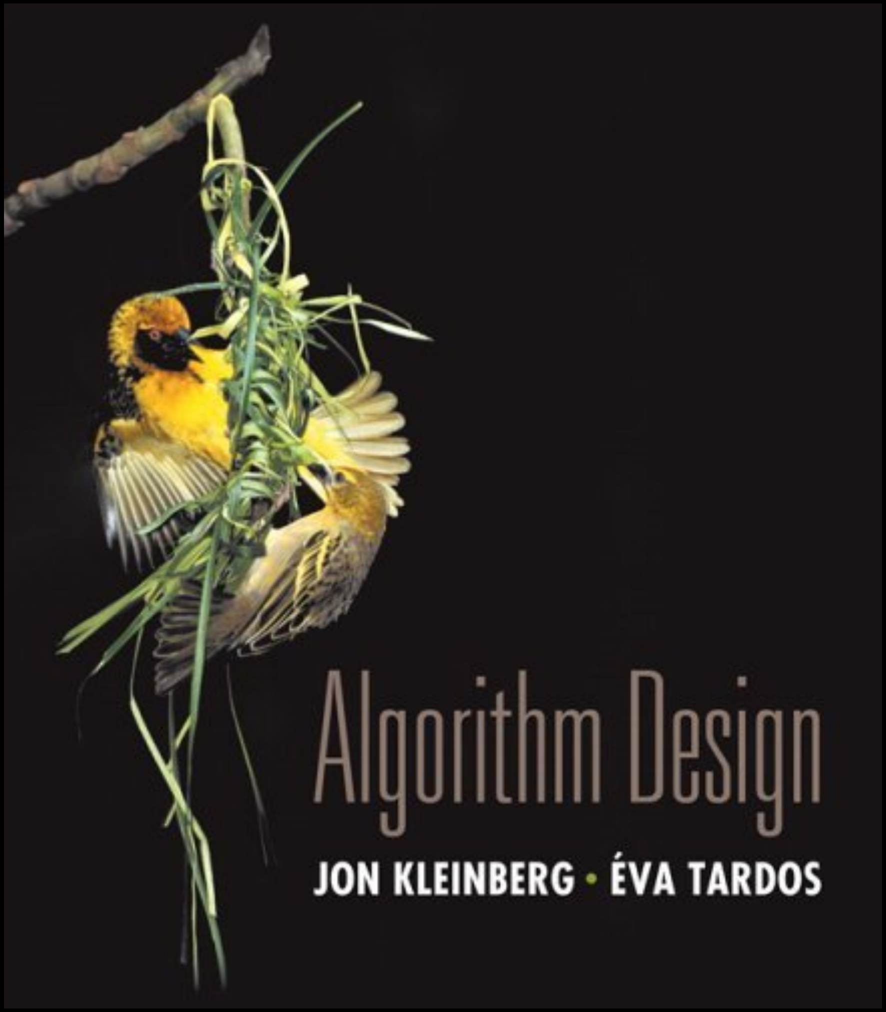


Chapter 5

Divide and Conquer

Slides by Kevin Wayne.

Divide-and-Conquer

Divide-and-conquer.

- Break up problem into several parts.
- Solve each part recursively.
- Combine solutions to sub-problems into overall solution.

Most common usage.

- Break up problem of size n into two equal parts of size $\frac{1}{2}$n.
- Solve two parts recursively.
- Combine two solutions into overall solution in linear time.

Consequence.

- Brute force: n^2.
- Divide-and-conquer: n log n.

5.1 Mergesort

Sorting

Sorting. Given n elements, rearrange in ascending order.

Applications.

- Sort a list of names.
- Organize an MP3 library.
- Display Google PageRank results.
- List RSS news items in reverse chronological order.

obvious applications

- Find the median.
- Find the closest pair.
- Binary search in a database.
- Identify statistical outliers.
- Find duplicates in a mailing list.

problems become easy once items are in sorted order

- Data compression.
- Computer graphics.
- Computational biology.
- Supply chain management.
- Book recommendations on Amazon.
- Load balancing on a parallel computer.

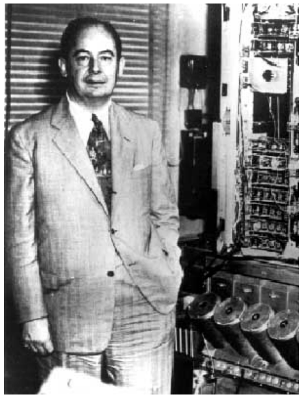
. . .

non-obvious applications

Mergesort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.



Jon von Neumann (1945)

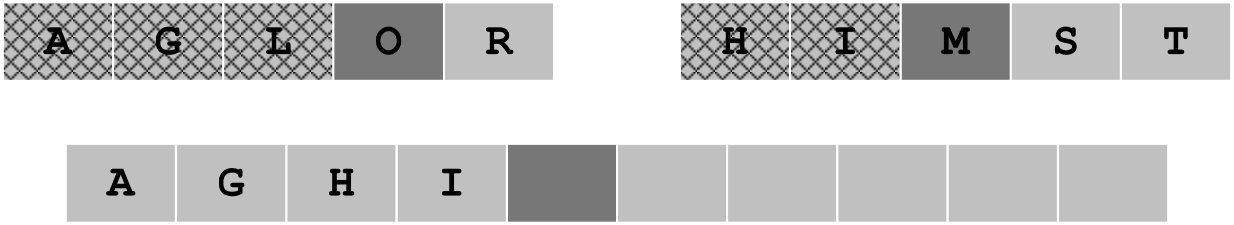
A	L	G	O	R	I	T	H	M	S		
A	L	G	O	R	I	T	H	M	S	divide	O(1)
A	G	L	O	R	H	I	M	S	T	sort	2T(n/2)
A	G	H	I	L	M	O	R	S	T	merge	O(n)

Merging

Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently?

- Linear number of comparisons.
- Use temporary array.



Challenge for the bored. In-place merge. [Kronrud, 1969]

using only a constant amount of extra storage

A Useful Recurrence Relation

Def. T(n) = number of comparisons to mergesort an input of size n.

Mergesort recurrence.

