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Inverse optimal control for strict-feedforward nonlinear systems with input delays

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Summary
We consider inverse optimal control for strict-feedforward systems with input delays. A basic predictor control is designed for compensation for this class of nonlinear systems. Furthermore, the proposed predictor control is inverse optimal with respect to a meaningful differential game problem. For a class of linearizable strict-feedforward system, an explicit formula for compensation for input delay, which is also inverse optimal with respect to a meaningful differential game problem, is also acquired. A cart with an inverted pendulum system is given to illustrate the validity of the proposed method.

KEYWORDS
actuator delay, explicit formula, inverse optimality, predictor feedback, strict-feedforward systems

1 | INTRODUCTION

The major progress on feedforward systems was in the work of Mazenc and Praly,[1] which introduced a Lyapunov approach for stabilization of feedforward systems. Further developments on feedforward systems have been acquired by other works.[2-5] For strict-feedforward systems with actuator delay, not only global stability was obtained but also an explicit formula for the predictor state was presented in the work of Krstic.[6]

Predictor-based controls for linear systems with input delays were developed in other works.[7-11] For nonlinear systems with time-varying input delays,[12-15] as well as wave actuator dynamics with moving boundaries,[16-18] predictor controls have also been achieved. The implementation and approximation issues of predictor-feedback law can be found in the work of Karafyllis and Krstic.[19]

The inverse optimality concept is of significant practical importance because it allows the design of optimal control laws without the need to solve a Hamilton-Jacobi-Isaacs partial differential equation that may not be possible to solve.[20]

In this paper, we extend the results in the work of Krstic[6] to inverse the optimal control design for strict-feedforward systems. A basic predictor control is designed for compensation for input delay of this class of nonlinear systems first. Furthermore, it is shown that it is inverse optimal with respect to a meaningful differential game problem. An explicit formula for compensation for input delay of a class of linearizable strict-feedforward system, which is also inverse optimal with respect to a meaningful differential game problem, is also acquired.

Notation . We use the common definitions of class $\mathcal{K}, \mathcal{K}_\infty, \mathcal{KL}$ functions from the aforementioned work.[6] For a vector $X \in R^n$,$|X|$ denotes its usual Euclidean norm. For a scalar function $u(\cdot, t) \in L_2(0,1)$, $\|u(t)\|$ denotes the norm given by $\left(\int_0^1 u^2(x,t)\mathrm{d}x\right)^{1/2}$.

2 | GENERAL STRICT-FEEDFORWARD NONLINEAR SYSTEMS

Consider a strict-feedforward nonlinear system with actuator delay

$$\dot{Z}_1(t) = Z_2(t) + \varphi_1(Z_2(t), Z_3(t), \dots, Z_n(t)) + \phi_1(Z_2(t), Z_3(t), \dots, Z_n(t))\, U(t-D) \tag{1}$$

$$\vdots \tag{2}$$

$$\dot{Z}_{n-2}(t) = Z_{n-1}(t) + \varphi_{n-2}(Z_{n-1}(t), Z_n(t)) + \phi_{n-2}(Z_{n-1}(t), Z_n(t))U(t-D) \tag{3}$$

$$\dot{Z}_{n-1}(t) = Z_n(t) + \phi_{n-1}(Z_n(t))U(t-D) \tag{4}$$

$$\dot{Z}_n(t) = U(t-D), \tag{5}$$

for short,

$$\dot{Z}_i(t) = Z_{i+1}(t) + \varphi_i\left(\underline{Z}_{i+1}(t)\right) + \phi_i\left(\underline{Z}_{i+1}(t)\right) U(t-D), \tag{6}$$

where $i = 1, 2, \dots, n$, $\underline{Z}_j = [Z_j, Z_{j+1}, \dots, Z_n]^T$, $Z_{n+1}(t) = U(t-D)$, $\phi_n = 1$, $\phi_i(0) = 0$, $(\partial\varphi_i(0)/\partial Z_j) = 0$, $\varphi_i(Z_{i+1}, 0, \dots, 0) = 0$, for $i = 1, 2, \dots, n-1$, $j = i+1, \dots, n$, and $\underline{Z}_1 \in R^n$ is the state vector, U is a scalar control input, and $D \in R^+$ is an actuator delay.

