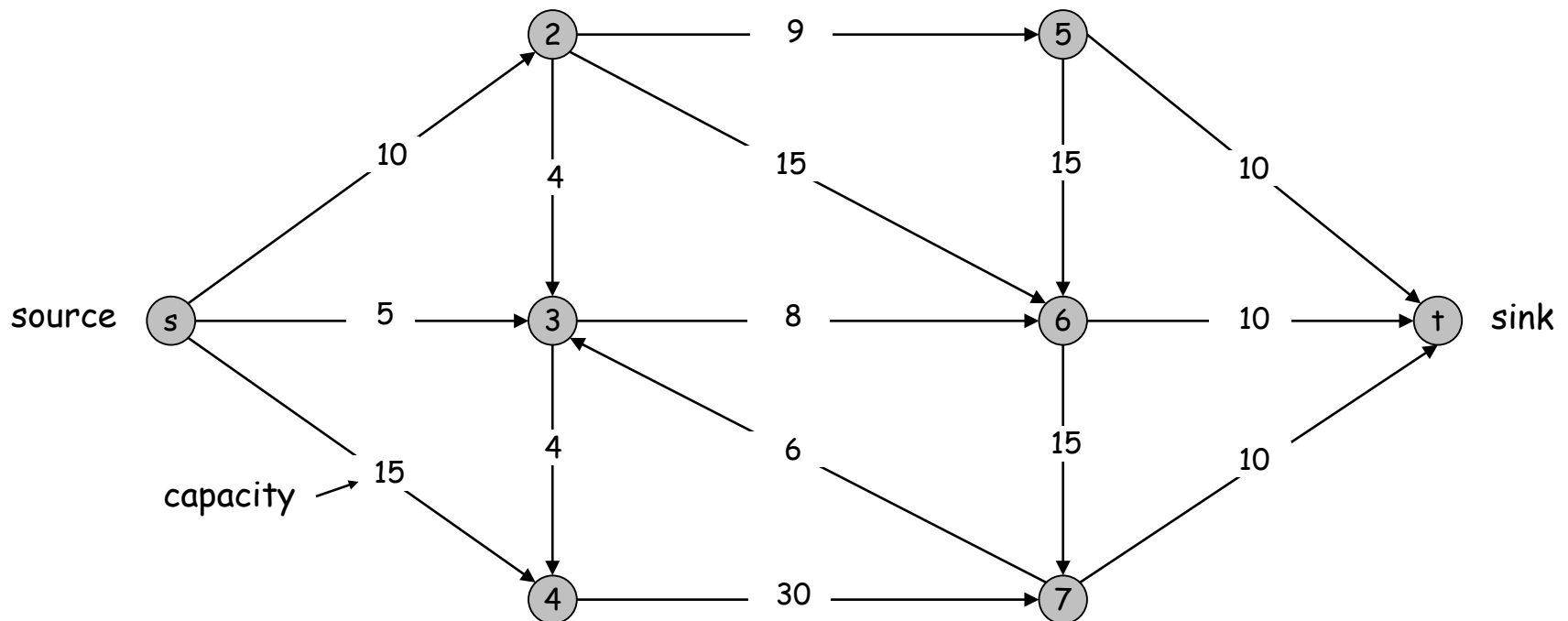
Flow network.

Abstraction for material **flowing** through the edges.

$G = (V, E)$ = directed graph, no parallel edges.

Two distinguished nodes: s = source, t = sink.

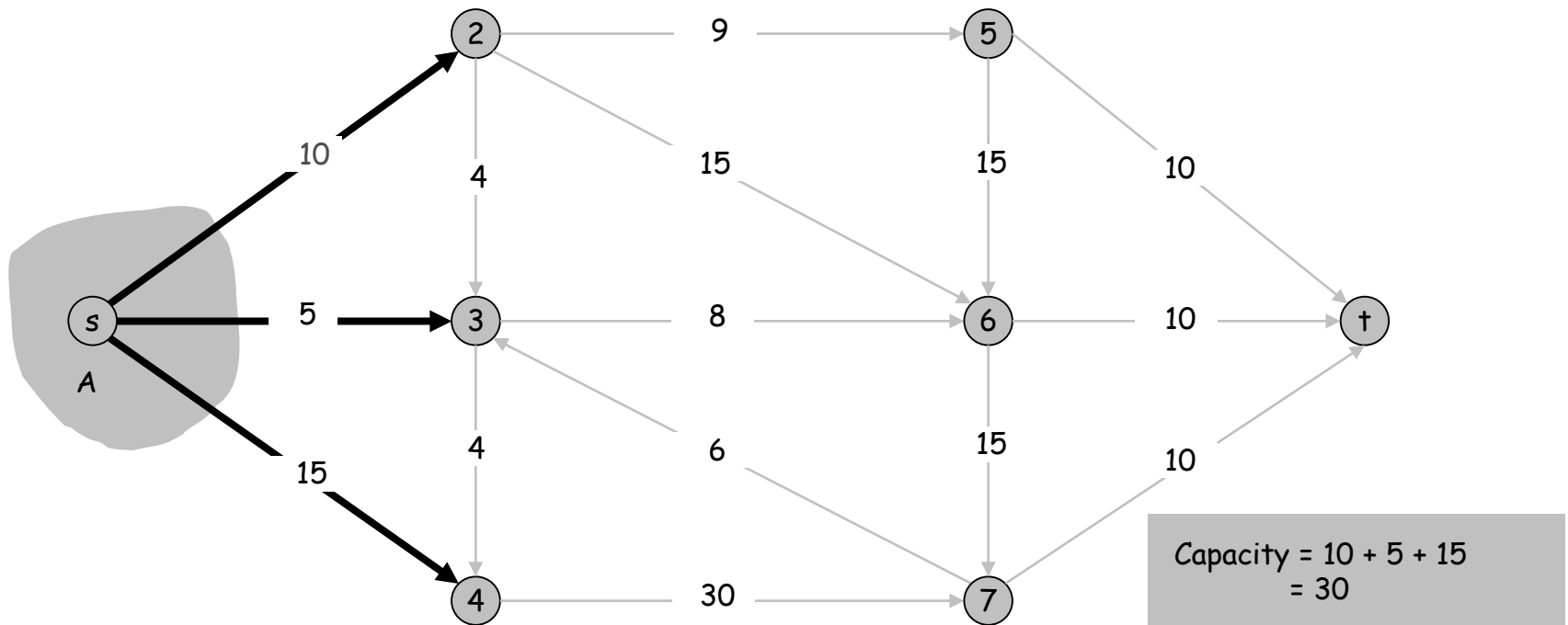
$c(e)$ = capacity of edge e .



Cuts

Def. An **s-t cut** is a partition (A, B) of V with $s \in A$ and $t \in B$.

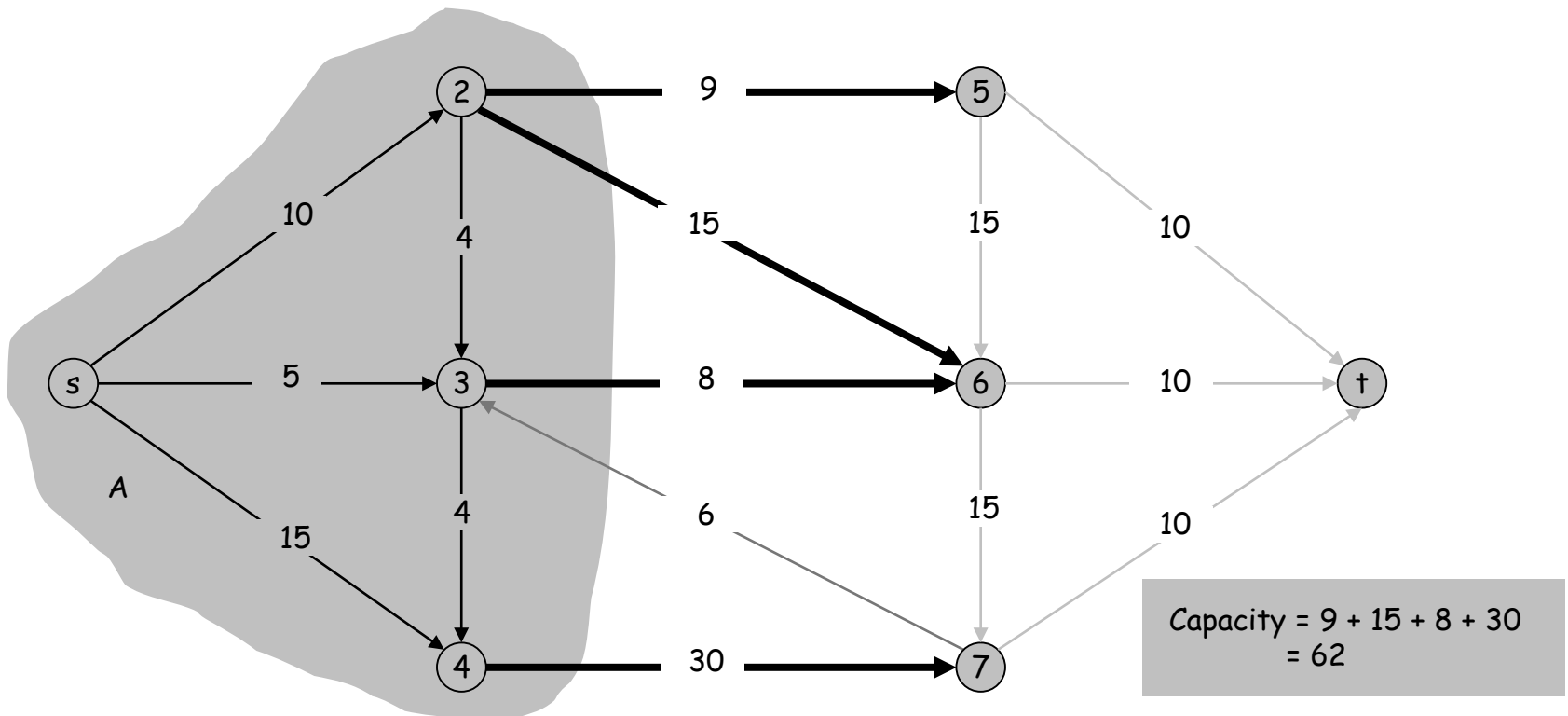
Def. The **capacity** of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