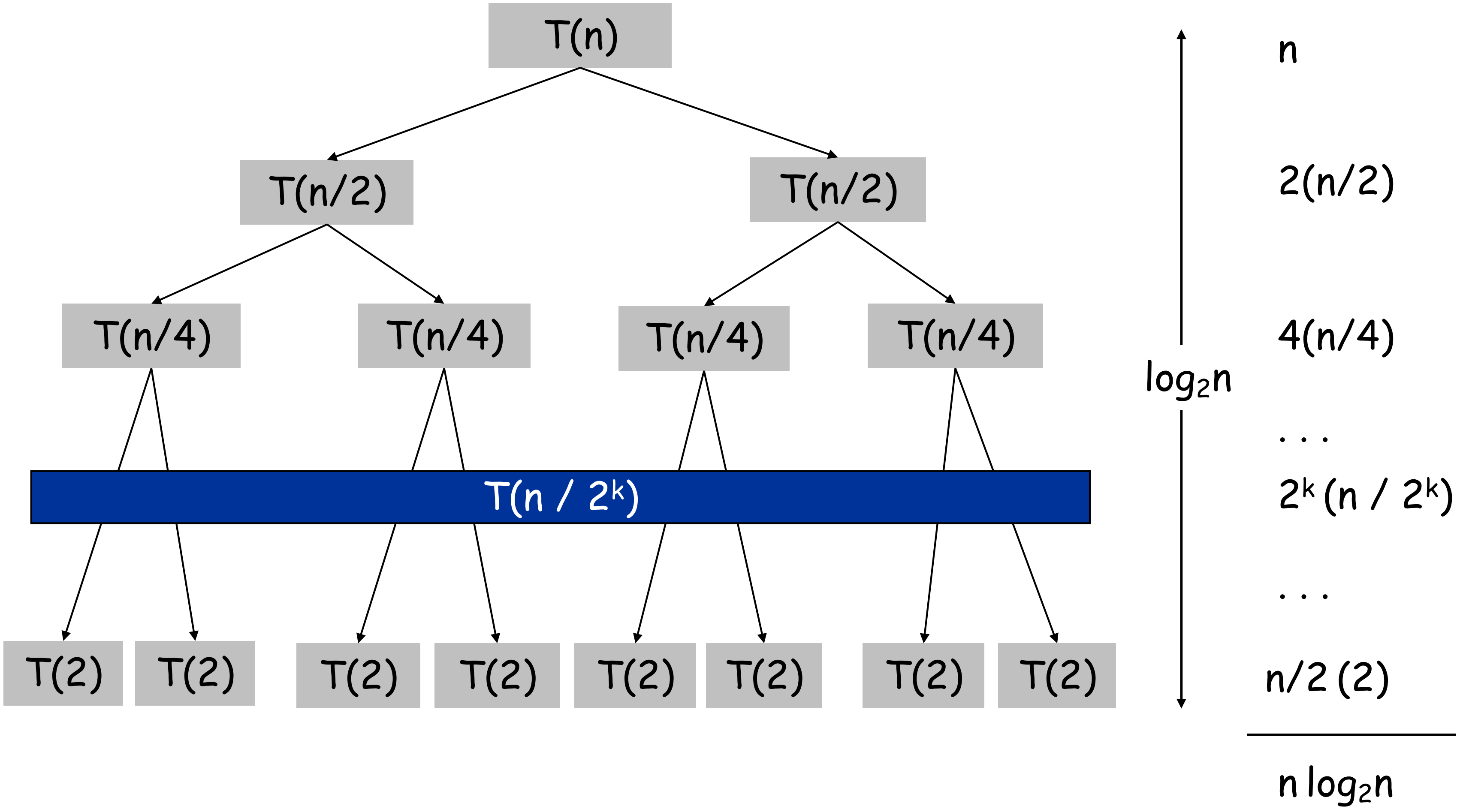
$$\mathrm{T}(n) \le \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Solution. $T(n) = O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence. Initially we assume n is a power of 2 and replace ≤ with =.

Proof by Recursion Tree

$$\mathrm{T}(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$



Proof by Telescoping

Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

↑ assumes n is a power of 2

$$\mathrm{T}(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. For n > 1:

$$\begin{aligned} \frac{T(n)}{n} &= \frac{2T(n/2)}{n} + 1 \\ &= \frac{T(n/2)}{n/2} + 1 \\ &= \frac{T(n/4)}{n/4} + 1 + 1 \\ &\cdots \\ &= \frac{T(n/n)}{n/n} + \underbrace{1 + \cdots + 1}_{\log_2 n} \\ &= \log_2 n \end{aligned}$$

Proof by Induction

Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

↑ assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: n = 1.
- Inductive hypothesis: T(n) = n $\log_2$ n.
- Goal: show that T(2n) = 2n $\log_2$ (2n).

$$\begin{aligned} T(2n) &= 2T(n) + 2n \\ &= 2n \log_2 n + 2n \\ &= 2n\big(\log_2 (2n) - 1\big) + 2n \\ &= 2n \log_2 (2n) \end{aligned}$$

Analysis of Mergesort Recurrence

Claim. If T(n) satisfies the following recurrence, then T(n) $\leq n \lceil \lg n \rceil$.

↑ $\log_2 n$

$$T(n) \;\leq\; \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: n = 1.
- Define $n_1 = \lfloor n / 2 \rfloor$, $n_2 = \lceil n / 2 \rceil$.
- Induction step: assume true for 1, 2, ... , n–1.

$$\begin{aligned} T(n) &\leq T(n_1) + T(n_2) + n \\ &\leq n_1 \lceil \lg n_1 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &\leq n_1 \lceil \lg n_2 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &= n \lceil \lg n_2 \rceil + n \\ &\leq n (\lceil \lg n \rceil - 1) + n \\ &= n \lceil \lg n \rceil \end{aligned}$$

$$\begin{aligned} n_2 &= \left| n/2 \right| \\ &\leq \left\lceil 2^{\lceil \lg n \rceil} / 2 \right\rceil \\ &= 2^{\lceil \lg n \rceil} / 2 \\ \Rightarrow \lg n_2 &\leq \lceil \lg n \rceil - 1 \end{aligned}$$

Two Exercises

- Using recursion tree to guess a result, and then, applying induction to prove.