2.1 | Predictor control for general strict-feedforward nonlinear systems

The nominal control design ($D = 0$) for system (6) is given by Krstic[6] as

$$U(t) = \alpha_1(Z(t)), \tag{7}$$

where

$$\vartheta_{n+1} = 0, \quad \alpha_{n+1} = 0, \tag{8}$$

and

$$h_i\left(\underline{Z}_i\right) = Z_i - \vartheta_{i+1}\left(\underline{Z}_{i+1}\right), \tag{9}$$

$$\varpi_i\left(\underline{Z}_{i+1}\right) = \phi_i - \sum_{j=i+1}^{n-1} \frac{\partial\vartheta_{i+1}}{\partial Z_j}\phi_j - \frac{\partial\vartheta_{i+1}}{\partial Z_n}, \tag{10}$$

$$\alpha_i\left(\underline{Z}_i\right) = \alpha_{i+1} - \varpi_i h_i, \tag{11}$$

$$\vartheta_i\left(\underline{Z}_i\right) = -\int_0^\infty \left[\zeta_i^{[i]}\left(\tau, \underline{Z}_i\right) + \varphi_{i-1}\left(\underline{\zeta}_i^{[i]}\left(\tau, \underline{Z}_i\right)\right) + \phi_{i-1}\left(\underline{\zeta}_i^{[i]}\left(\tau, \underline{Z}_i\right)\right)\alpha_i\left(\underline{\zeta}_i^{[i]}\left(\tau, \underline{Z}_i\right)\right)\right]\mathrm{d}\tau, \tag{12}$$

for $i = n, n-1, \dots, 2, 1$, and the notation in the integrand of (12) refers to the solutions of the subsystem(s)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\zeta_j^{[i]} = \zeta_{j+1}^{[i]} + \varphi_j\left(\underline{\zeta}_{j+1}^{[i]}\right) + \phi_j\left(\underline{\zeta}_{j+1}^{[i]}\right)\alpha_i\left(\underline{\zeta}_i^{[i]}\right), \tag{13}$$

for $j = i, i+1, \dots, n$ at time τ, starting from the initial condition $\underline{X}_i$. Note that the last of the $\vartheta's$ that need to be computed is ϑ_2 (ϑ_1 is not defined).

Using a transport partial differential equation for representing the actuator state, we represent system (6) as

$$\dot{Z}_i(t) = Z_{i+1}(t) + \varphi_i\left(\underline{Z}_{i+1}(t)\right) + \varphi_i\left(\underline{Z}_{i+1}(t)\right) u(0,t), \tag{14}$$

$$u_t(x,t) = u_x(x,t), \tag{15}$$

$$u(D,t) = U(t), \tag{16}$$

where $i = 1, 2, \cdots, n$, and $u(x,t) = U(t + x - D)$.

The backstepping transformation is given as

$$w(x,t) = u(x,t) - \alpha_1(p(x,t)), \tag{17}$$

where $p(x,t) = [p_1(x,t), p_2(x,t), \cdots, p_n(x,t)]^T, x \in [0, D]$ is defined by

$$\frac{\partial p_1(x,t)}{\partial x} = p_2(x,t) + \varphi_1\left(p_2(x,t), p_3(x,t), \cdots, p_n(x,t)\right) + \phi_1\left(p_2(x,t), p_3(x,t), \cdots, p_n(x,t)\right) u(x,t) \tag{18}$$

$$\vdots \tag{19}$$

$$\frac{\partial p_{n-2}(x,t)}{\partial x} = p_{n-1}(x,t) + \varphi_{n-2}\left(p_{n-1}(x,t), p_n(x,t)\right) + \phi_{n-2}(p_{n-1}(x,t), p_n(x,t))u(x,t) \tag{20}$$

$$\frac{\partial p_{n-1}(x,t)}{\partial x} = p_n(x,t) + \phi_{n-1}\left(p_n(x,t)\right) u(x,t) \tag{21}$$

$$\frac{\partial p_n(x,t)}{\partial x} = u(x,t) \tag{22}$$

with an initial condition

$$p_i(0,t) = Z_i(t), i = 1, 2, \cdots, n. \tag{23}$$

From (18)-(23), we have

$$p_n(x,t) = Z_n(t) + \int_0^x u(y,t)\mathrm{d}y, \tag{24}$$

$$p_{n-1}(x,t) = Z_{n-1}(t) + \int_0^x \left(p_n(y,t) + \phi_{n-1}\left(p_n(y,t)\right) u(y,t)\right) \mathrm{d}y, \tag{25}$$

for $i = n-2, n-3, \cdots, 2, 1$, and the predictor solution is obtained recursively as

$$p_i(x,t) = Z_i(t) + \int_0^x \left(p_{i+1}(y,t) + \varphi_i\left(p_{i+1}(y,t), \cdots, p_n(y,t)\right) + \phi_i\left(p_{i+1}(y,t), \cdots, p_n(y,t)\right) u(y,t)\right) \mathrm{d}y. \tag{26}$$