Cuts

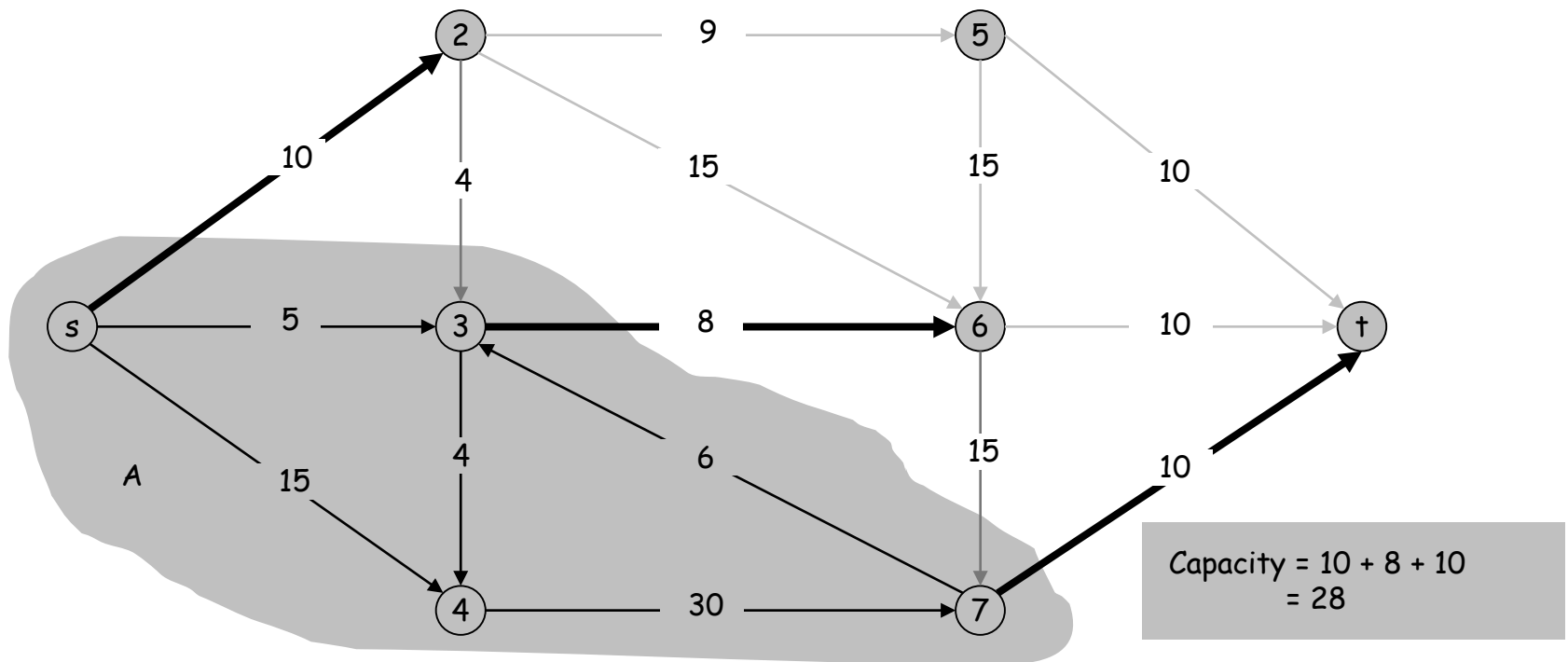
Def. An **s-t cut** is a partition (A, B) of V with $s \in A$ and $t \in B$.

Def. The **capacity** of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



Minimum Cut Problem

Min s-t cut problem. Find an s-t cut of minimum capacity.



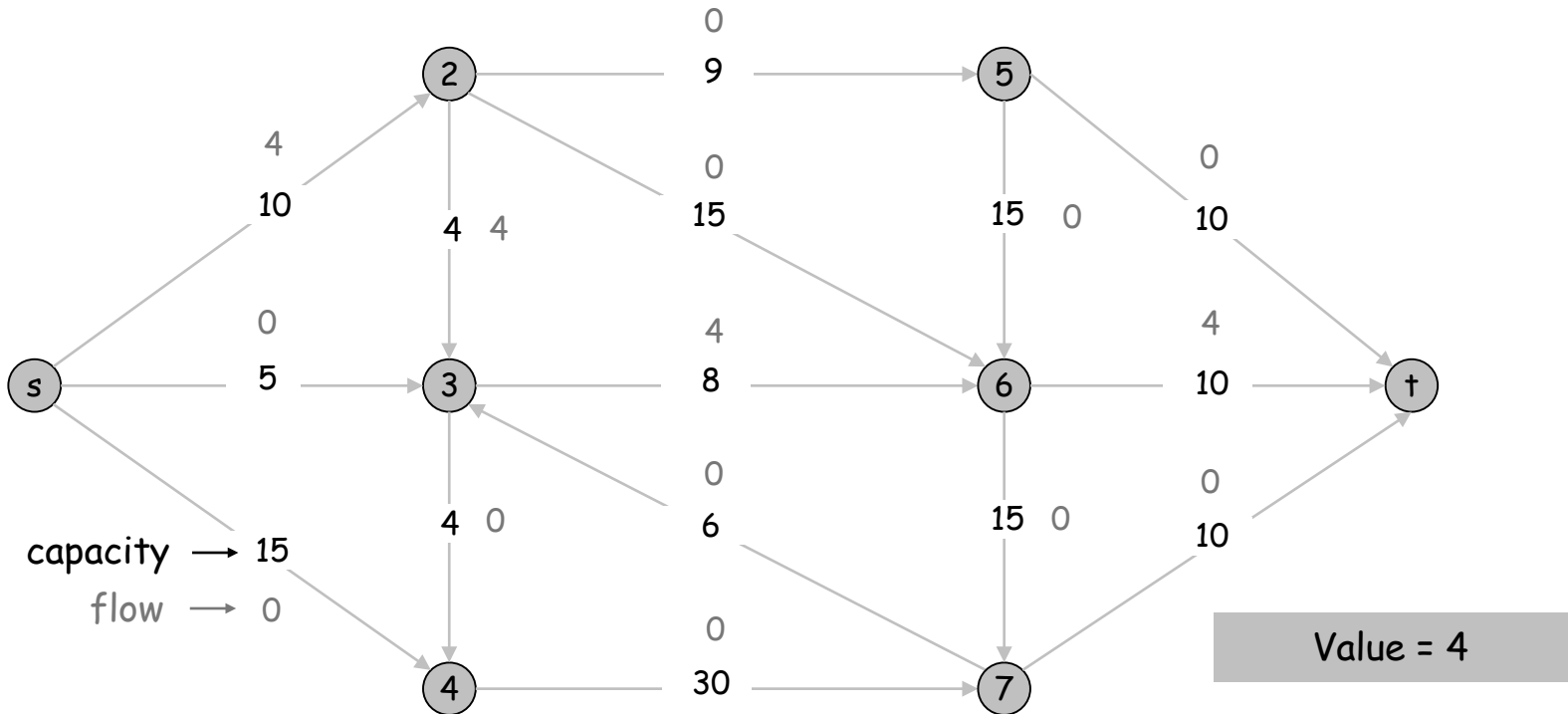
Flows

Def. An **s-t flow** is a function that satisfies:

For each $e \in E$: $0 \leq f(e) \leq c(e)$ [capacity]

For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [conservation]

Def. The **value** of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.