(1) $T(n) = 3\, T(\lfloor n/4 \rfloor) + \Theta(n^2)$

Use cn^2 to replace $\Theta(n^2)$ for $c > 0$ in recursion tree

Apply $T(n) \le dn^2$ for $d > 0$, the guess result, in induction prove

Determine the constraint associated with d and c

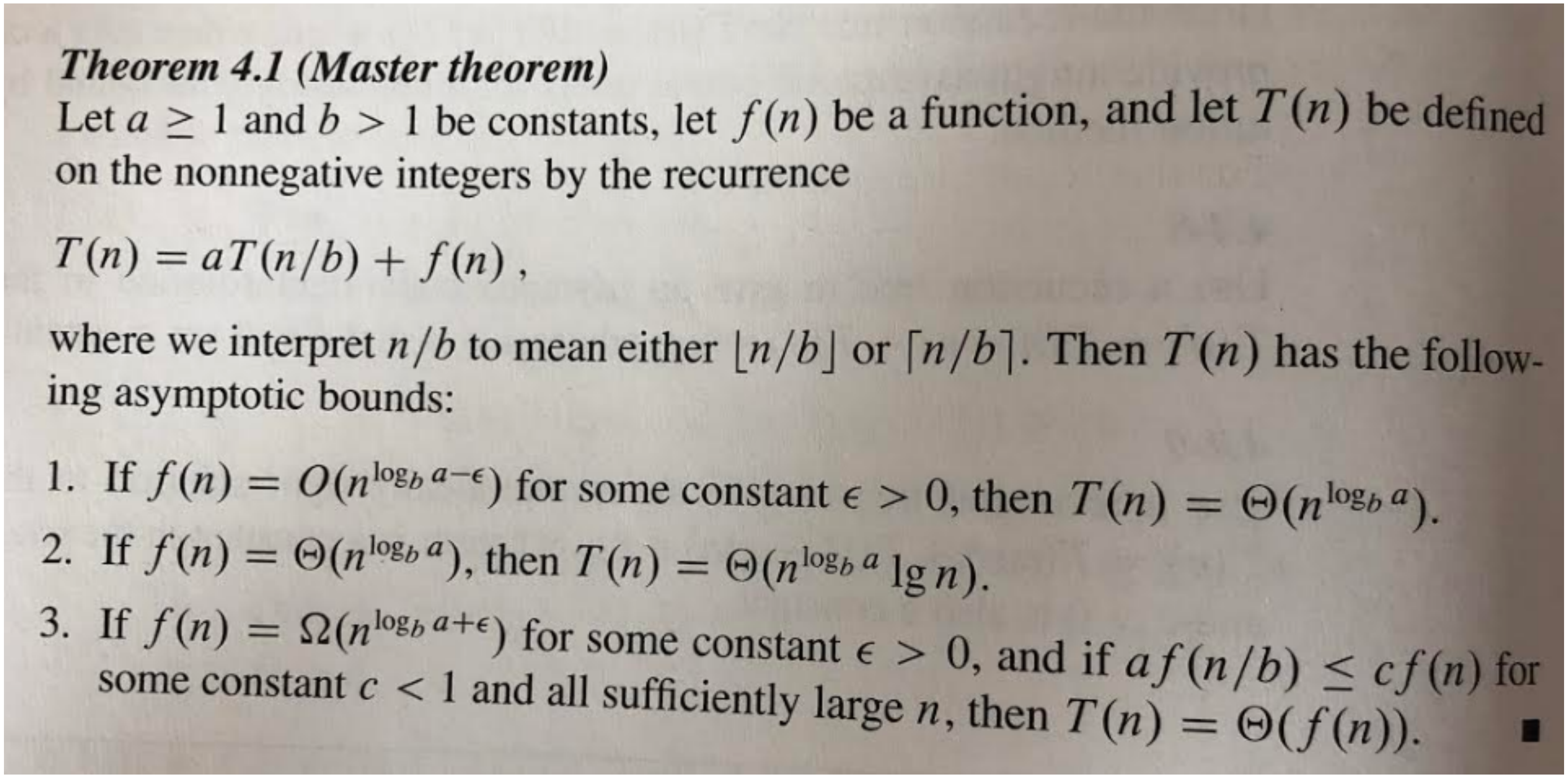
(2) $T(n) = T(n/3) + T(2n/3) + O(n)$

Use c to represent the constant factor in $O(n)$ in recursion tree

Apply $T(n) \le d\, n \lg n$ for $d > 0$, the guess result, in induction prove

Determine the constraint associated with d and c

Master Theorem



Theorem 4.1 (Master theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n) ,$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ∎

$T(n) = 9\ T(n/3) + n$, $T(n) = \Theta(n^2)$;

$T(n) = T(2n/3) + 1$, $T(n) = \Theta(\log n)$;

$T(n) = 8T(n/2) + \Theta(n^2)$, $T(n) = \Theta(n^3)$;

$T(n) = 3T(n/4) + n \log n$, $T(n) = \Theta(n \log n)$

$T(n) = 2T(n/2) + \Theta(n)$, $T(n) = \Theta(n \log n)$

$T(n) = 7T(n/2) + \Theta(n2)$, $T(n) = \Theta(n^{\log 7})$

5.3 Counting Inversions

Counting Inversions

Music site tries to match your song preferences with others.

- You rank n songs.
- Music site consults database to find people with similar tastes.

Similarity metric: number of inversions between two rankings.

- My rank: 1, 2, …, n.
- Your rank: $a_1, a_2, \ldots, a_n$.
- Songs i and j inverted if $i < j$, but $a_i > a_j$.

Songs

	A	B	C	D	E
Me	1	2	3	4	5
You	1	3	4	2	5

Inversions

3-2, 4-2

Brute force: check all $\Theta(n^2)$ pairs i and j.

Applications

Applications.

- Voting theory.
- Collaborative filtering.
- Measuring the "sortedness" of an array.
- Sensitivity analysis of Google's ranking function.
- Rank aggregation for meta-searching on the Web.
- Nonparametric statistics (e.g., Kendall's Tau distance).

Counting Inversions: Divide-and-Conquer

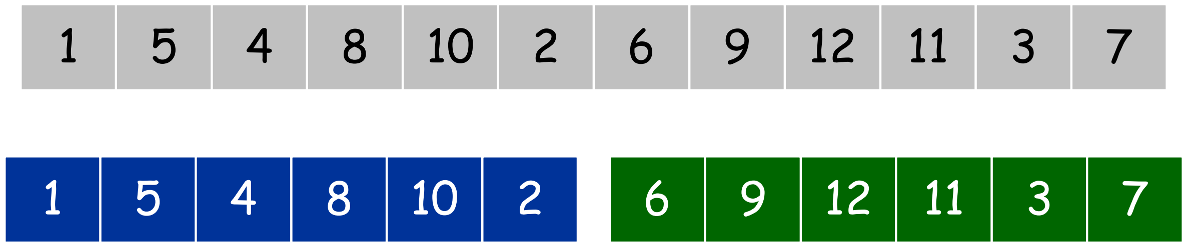
Divide-and-conquer.

1	5	4	8	10	2	6	9	12	11	3	7

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.



1	5	4	8	10	2	6	9	12	11	3	7

1	5	4	8	10	2

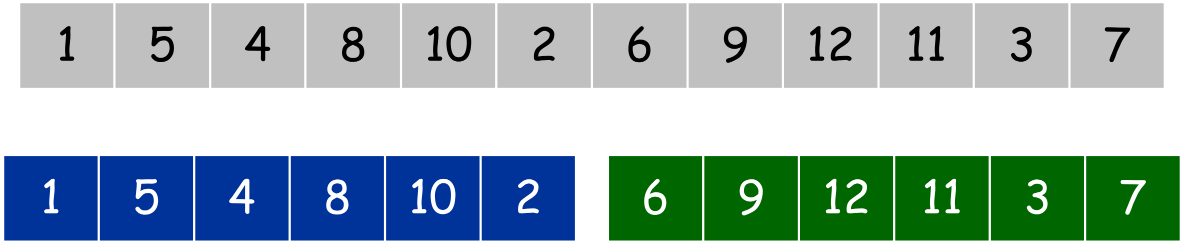
6	9	12	11	3	7

Divide: $O(1)$.

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.
- Conquer: recursively count inversions in each half.



1	5	4	8	10	2	6	9	12	11	3	7

Divide: O(1).

1	5	4	8	10	2

6	9	12	11	3	7

Conquer: 2T(n / 2)

5 blue-blue inversions

5-4, 5-2, 4-2, 8-2, 10-2

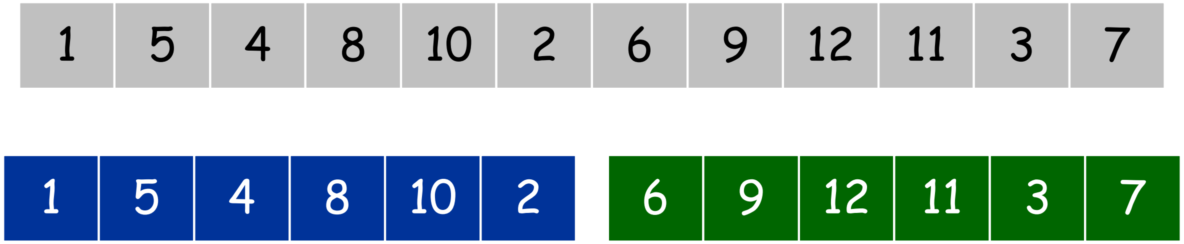
8 green-green inversions

6-3, 9-3, 9-7, 12-3, 12-7, 12-11, 11-3, 11-7

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.
- Conquer: recursively count inversions in each half.
- Combine: count inversions where a_i and a_j are in different halves, and return sum of three quantities.



1	5	4	8	10	2	6	9	12	11	3	7

Divide: O(1).

1	5	4	8	10	2

6	9	12	11	3	7

5 blue-blue inversions

8 green-green inversions

Conquer: $2T(n / 2)$

9 blue-green inversions
5-3, 4-3, 8-6, 8-3, 8-7, 10-6, 10-9, 10-3, 10-7

Combine: ???

Total = 5 + 8 + 9 = 22.

Counting Inversions: Combine

Combine: count blue-green inversions

- Assume each half is sorted.
- Count inversions where a_i and a_j are in different halves.
- Merge two sorted halves into sorted whole.

to maintain sorted invariant

3	7	10	14	18	19

2	11	16	17	23	25
6	3	2	2	0	0

13 blue-green inversions: 6 + 3 + 2 + 2 + 0 + 0

Count: O(n)

2	3	7	10	11	14	16	17	18	19	23	25

Merge: O(n)

$$T(n) \le T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) \quad \Rightarrow \quad \mathrm{T}(n) = O(n \log n)$$

Counting Inversions: Implementation

Pre-condition. [Merge-and-Count] A and B are sorted.