A basic predictor feedback control law for system (14)-(16) is given as

$$U(t) = \frac{c}{c+1}\alpha_1\left(P(t)\right) = U^*(t), \tag{27}$$

where $c > 0$ is sufficiently large, and $P(t) = [p_1(D,t), p_2(D,t), \cdots, p_n(D,t)]^T$ is acquired by (24)-(26) for $x = D$.

Under the backstepping transformation (17), system (14)-(16) is transferred to a target system as

$$\dot{Z}_i(t) = Z_{i+1}(t) + \varphi_i\left(\underline{Z}_{i+1}(t)\right) + \phi_i\left(\underline{Z}_{i+1}(t)\right)\left(w(0,t) + \alpha_1\left(Z(t)\right)\right) \tag{28}$$

$$w_t(x,t) = w_x(x,t) \tag{29}$$

$$w(D,t) = U(t) - \alpha_1\left(p(D,t)\right). \tag{30}$$

Noting that $p(D,t) = [p_1(D,t), p_2(D,t), \cdots, p_n(D,t)]^T$ with the control law (27), (30) can be rewritten as

$$w(D,t) = -\frac{1}{c+1}\alpha_1(P(t)). \tag{31}$$

The inverse transformation of (17) is given for all $x \in [0, D]$ by

$$u(x,t) = w(x,t) + \alpha_1\left(q(x,t)\right), \tag{32}$$

where $q(x,t) = [q_1(x,t), q_2(x,t), \dots, q_n(x,t)]^T, x \in [0, D]$ is defined by

$$\frac{\partial q_1(x,t)}{\partial x} = q_2(x,t) + \varphi_1\left(q_2(x,t), q_3(x,t), \cdots, q_n(x,t)\right) + \phi_1\left(q_2(x,t), q_3(x,t), \cdots, q_n(x,t)\right)\left(w(x,t) + \alpha_1\left(q(x,t)\right)\right) \tag{33}$$

$$\vdots \tag{34}$$

$$\frac{\partial q_{n-2}(x,t)}{\partial x} = q_{n-1}(x,t) + \varphi_{n-2}\left(q_{n-1}(x,t), q_n(x,t)\right) + \phi_{n-2}\left(q_{n-1}(x,t), q_n(x,t)\right)\left(w(x,t) + \alpha_1\left(q(x,t)\right)\right) \tag{35}$$

$$\frac{\partial q_{n-1}(x,t)}{\partial x} = q_n(x,t) + \phi_{n-1}\left(q_n(x,t)\right)\left(w(x,t) + \alpha_1\left(q(x,t)\right)\right) \tag{36}$$

$$\frac{\partial q_n(x,t)}{\partial x} = w(x,t) + \alpha_1\left(q(x,t)\right) \tag{37}$$

with an initial condition

$$q_i(0,t) = Z_i(t), i = 1, 2, \cdots, n. \tag{38}$$

Under the inverse transformation (32), the target system (28), (29), (31) is transferred to system (14)-(16).

2.2 | Stability analysis of the closed-loop system

Denote the diffeomorphic transformation defined by (9)-(13) as

$$\xi(t) = H\left(Z(t)\right). \tag{39}$$

Lemma 1. *There exists a class $\mathcal{K}$ function σ^* such that*

$$\|p(t)\|_{L_\infty[0,D]} \leq \sigma^*(|Z(t)| + \|u(t)\|) \tag{40}$$

for all $t \geq 0$.

Proof. Using similar arguments to the proof in the work of Krstic,[6] it can be deduced. □

Lemma 2. *There exists a class $\mathcal{K}_\infty$ function $\underline{\sigma}$ such that*

$$|Z(t)| + \|u(t)\| \leq \underline{\sigma}(|Z(t)| + \|w(t)\|) \tag{41}$$

for all $t \geq 0$.