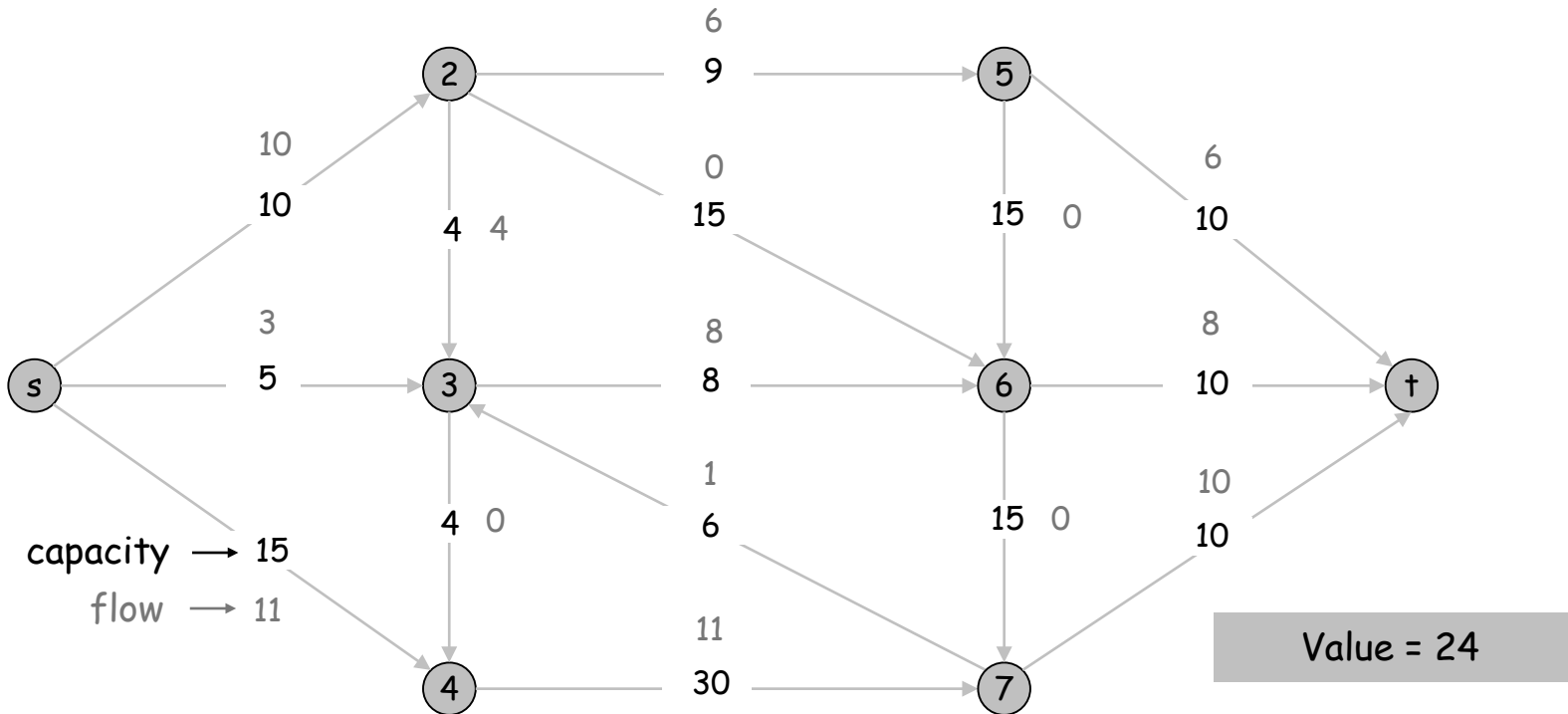
Flows

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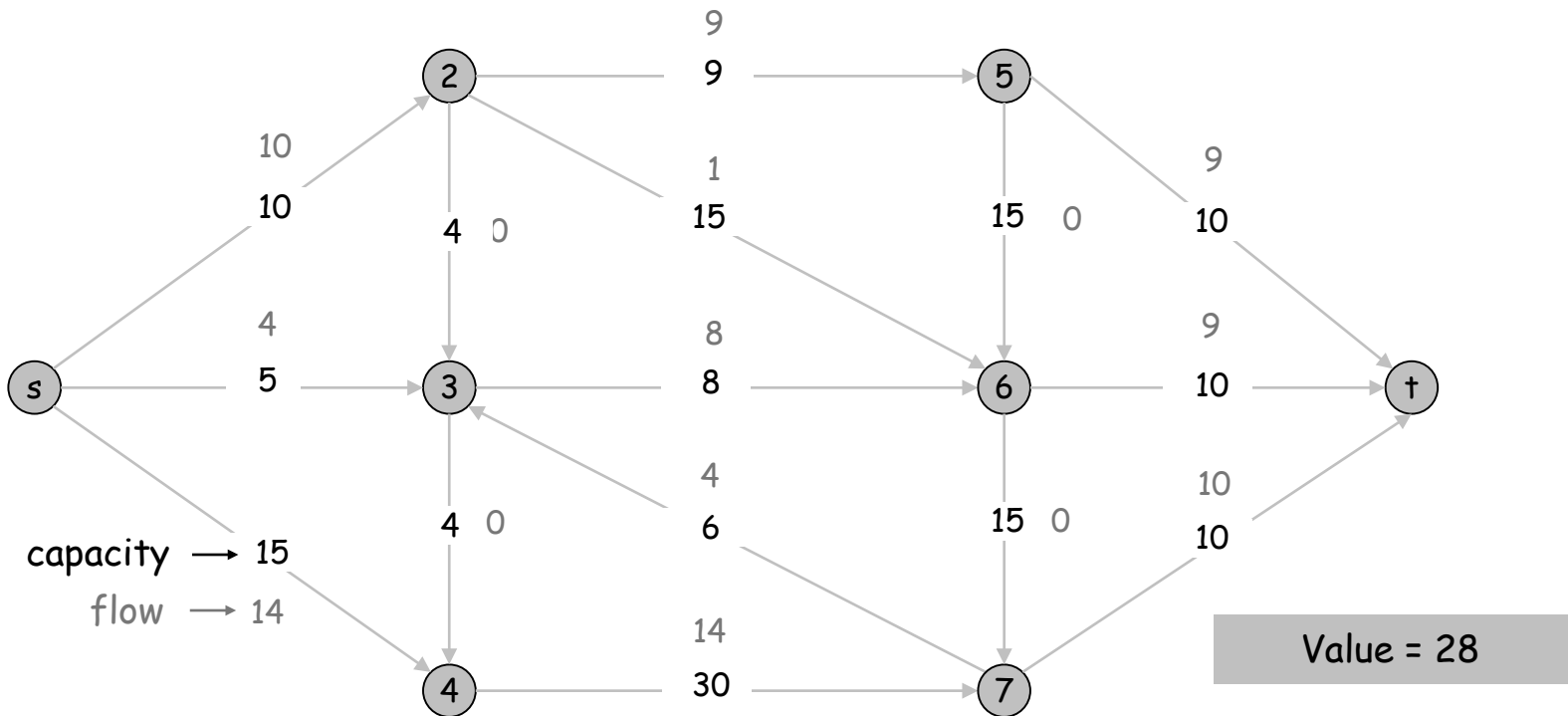
For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [conservation]

Def. The **value** of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.



Maximum Flow Problem

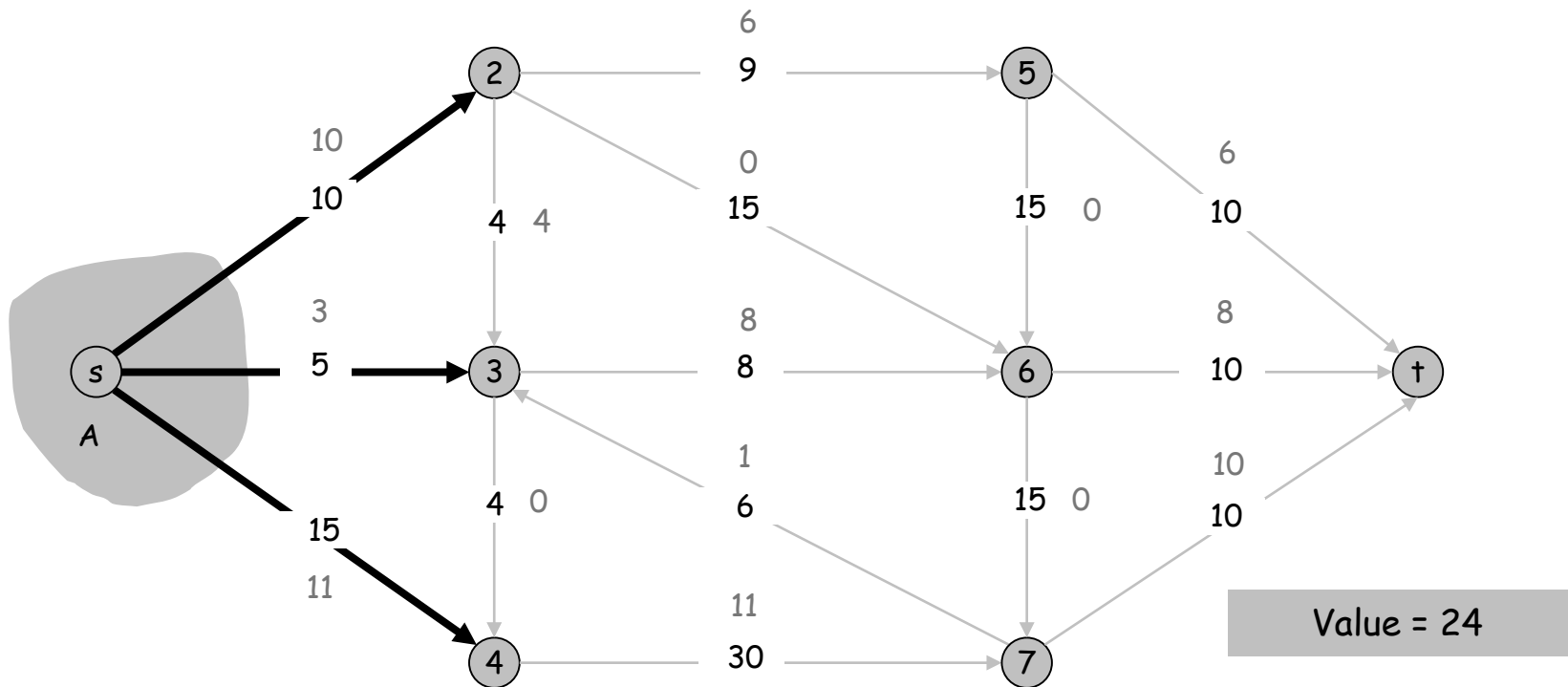
Max flow problem. Find s-t flow of maximum value.