Post-condition. [Sort-and-Count] L is sorted.

```
Sort-and-Count(L) {
   if list L has one element
      return 0 and the list L

   Divide the list into two halves A and B
   ($r_A$, A) ← Sort-and-Count(A)
   ($r_B$, B) ← Sort-and-Count(B)
   (r , L) ← Merge-and-Count(A, B)

   return $r = r_A + r_B + r$ and the sorted list L
}
```

5.4 Closest Pair of Points

Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

↑ fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

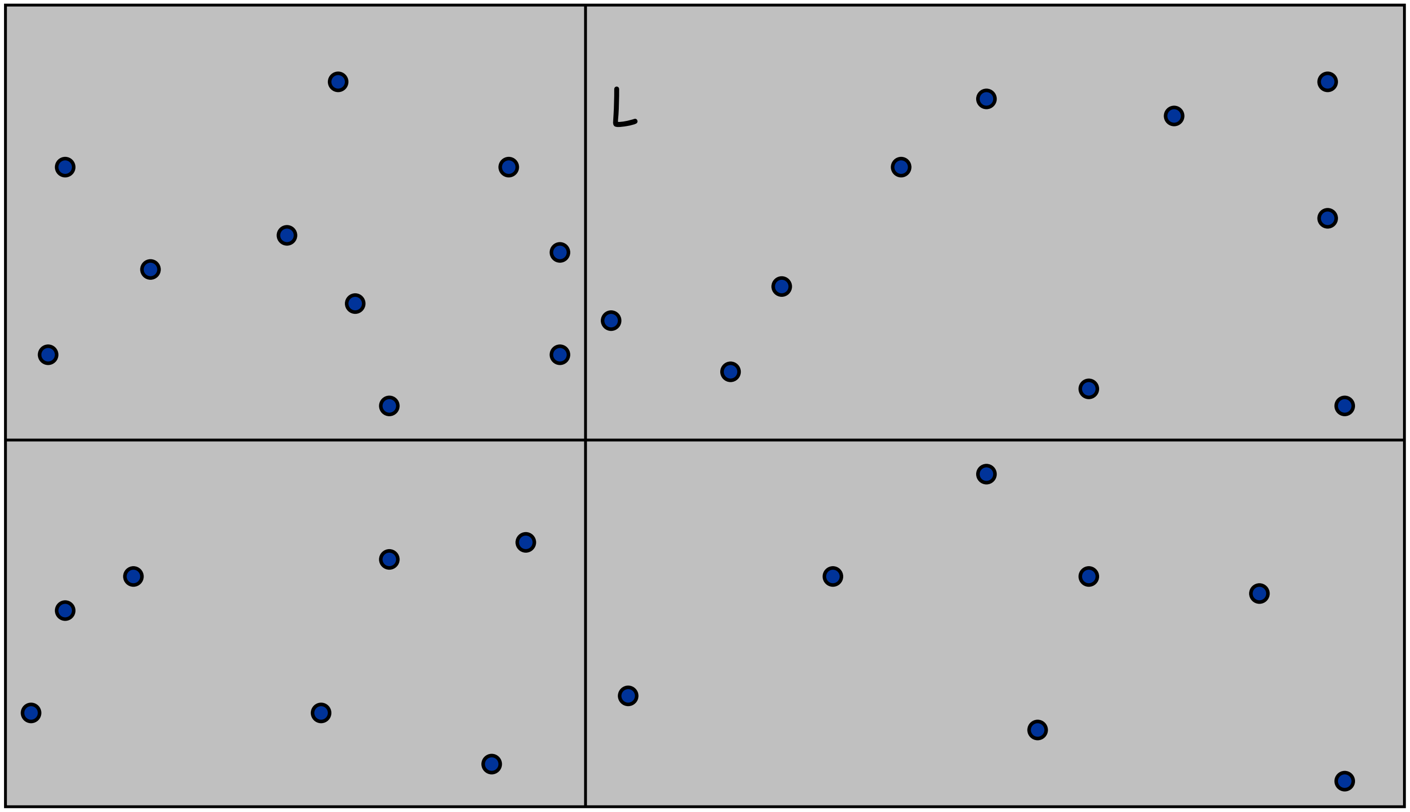
1-D version. O(n log n) easy if points are on a line.

Assumption. No two points have same x coordinate.

↑ to make presentation cleaner

Closest Pair of Points: First Attempt

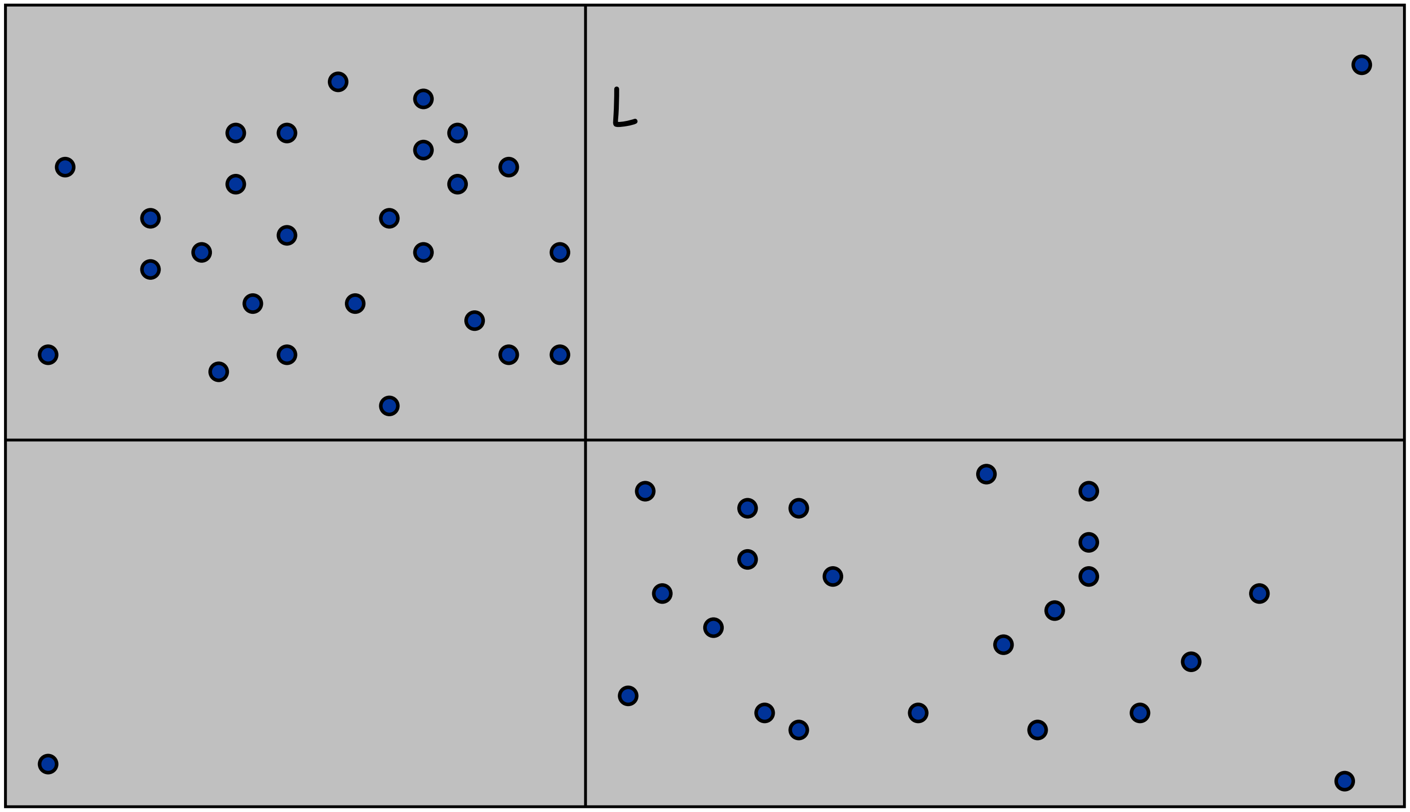
Divide. Sub-divide region into 4 quadrants.



Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.

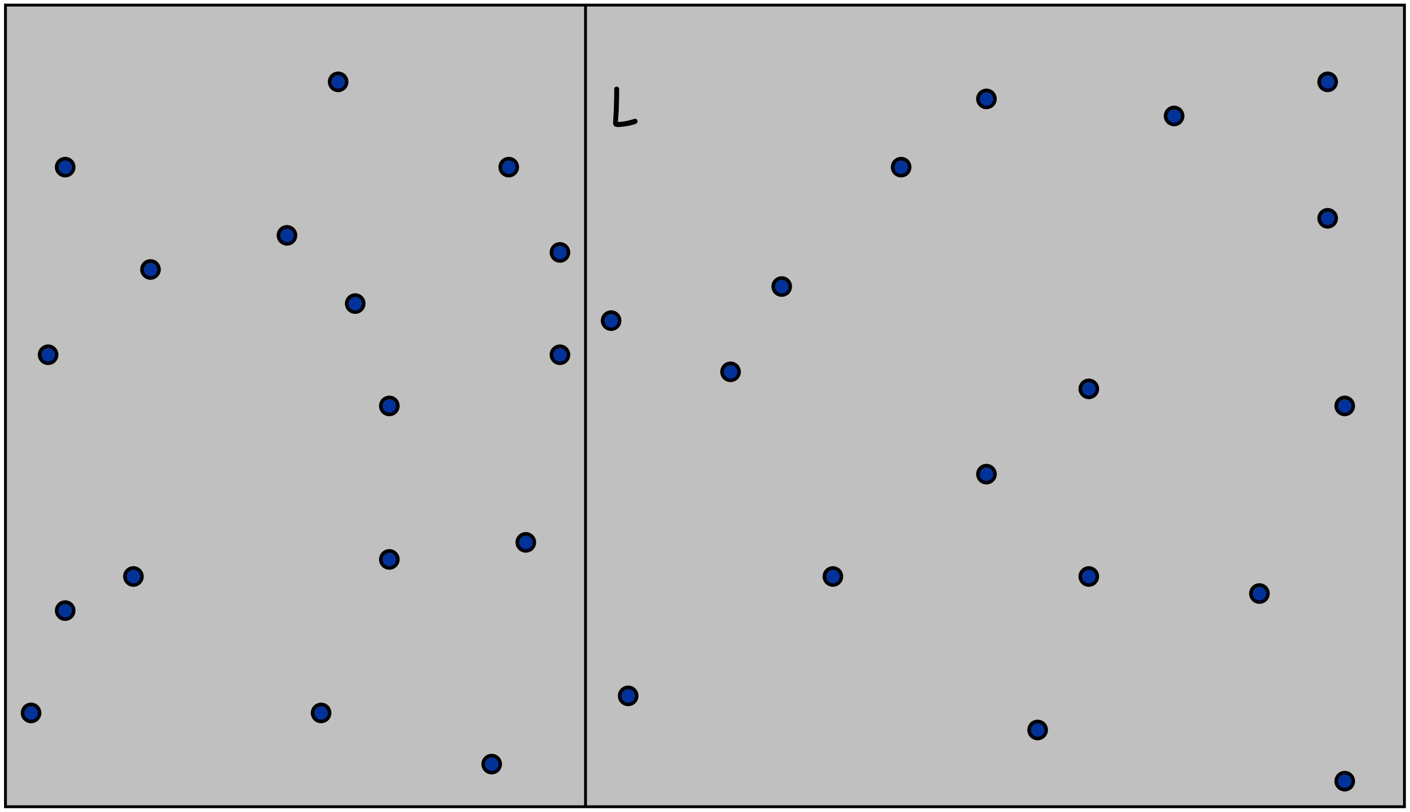
Obstacle. Impossible to ensure n/4 points in each piece.



Closest Pair of Points

Algorithm.

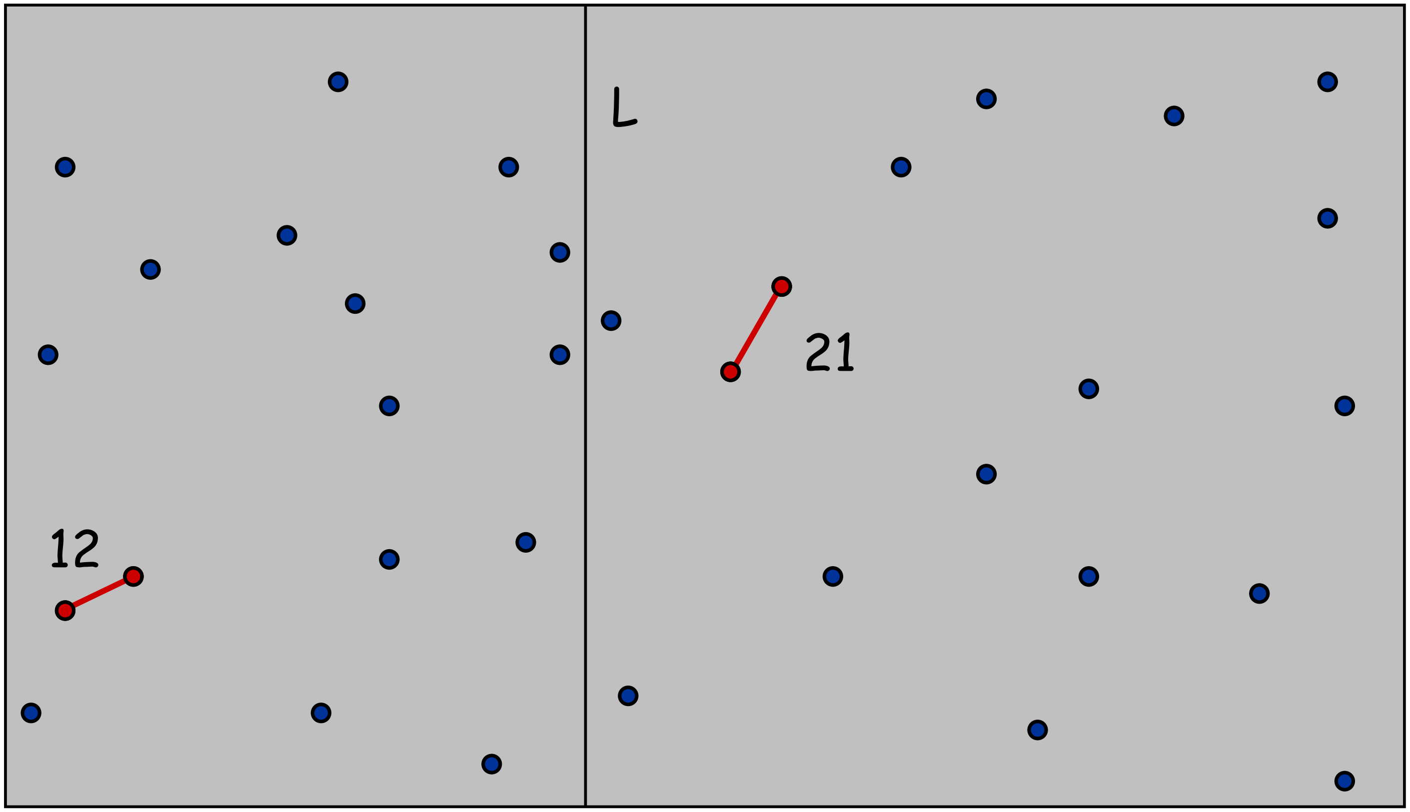
- **Divide**: draw vertical line L so that roughly $\frac{1}{2}$n points on each side.



