

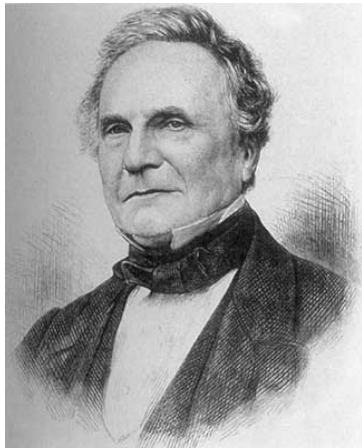
Chapter 2

Basics of Algorithm Analysis

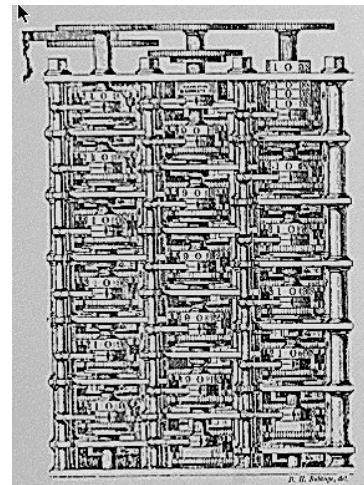
2.1 Computational Tractability

Computational Tractability

As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time? - *Charles Babbage*



Charles Babbage (1864)



Analytic Engine (schematic)

Polynomial-Time

Brute force. For many non-trivial problems, there is a natural brute force search algorithm that checks every possible solution.

- Typically takes 2^N time or worse for inputs of size N .
- Unacceptable in practice.

↖
 $n!$ for stable matching
with n men and n women

Desirable scaling property. When the input size doubles, the algorithm should only slow down by some constant factor C .

There exists constants $c > 0$ and $d > 0$ such that on every input of size N , its running time is bounded by $c N^d$ steps.

Def. An algorithm is **poly-time** if the above scaling property holds.

↖
choose $C = 2^d$

Worst-Case Analysis

Worst case running time. Obtain bound on **largest possible** running time of algorithm on input of a given size N .

- Generally captures efficiency in practice.
- Draconian view, but hard to find effective alternative.

Average case running time. Obtain bound on running time of algorithm on **random** input as a function of input size N .

- Hard (or impossible) to accurately model real instances by random distributions.
- Algorithm tuned for a certain distribution may perform poorly on other inputs.

Worst-Case Polynomial-Time

Def. An algorithm is **efficient** if its running time is polynomial.

Justification: **It really works in practice!**

- Although $6.02 \times 10^{23} \times N^{20}$ is technically poly-time, it would be useless in practice.
- In practice, the poly-time algorithms that people develop almost always have low constants and low exponents.
- Breaking through the exponential barrier of brute force typically exposes some crucial structure of the problem.

Exceptions.

- Some poly-time algorithms do have high constants and/or exponents, and are useless in practice.
- Some exponential-time (or worse) algorithms are widely used because the worst-case instances seem to be rare.



simplex method

Why It Matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1,000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10,000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100,000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1,000,000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

2.2 Asymptotic Order of Growth

Asymptotic Order of Growth

Upper bounds. $T(n)$ is $O(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$ we have $T(n) \leq c \cdot f(n)$.

Lower bounds. $T(n)$ is $\Omega(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$ we have $T(n) \geq c \cdot f(n)$.

Tight bounds. $T(n)$ is $\Theta(f(n))$ if $T(n)$ is both $O(f(n))$ and $\Omega(f(n))$.

Ex: $T(n) = 32n^2 + 17n + 32$.

- $T(n)$ is $O(n^2)$, $O(n^3)$, $\Omega(n^2)$, $\Omega(n)$, and $\Theta(n^2)$.
- $T(n)$ is not $O(n)$, $\Omega(n^3)$, $\Theta(n)$, or $\Theta(n^3)$.

Notation

Slight abuse of notation. $T(n) = O(f(n))$.

- Not transitive:
 - $f(n) = 5n^3$; $g(n) = 3n^2$
 - $f(n) = O(n^3) = g(n)$
 - but $f(n) \neq g(n)$.
- Better notation: $T(n) \in O(f(n))$.

Meaningless statement. Any comparison-based sorting algorithm requires at least $O(n \log n)$ comparisons.

- Statement doesn't "type-check."
- Use Ω for lower bounds.

Properties

Transitivity.

- If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.

Additivity.

- If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.
- If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.
- If $f = \Theta(h)$ and $g = \Theta(h)$ then $f + g = \Theta(h)$.

Asymptotic Bounds for Some Common Functions

Polynomials. $a_0 + a_1n + \dots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$.