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s - t cut. Then, the net flow sent across the cut is equal to the amount leaving s .

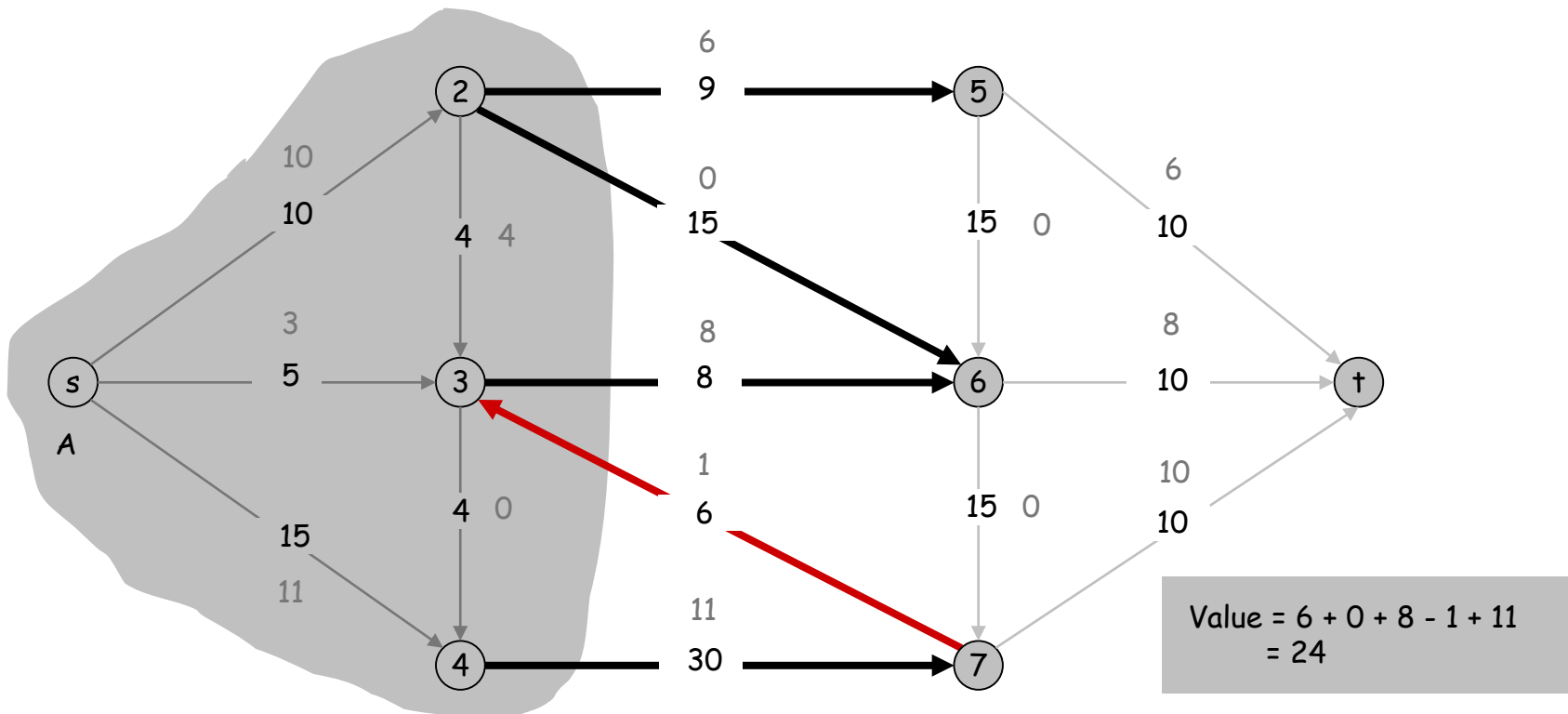
$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s - t cut. Then, the net flow sent across the cut is equal to the amount leaving s .

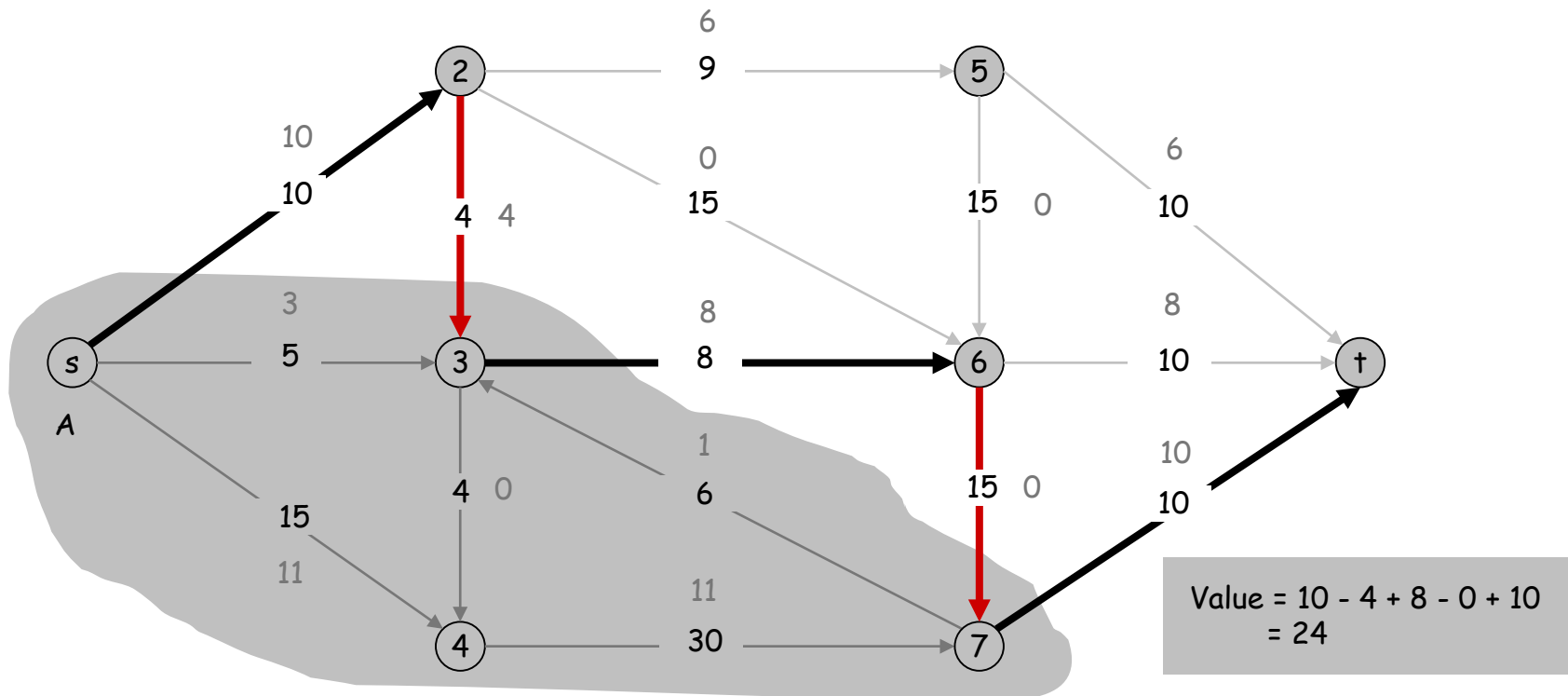
$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s - t cut. Then, the net flow sent across the cut is equal to the amount leaving s .