Closest Pair of Points

Algorithm.

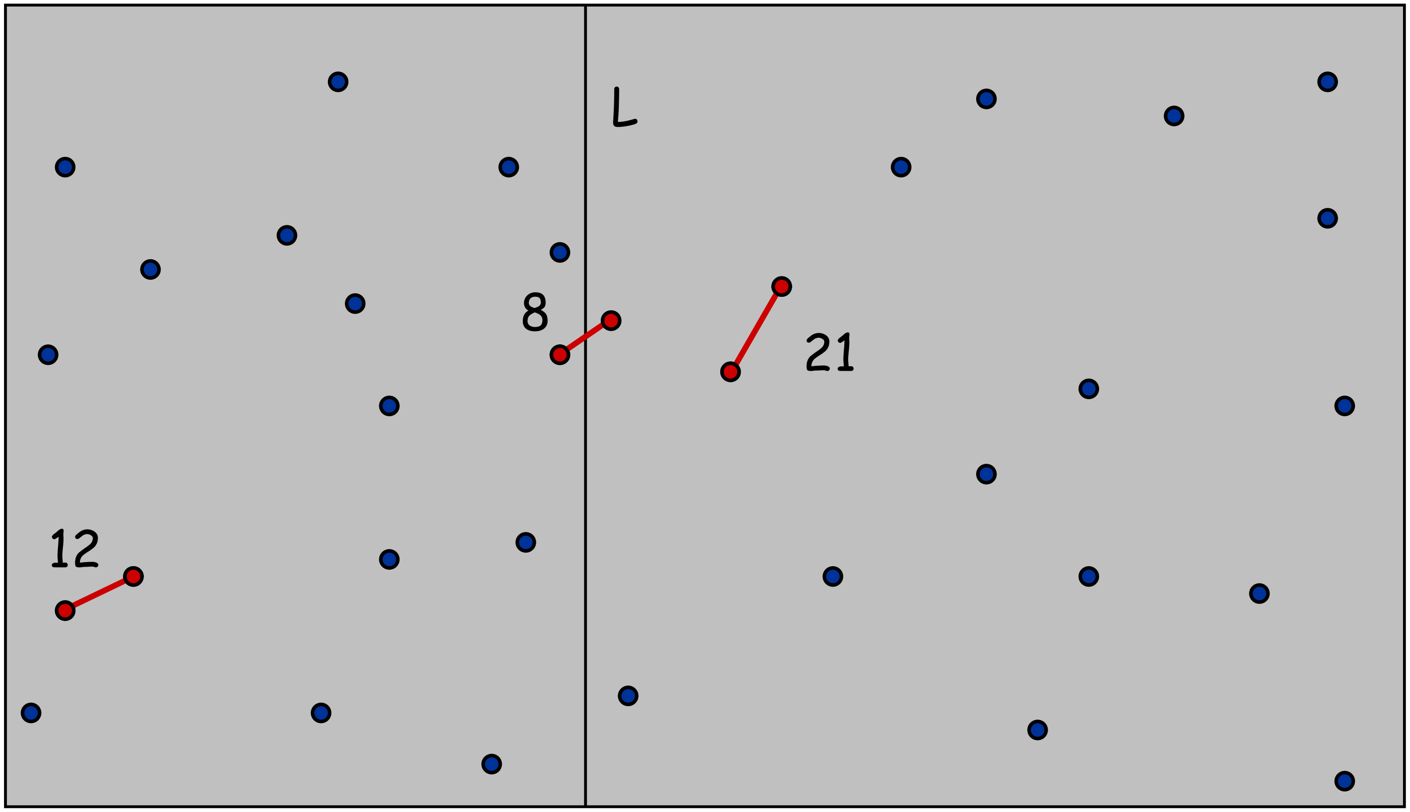
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.



Closest Pair of Points

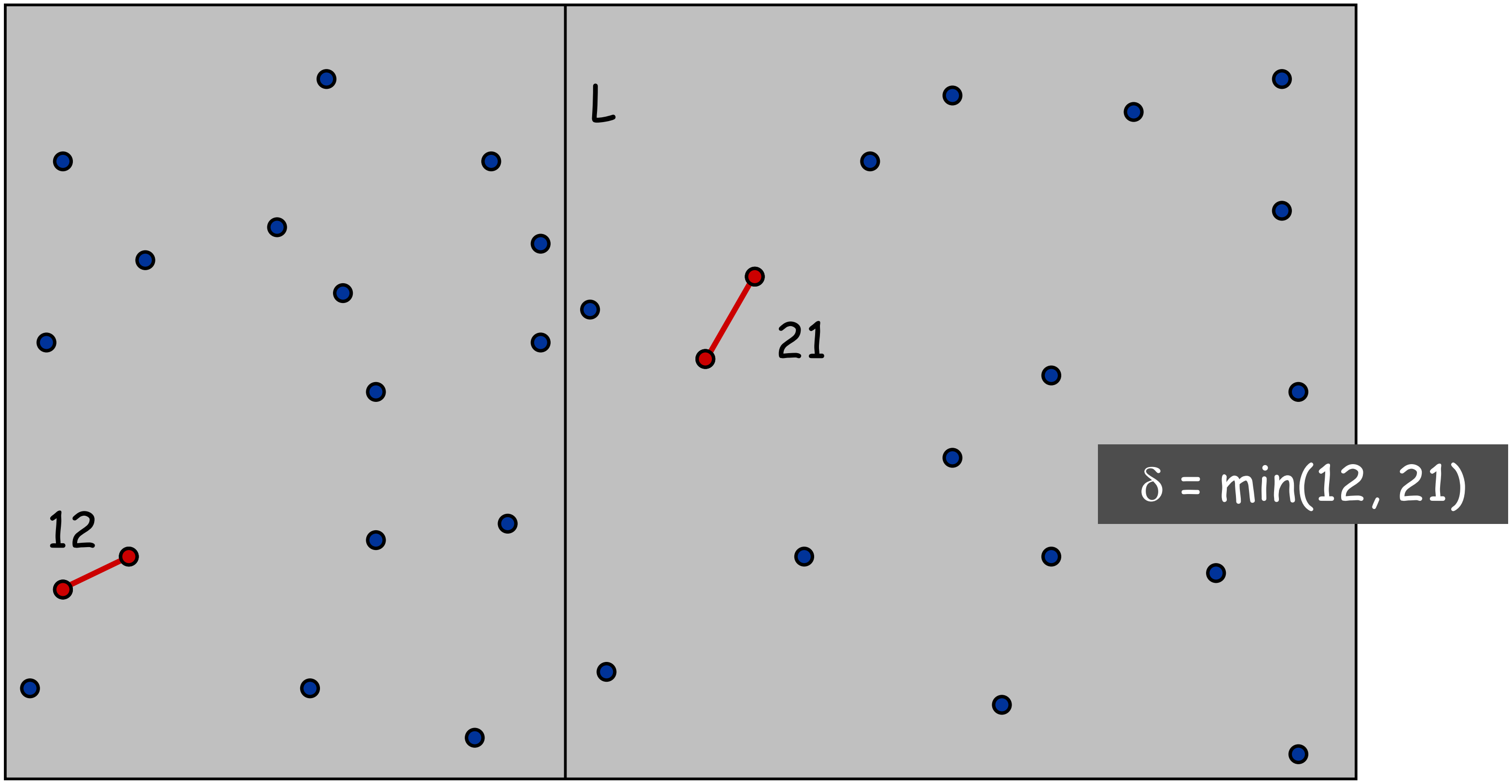
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.