Polynomial time. Running time is $O(n^d)$ for some constant d independent of the input size n .

Logarithms. $O(\log_a n) = O(\log_b n)$ for any constants $a, b > 0$.

↑
can avoid specifying the
base

Logarithms. For every $x > 0$, $\log n = O(n^x)$.

↑
log grows slower than every polynomial

Exponentials. For every $r > 1$ and every $d > 0$, $n^d = O(r^n)$.

↑
every exponential grows faster than every polynomial

2.4 A Survey of Common Running Times

Linear Time: $O(n)$

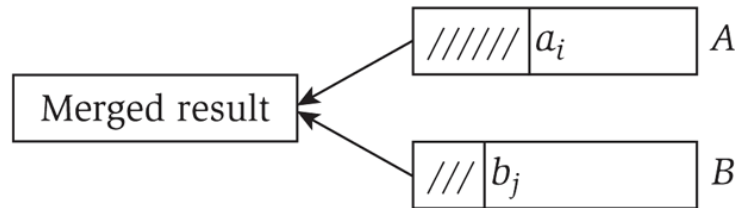
Linear time. Running time is proportional to input size.

Computing the maximum. Compute maximum of n numbers a_1, \dots, a_n .

```
max ← a1
for i = 2 to n {
  if (ai > max)
    max ← ai
}
```

Linear Time: $O(n)$

Merge. Combine two sorted lists $A = a_1, a_2, \dots, a_n$ with $B = b_1, b_2, \dots, b_n$ into sorted whole.



```
i = 1, j = 1
while (both lists are nonempty) {
    if (ai ≤ bj) append ai to output list and increment i
    else          append bj to output list and increment j
}
append remainder of nonempty list to output list
```

Claim. Merging two lists of size n takes $O(n)$ time.

Pf. After each comparison, the length of output list increases by 1.

$O(n \log n)$ Time

$O(n \log n)$ time. Arises in divide-and-conquer algorithms.



also referred to as linearithmic time

Sorting. Mergesort, heapsort, quicksort are sorting algorithms that perform $O(n \log n)$ comparisons (on average for quicksort).

Largest empty interval. Given n time-stamps x_1, \dots, x_n on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?

$O(n \log n)$ solution. Sort the time-stamps. Scan the sorted list in order, identifying the maximum gap between successive time-stamps.

Quadratic Time: $O(n^2)$

Quadratic time. Enumerate all pairs of elements.

Closest pair of points. Given a list of n points in the plane $(x_1, y_1), \dots, (x_n, y_n)$, find the pair that is closest.

$O(n^2)$ solution. Try all pairs of points.

```
min ← (x1 - x2)2 + (y1 - y2)2
for i = 1 to n {
  for j = i+1 to n {
    d ← (xi - xj)2 + (yi - yj)2
    if (d < min)
      min ← d
  }
}
```

← don't need to
take square roots

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion. ← see chapter 5

Cubic Time: $O(n^3)$

Cubic time. Enumerate all triples of elements.

Set disjointness. Given n sets S_1, \dots, S_n each of which is a subset of $1, 2, \dots, n$, is there some pair of these which are disjoint?

$O(n^3)$ solution. For each pairs of sets, determine if they are disjoint.

```
foreach set  $S_i$  {
  foreach other set  $S_j$  {
    foreach element  $p$  of  $S_i$  {
      determine whether  $p$  also belongs to  $S_j$ 
    }
    if (no element of  $S_i$  belongs to  $S_j$ )
      report that  $S_i$  and  $S_j$  are disjoint
  }
}
```

Polynomial Time: $O(n^k)$ Time

Independent I (dominating D) set of size k . Given a graph, are there k nodes such that no two are joined by an edge (all nodes not in the set have a neighbor in the set)?

k is a constant

$O(n^k)$ solution. Enumerate all subsets of k nodes.

```
foreach subset S of k nodes {  
  check whether S is I (D)  
  if (S is I (D))  
    report S is I (D)  
}  
}
```

- Check whether S is an independent set = $O(k^2)$.
- Number of k element subsets = $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots(2)(1)} \leq \frac{n^k}{k!}$
- $O(k^2 n^k / k!) = O(n^k)$.

poly-time for $k=17$,
but not practical

Exponential Time

Independent/Dominating set: maximum I (minimum D) Given a graph, what is maximum (minimum) size of an independent/dominating set?

$O(n^2 2^n)$ solution. Enumerate all subsets.

```
S* ← ∅  
foreach subset S of nodes {  
    check whether S is I (D)  
    if (S is largest I (smallest D) seen so far)  
        update S* ← S  
}  
}
```