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s - t cut. Then

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$$

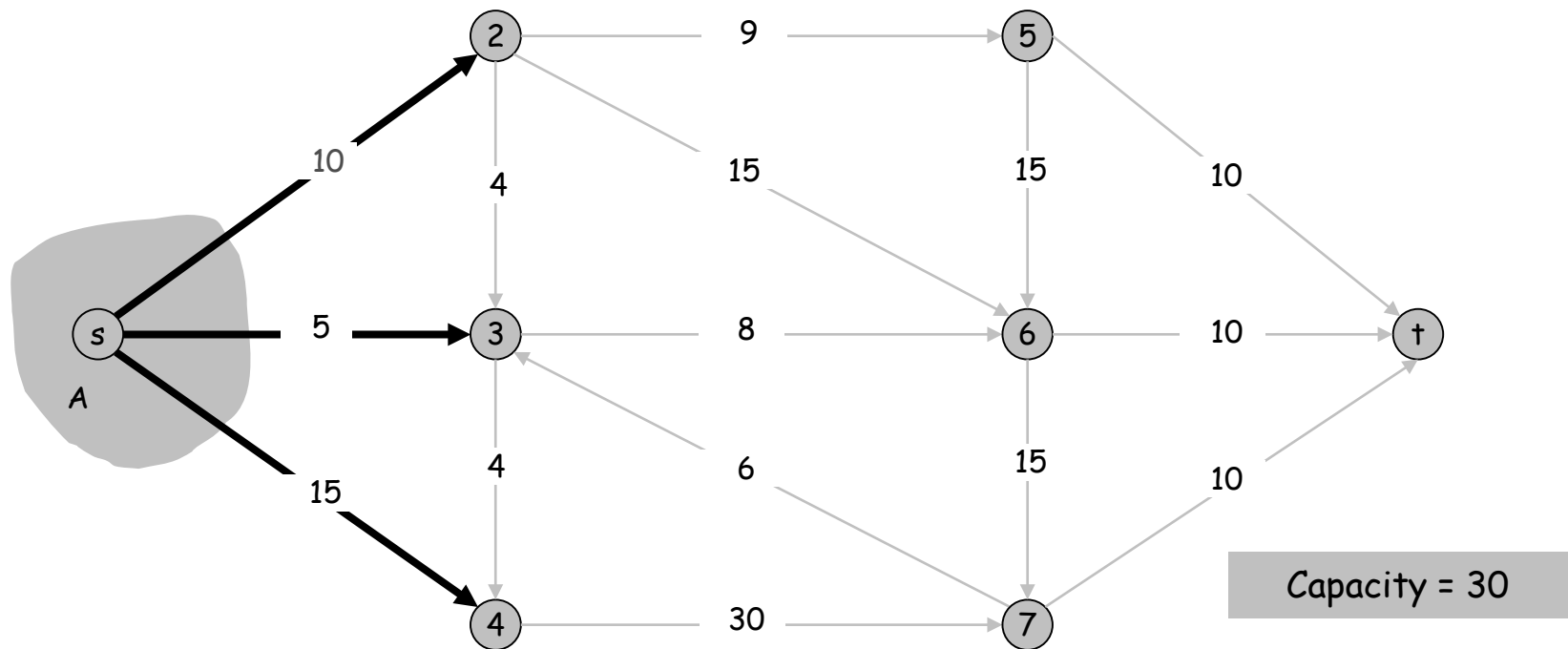
Pf.

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } s} f(e) \\ \text{by flow conservation, all terms} &\rightarrow = \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right) \\ \text{except } v = s \text{ are } 0 & \\ &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e). \end{aligned}$$

Flows and Cuts

Weak duality. Let f be any flow, and let (A, B) be any s - t cut. Then the value of the flow is at most the capacity of the cut.

Cut capacity = 30 \Rightarrow Flow value \leq 30

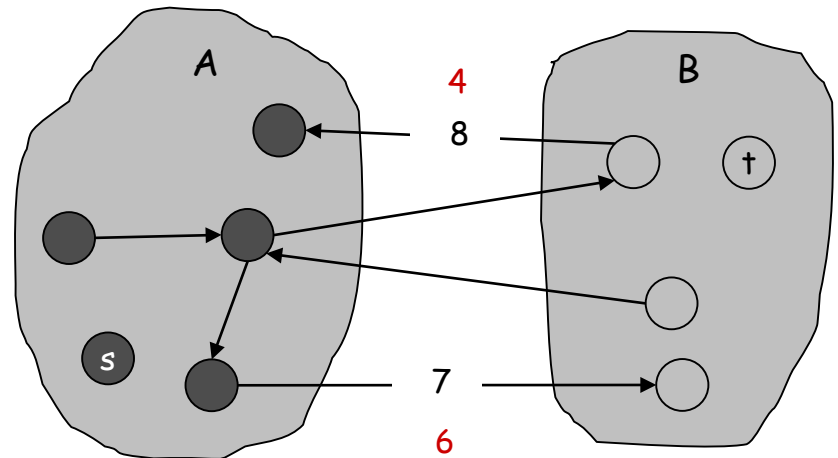


Flows and Cuts

Weak duality. Let f be any flow. Then, for any s - t cut (A, B) we have $v(f) \leq \text{cap}(A, B)$.

Pf.

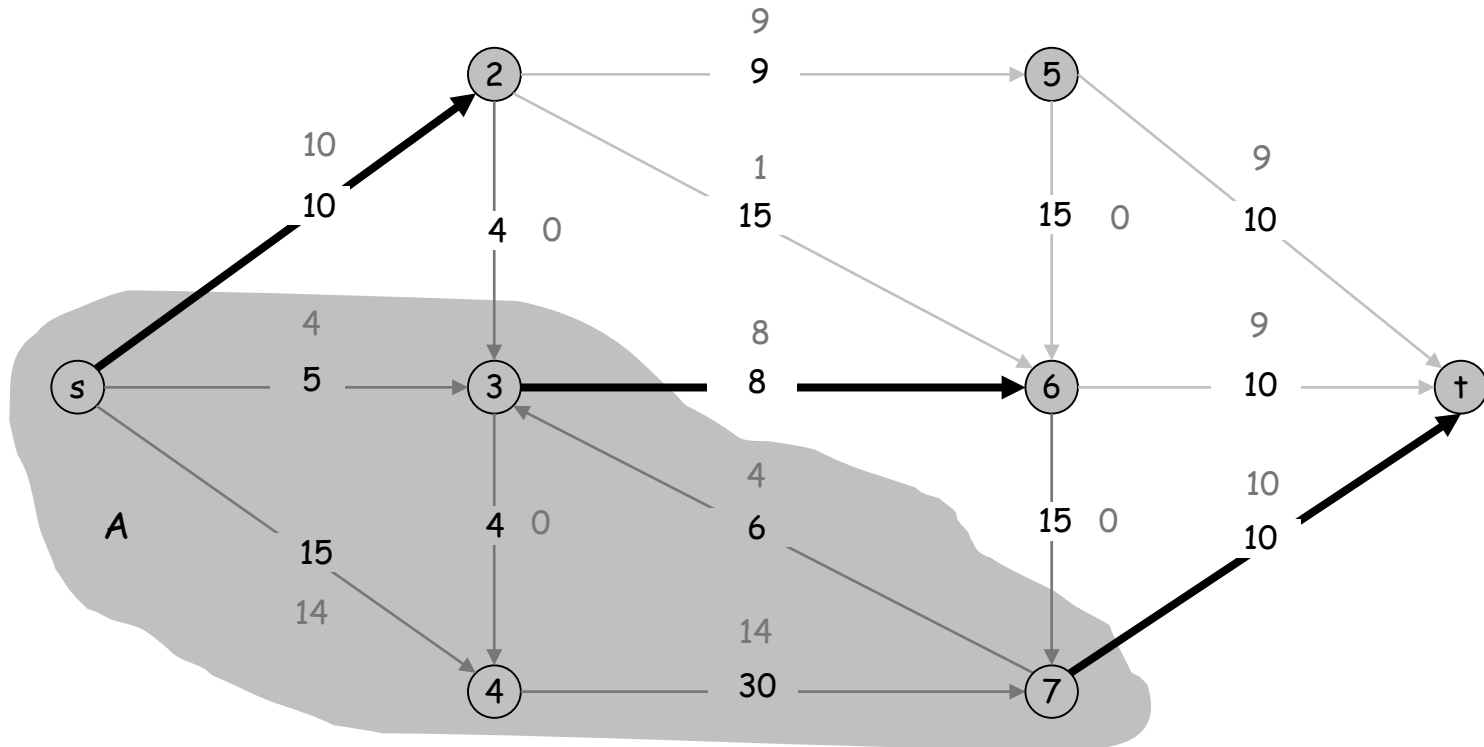
$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &\leq \sum_{e \text{ out of } A} f(e) \\ &\leq \sum_{e \text{ out of } A} c(e) \\ &= \text{cap}(A, B) \quad \blacksquare \end{aligned}$$



Certificate of Optimality

Corollary. Let f be any flow, and let (A, B) be any cut. If $v(f) = \text{cap}(A, B)$, then f is a max flow and (A, B) is a min cut.

Value of flow = 28
 Cut capacity = 28 \Rightarrow Flow value \leq 28



Towards a Max Flow Algorithm

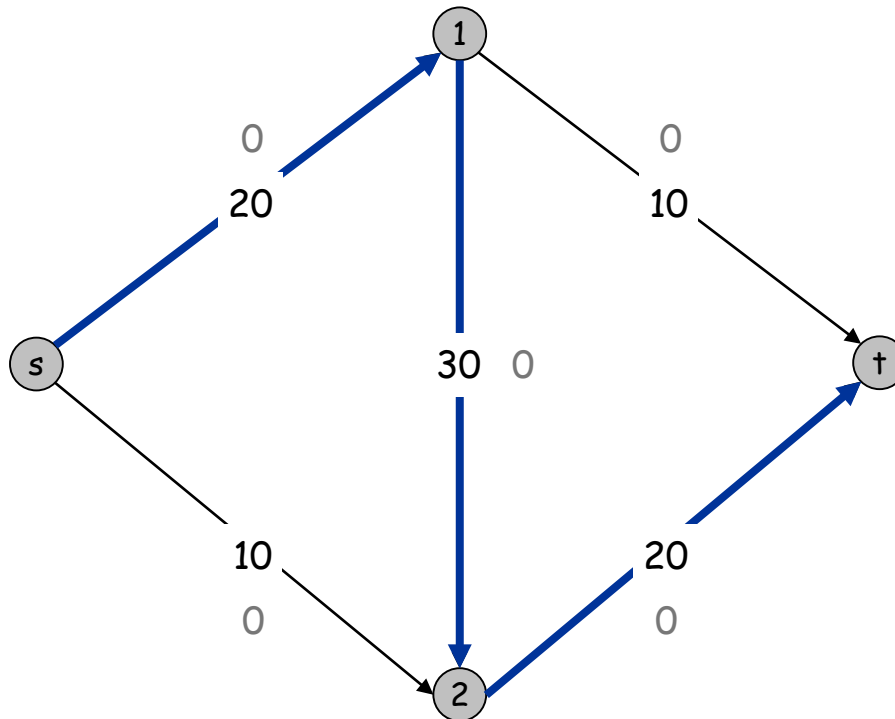
Greedy algorithm.