Closest Pair of Points

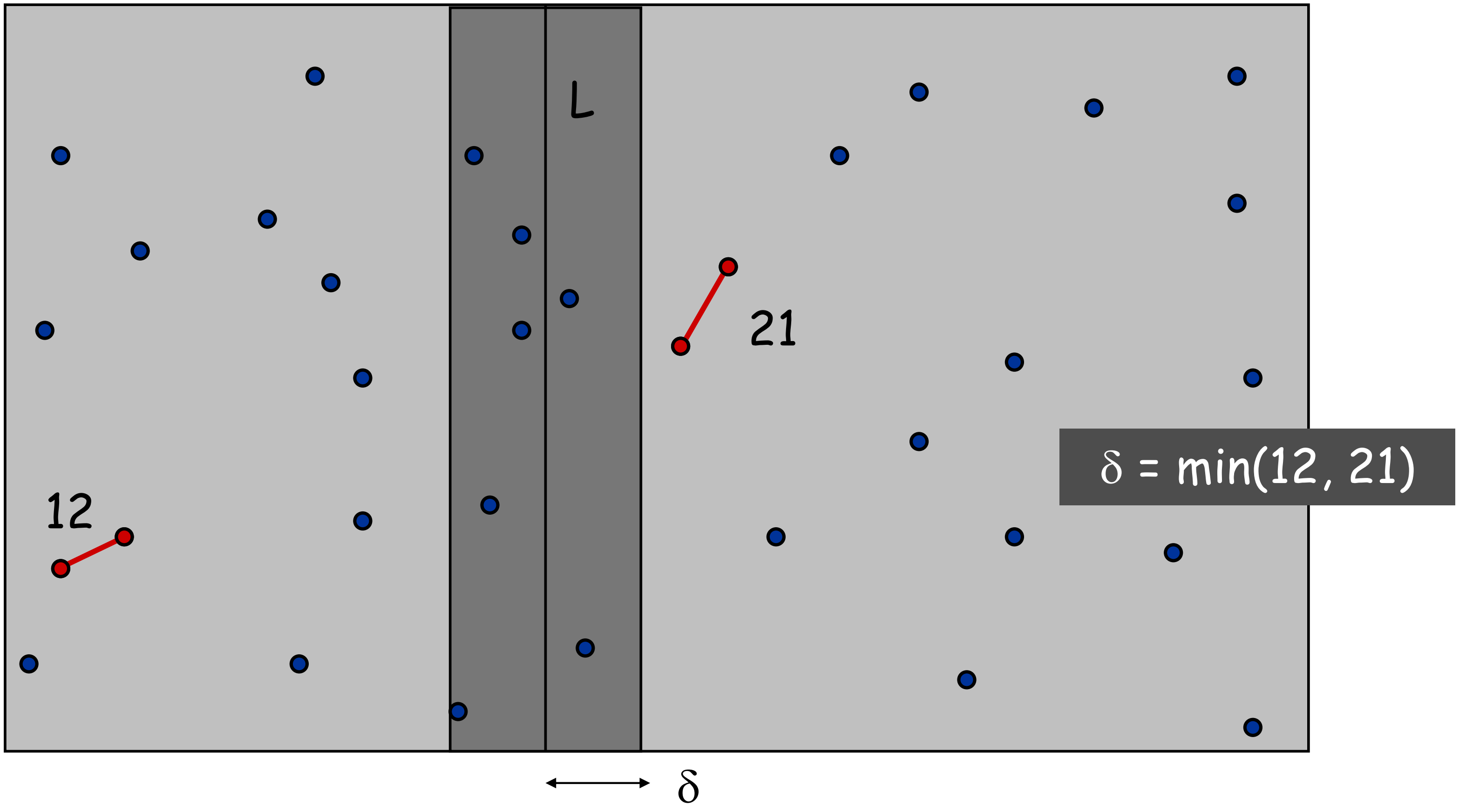
Find closest pair with one point in each side, assuming that distance < δ.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

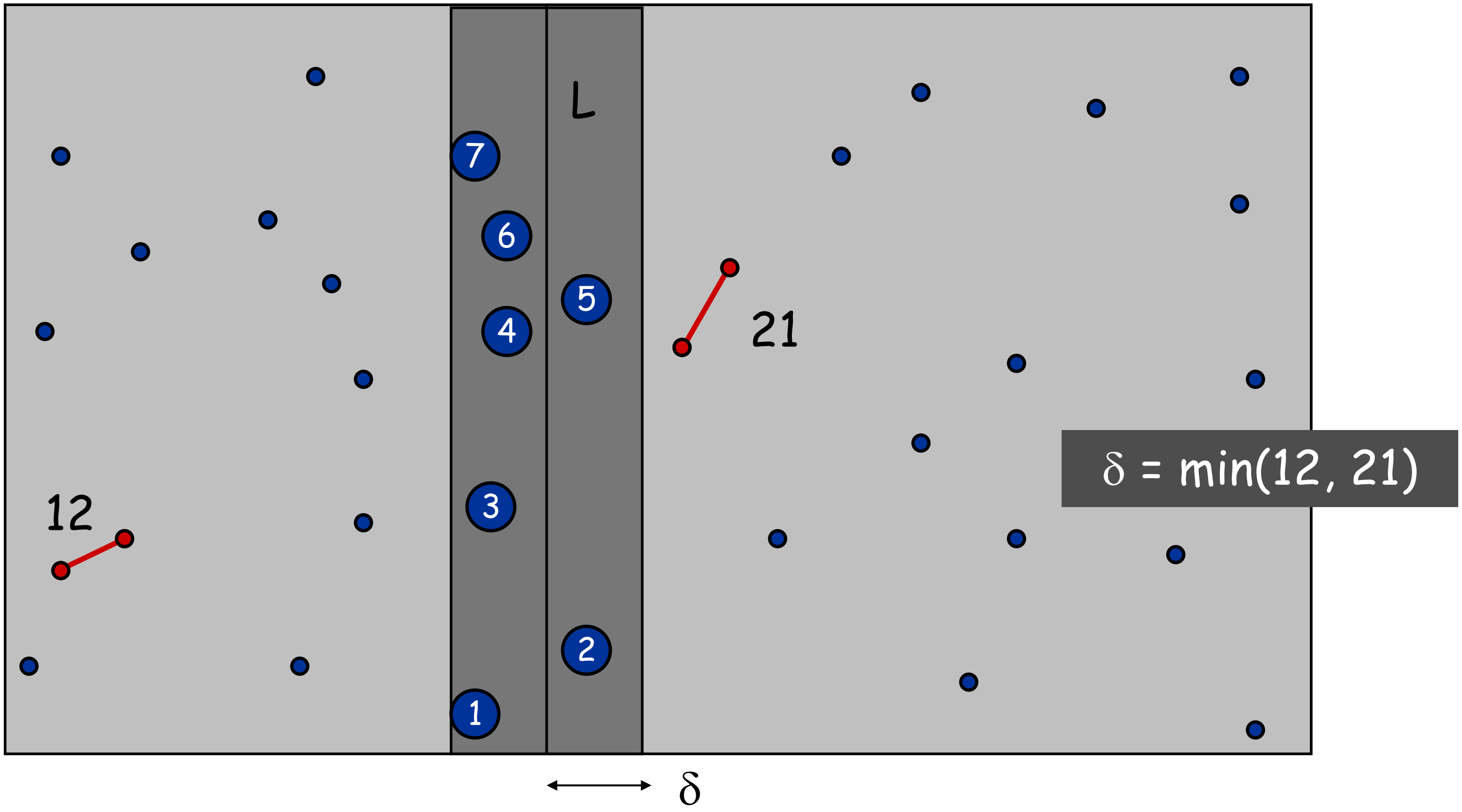
- Observation: only need to consider points within δ of line L.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

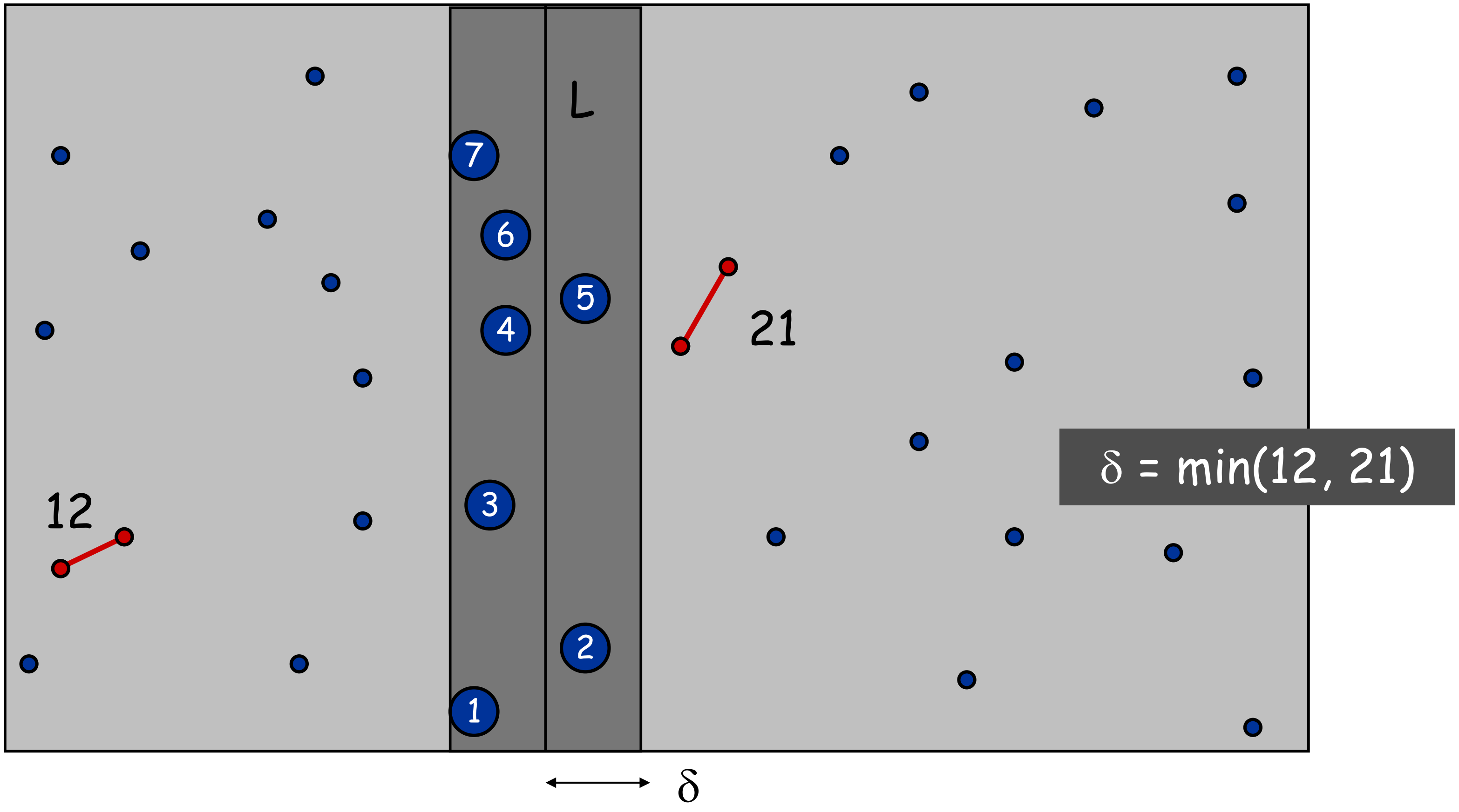
- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.
- Only check distances of those within 11 positions in sorted list!



Closest Pair of Points

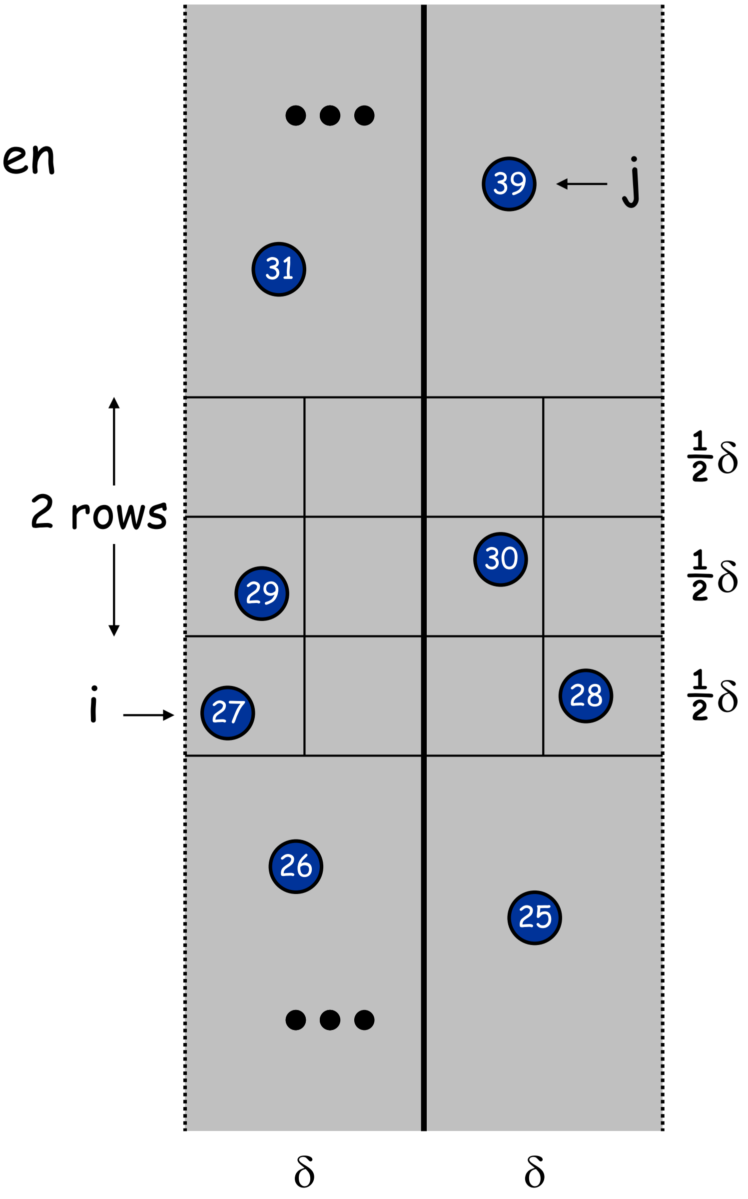
Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

Fact. Still true if we replace 12 with 7.



Closest Pair Algorithm

```
Closest-Pair($p_1$, …, $p_n$) {
   Compute separation line L such that half the points          O(n log n)
   are on one side and half on the other side.

   $\delta_1$ = Closest-Pair(left half)                              2T(n / 2)
   $\delta_2$ = Closest-Pair(right half)
   $\delta$  = min($\delta_1$, $\delta_2$)

   Delete all points further than $\delta$ from separation line L    O(n)

   Sort remaining points by y-coordinate.                          O(n log n)

   Scan points in y-order and compare distance between
   each point and next 11 neighbors. If any of these               O(n)
   distances is less than $\delta$, update $\delta$.

   return $\delta$.
}
```

Closest Pair of Points: Analysis

Running time.

$$T(n) \le 2T(n/2) + O(n \log n) \Rightarrow T(n) = O(n \log^2 n)$$

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points in strip from scratch each time.

- Each recursive returns two lists: all points sorted by y coordinate, and all points sorted by x coordinate.
- Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$$