Start with $f(e) = 0$ for all edge $e \in E$.

Find an s - t path P where each edge has $f(e) < c(e)$.

Augment flow along path P .

Repeat until you get stuck.



Flow value = 0

Towards a Max Flow Algorithm

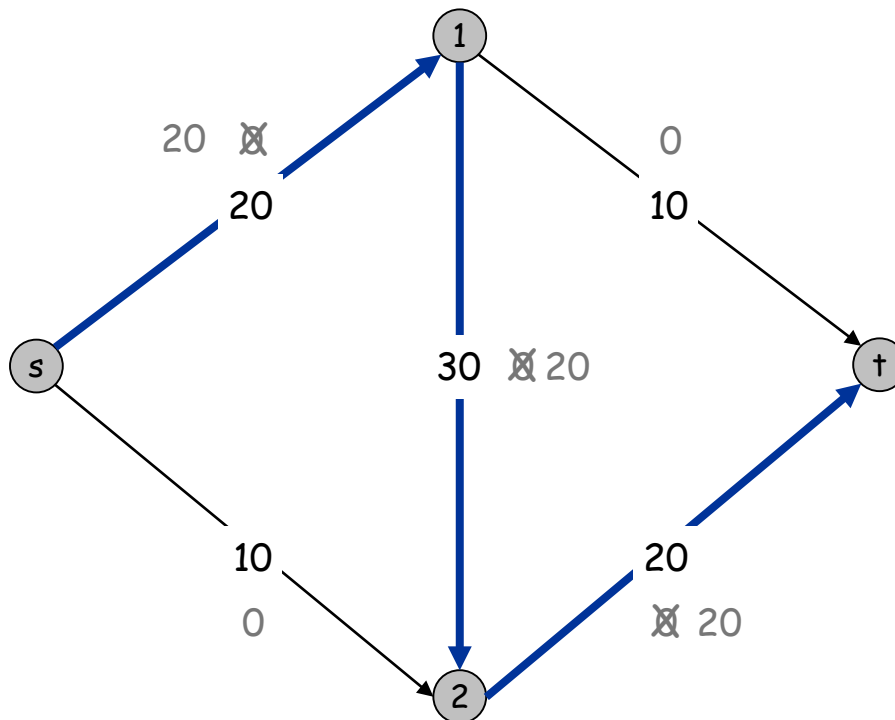
Greedy algorithm.

Start with $f(e) = 0$ for all edge $e \in E$.

Find an s - t path P where each edge has $f(e) < c(e)$.

Augment flow along path P .

Repeat until you get stuck.



Flow value = 20

Towards a Max Flow Algorithm

Greedy algorithm.

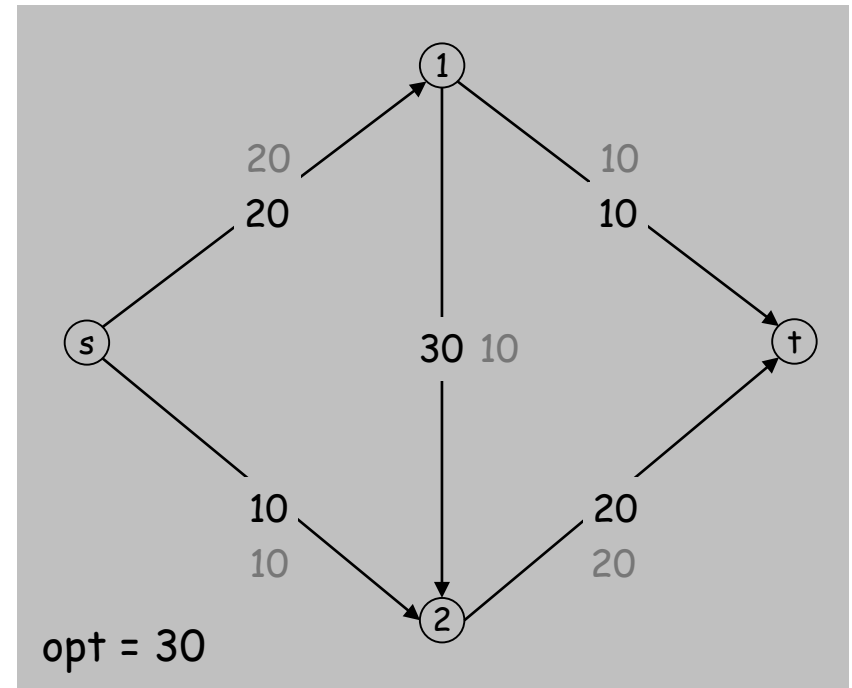
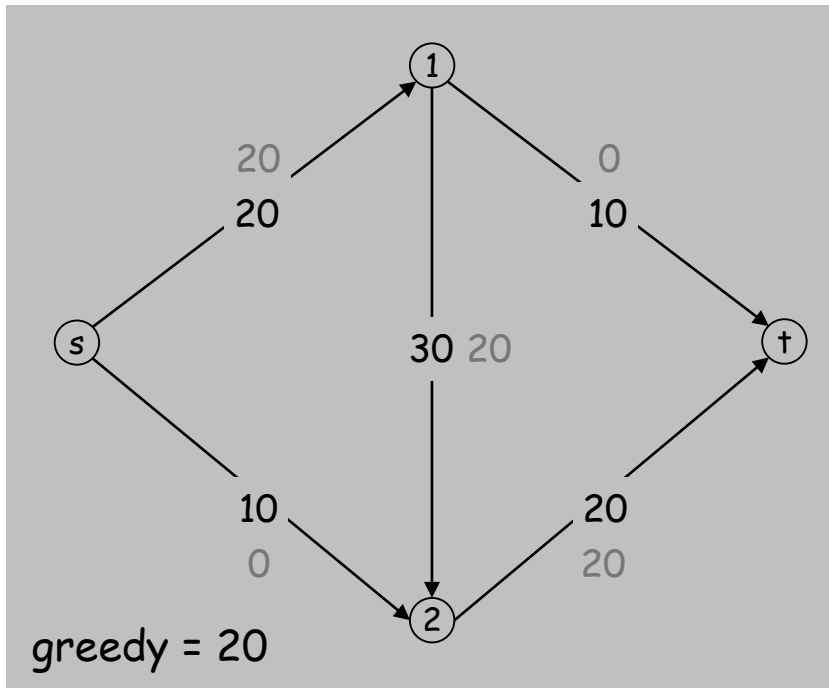
Start with $f(e) = 0$ for all edge $e \in E$.

Find an s - t path P where each edge has $f(e) < c(e)$.

Augment flow along path P .

Repeat until you get **stuck**.

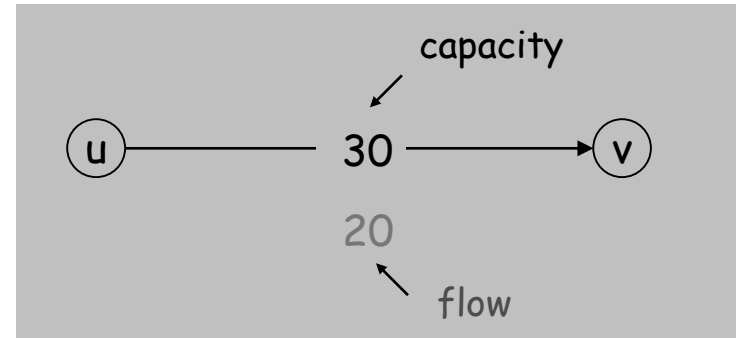
← locally optimality $\not\Rightarrow$ global optimality



Residual Graph

Original edge: $e = (u, v) \in E$.

Flow $f(e)$, capacity $c(e)$.



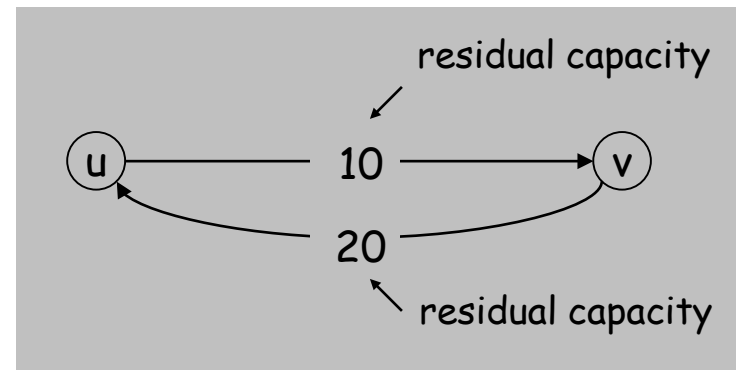
Residual edge.

"Undo" flow sent.

$e = (u, v)$ and $e^R = (v, u)$.

Residual capacity:

$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^R \in E \end{cases}$$



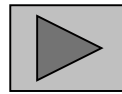
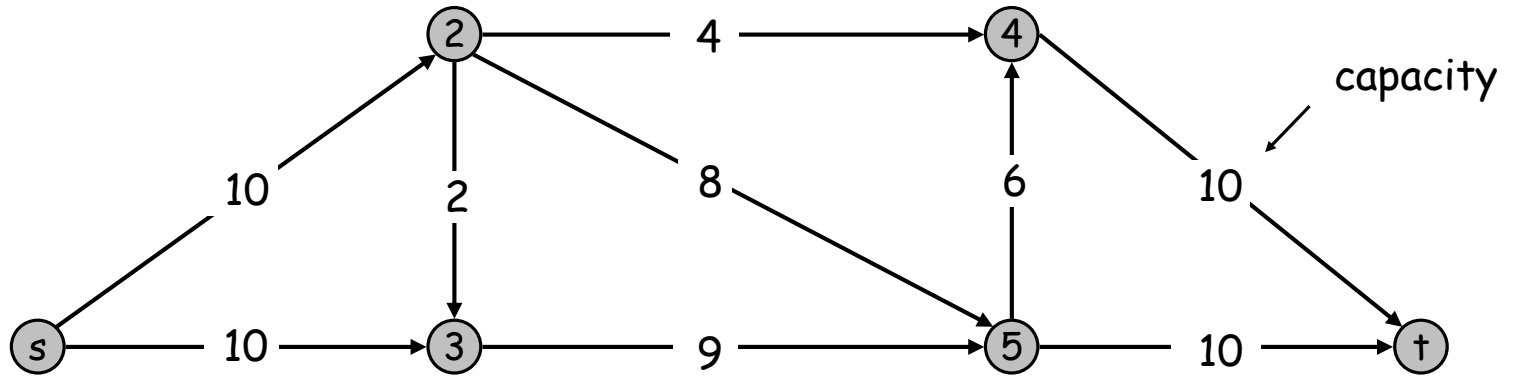
Residual graph: $G_f = (V, E_f)$.

Residual edges with positive residual capacity.

$$E_f = \{e : f(e) < c(e)\} \cup \{e^R : f(e) > 0\}.$$

Ford-Fulkerson Algorithm

G :



Augmenting Path Algorithm

```
Augment(f, c, P) {  
  b ← bottleneck(P)  
  foreach e ∈ P {  
    if (e ∈ E) f(e) ← f(e) + b  
    else      f(eR) ← f(eR) - b  
  }  
  return f  
}
```

forward edge
reverse edge

```
Ford-Fulkerson(G, s, t, c) {  
  foreach e ∈ E f(e) ← 0  
  Gf ← residual graph  
  
  while (there exists augmenting path P) {  
    f ← Augment(f, c, P)  
    update Gf  
  }  
  return f  
}
```

Max-Flow Min-Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Elias-Feinstein-Shannon 1956, Ford-Fulkerson 1956]
The value of the max flow is equal to the value of the min cut.

Pf. We prove both simultaneously by showing TFAE:

- (i) There exists a cut (A, B) such that $v(f) = \text{cap}(A, B)$.
- (ii) Flow f is a max flow.
- (iii) There is no augmenting path relative to f .

(i) \Rightarrow (ii) This was the corollary to weak duality lemma.

(ii) \Rightarrow (iii) We show contrapositive.

Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along path.

Proof of Max-Flow Min-Cut Theorem

(iii) \Rightarrow (i)

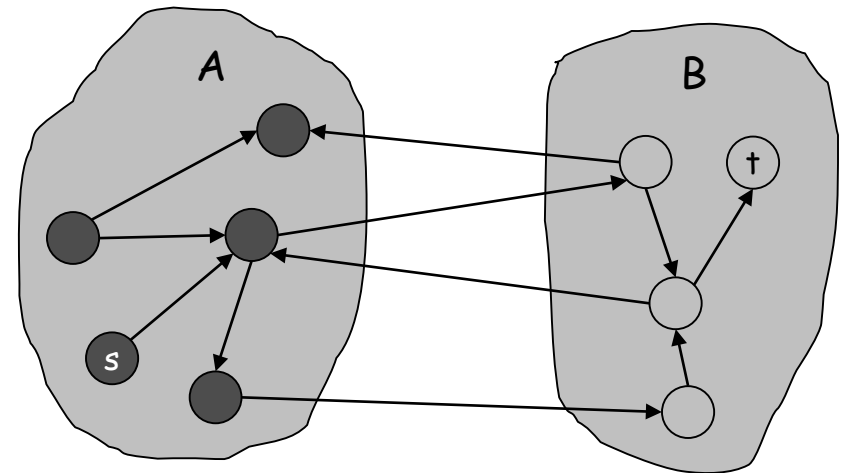
Let f be a flow with no augmenting paths.

Let A be set of vertices **reachable from s** in residual graph.

By definition of A , $s \in A$.

By definition of f , $t \notin A$.

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &= \sum_{e \text{ out of } A} c(e) \\ &= \text{cap}(A, B) \quad \blacksquare \end{aligned}$$



original network

Running Time

Assumption. All capacities are integers between 1 and C .

Invariant. Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \leq nC$ iterations.

Pf. Each augmentation increase value by at least 1. ▀

Corollary. If $C = 1$, Ford-Fulkerson runs in $O(mn)$ time.

Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value $f(e)$ is an integer.

Pf. Since algorithm terminates, theorem follows from invariant. ▀

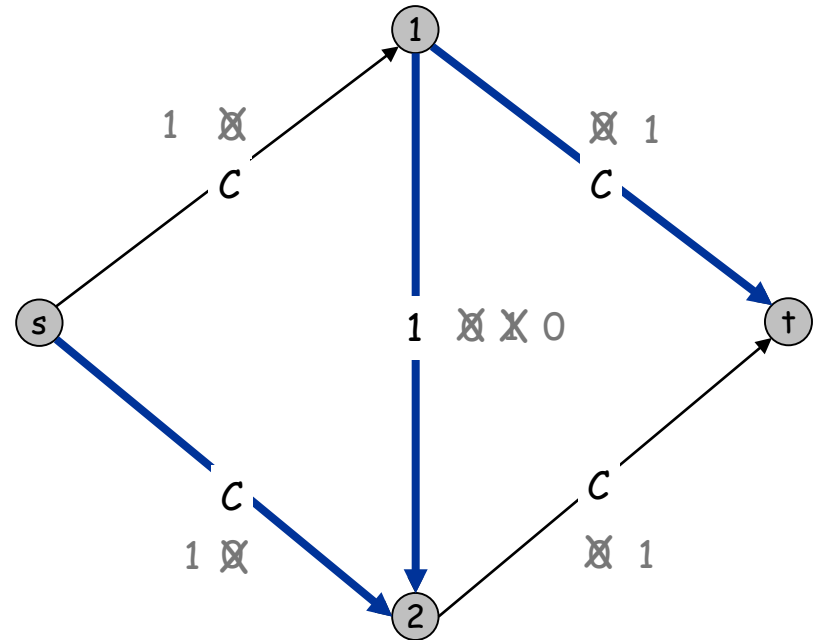
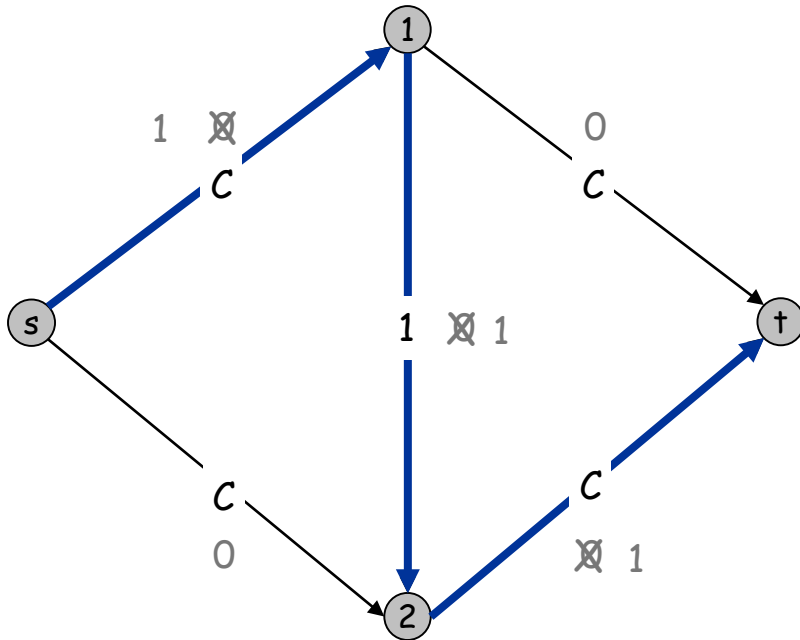
7.3 Choosing Good Augmenting Paths

Ford-Fulkerson: Exponential Number of Augmentations

Q. Is generic Ford-Fulkerson algorithm polynomial in input size?

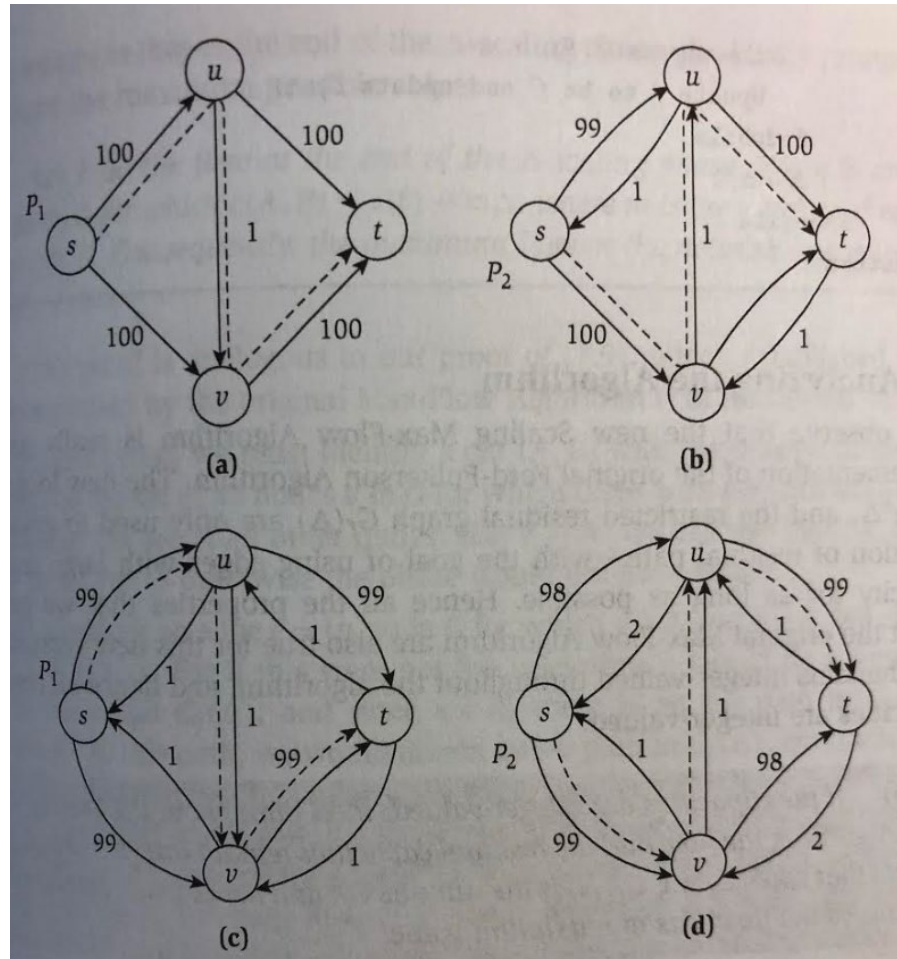
m (# of edges), n (# of nodes), and $\log C$

A. No. If max capacity is C , then algorithm can take C iterations.



Ford-Fulkerson: Exponential Number of Augmentations

$C=100$



Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

Some choices lead to exponential algorithms.

Clever choices lead to polynomial algorithms.

If capacities are irrational, algorithm not guaranteed to terminate!

Goal: choose augmenting paths so that:

Can find augmenting paths efficiently.

Few iterations.

Choose augmenting paths with:

Augmenting path with fewest number of edges [Edmonds-Karp 1972, Dinitz 1970]

Preflow-push maximum-flow (notion of node height) [Goldberg 1986]

Both are strongly polynomial algorithms: $O(mn)$

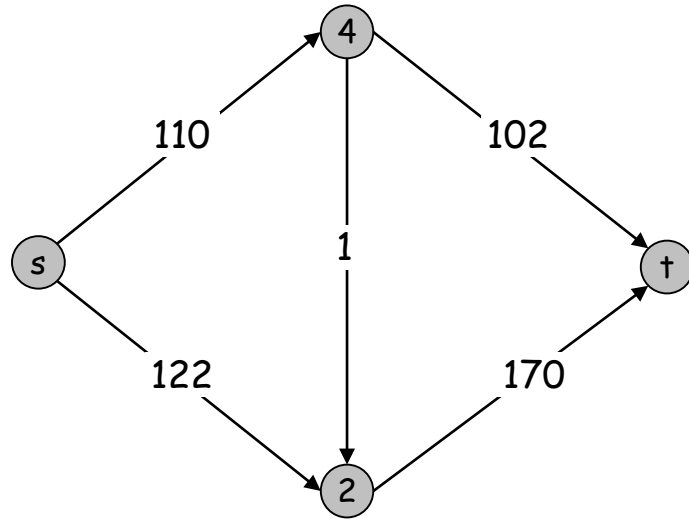
Capacity Scaling

Intuition. Choosing path with highest bottleneck capacity increases flow by max possible amount.

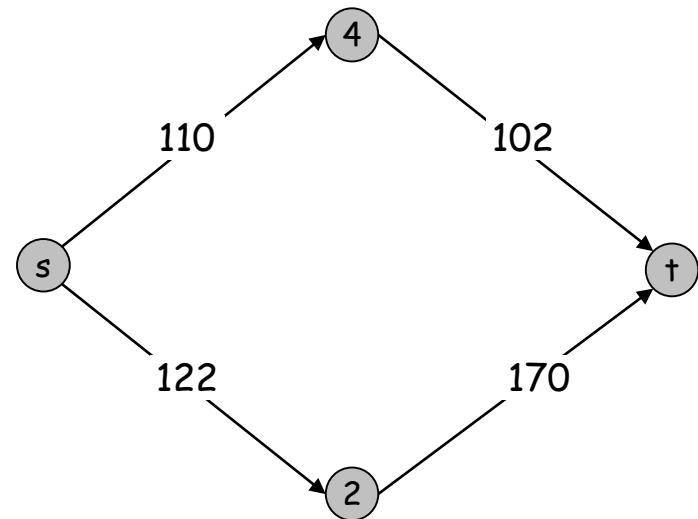
Don't worry about finding exact highest bottleneck path.

Maintain scaling parameter Δ .

Let $G_f(\Delta)$ be the subgraph of the residual graph consisting of only arcs with capacity at least Δ .



G_f



$G_f(100)$

Capacity Scaling

```
Scaling-Max-Flow( $G, s, t, c$ ) {  
  foreach  $e \in E$   $f(e) \leftarrow 0$   
   $\Delta \leftarrow$  smallest power of 2 greater than or equal to  $C$   
   $G_f \leftarrow$  residual graph  
  
  while ( $\Delta \geq 1$ ) {  
     $G_f(\Delta) \leftarrow \Delta$ -residual graph  
    while (there exists augmenting path  $P$  in  $G_f(\Delta)$ ) {  
       $f \leftarrow$  augment( $f, c, P$ )  
      update  $G_f(\Delta)$   
    }  
     $\Delta \leftarrow \Delta / 2$   
  }  
  return  $f$   
}
```


Capacity Scaling: Correctness

Assumption. All edge capacities are integers between 1 and C .

Integrality invariant. All flow and residual capacity values are integral.

Correctness. If the algorithm terminates, then f is a max flow.

Pf.

By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$.

Upon termination of $\Delta = 1$ phase, there are no augmenting paths. ▀

Capacity Scaling: Running Time

Lemma 1. The outer while loop repeats $1 + \lceil \log_2 C \rceil$ times.

Pf. Initially $C \leq \Delta < 2C$. Δ decreases by a factor of 2 each iteration. ▀

Lemma 2. Let f be the flow at the end of a Δ -scaling phase. Then the value of the maximum flow is at most $v(f) + m \Delta$. ← proof on next slide

Lemma 3. There are at most $2m$ augmentations per scaling phase.

Let f be the flow at the end of the previous scaling phase.

L2 $\Rightarrow v(f^*) \leq v(f) + m (2\Delta)$.

Each augmentation in a Δ -phase increases $v(f)$ by at least Δ . ▀

Theorem. The scaling max-flow algorithm finds a max flow in $O(m \log C)$ augmentations. It can be implemented to run in $O(m^2 \log C)$ time. ▀

Capacity Scaling: Running Time

Lemma 2. Let f be the flow at the end of a Δ -scaling phase. Then value of the maximum flow is at most $v(f) + m \Delta$.

Pf. (almost identical to proof of max-flow min-cut theorem)

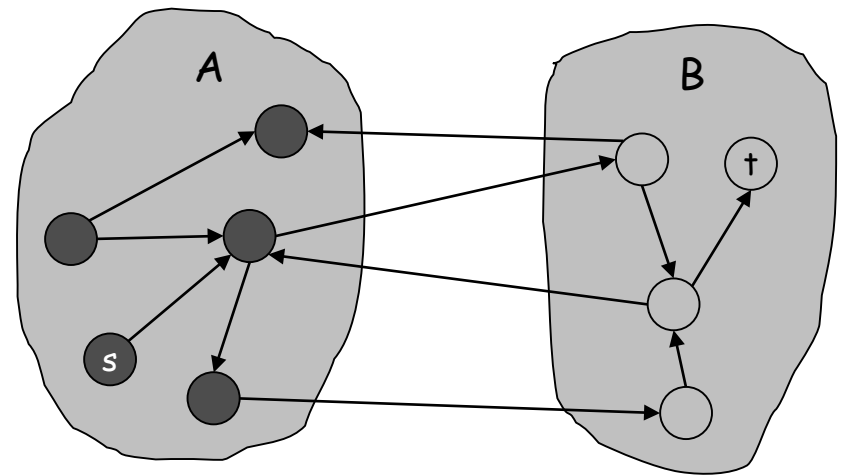
We show that at the end of a Δ -phase, there exists a cut (A, B) such that $\text{cap}(A, B) \leq v(f) + m \Delta$.

Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.

By definition of A , $s \in A$.

By definition of f , $t \notin A$.

$$\begin{aligned}
 v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
 &\geq \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\
 &= \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta \\
 &\geq \text{cap}(A, B) - m\Delta \quad \blacksquare
 \end{aligned}$$



original network