Proof. Using similar arguments to the proof in the work of Krstic,[6] it can be deduced. □

Lemma 3. *There exists a class $\mathcal{K}$ function $\overline{\sigma}$ such that*

$$|Z(t)| + \|w(t)\| \leq \overline{\sigma}(|Z(t)| + \|u(t)\|) \tag{42}$$

for all $t \geq 0$.

Proof. Using similar arguments to the proof in the work of Krstic,[6] it can be deduced. □

Note that α_1 is continuous with $\alpha_1(0) = 0$, and there exists a class $\mathcal{K}_\infty$ function ϱ_1 such that

$$\alpha_1^2\left(p(D,t)\right) \leq \varrho_1\left(|p(D,t)|\right). \tag{43}$$

Using Lemmas 1 and 2, we have

$$\begin{aligned} \alpha_1^2\left(p(D,t)\right) &\leq \varrho_1(|p(D,t)|) \\ &\leq \varrho_1\left(\sigma^*(|Z(t)| + \|u(t)\|)\right) \\ &\leq \varrho_1\left(\sigma^*\left(\underline{\sigma}\left(|Z(t)| + \|w(t)\|\right)\right)\right) \end{aligned} \tag{44}$$

for all $t \geq 0$.

Denote $\varphi = \varrho_1 \circ \sigma^* \circ \underline{\sigma}$, it is easy to know that

$$\alpha_1^2\left(p(D,t)\right) \leq \varphi(2|Z(t)|) + \varphi(2\|w(t)\|) \tag{45}$$

for all $t \geq 0$.

Now, we turn our attention to the target system and prove the following result on stability in the sense of its norm.

Lemma 4. *Consider the target system (28), (29), (31). If there exists an $M > 0$ such that*

$$\varphi\left(2|Z(t)|\right) \leq M\alpha_1^2\left(Z(t)\right), \tag{46}$$

$$\varphi\left(2\|w(t)\|\right) \leq M\|w(t)\|^2, \tag{47}$$

for all $t \geq 0$, then there exists $c_1^* > 0$, for all $c > c_1^*$, the target system (28), (29), (31) is asymptotically stable, that is, there exists a $\mathcal{KL}$ function β_1 such that

$$|Z(t)| + \|w(t)\| \leq \beta_1(|Z(0)| + \|w(0)\|, t) \tag{48}$$

for all $t \geq 0$.

Proof. Consider (28) along with the diffeomorphic transformation $\xi(t) = H(Z(t))$ defined by (39). With the observation that $Z_{i+1} + \varphi_i + \phi_i \alpha_{i+1} = \sum_{j=i+1}^{n} \frac{\partial \vartheta_{i+1}}{\partial Z_j}(Z_{j+1} + \varphi_j + \phi_j \alpha_{j+1})$, it is easy to verify that $\dot{\xi}_i = \varpi_i(\alpha_1 + w(0,t) + \sum_{j=i+1}^{n} \varpi_i \xi_i)$, noting from (11) that $\alpha_1 = -\sum_{j=1}^{n} \varpi_i \xi_i$, we get $\dot{\xi}_i = -\varpi_i^2 \xi_i - \sum_{j=1}^{i-1} \varpi_i \varpi_j \xi_j + \varpi_i w(0,t)$, and it implies that $\dot{\xi}_1 = -\varpi_1^2 \xi_1 + \varpi_1 w(0,t)$. Taking a Lyapunov function $S(t) = \frac{1}{2}\sum_{i=1}^{n} \xi_i^2(t) = \frac{1}{2}|H(Z)|^2$, we have that

$$\begin{aligned} \dot{S}(t) &= -\frac{1}{2}\sum_{i=1}^{n} \varpi_i^2 \xi_i^2 - \frac{1}{2}\left(\sum_{i=1}^{n} \xi_i \varpi_i\right)^2 + w(0,t)\sum_{i=1}^{n} \varpi_i \xi_i \\ &\leq -\frac{1}{4}\sum_{i=1}^{n} \varpi_i^2 \xi_i^2 - \frac{1}{2}\left(\sum_{i=1}^{n} \xi_i \varpi_i\right)^2 + n w^2(0,t). \end{aligned} \tag{49}$$

Consider system (28), (29), (31), an overall Lyapunov function is given as follows:

$$V(t) = S(t) + n\int_0^D e^x w^2(x,t)\mathrm{d}x. \tag{50}$$

With (49), we have that

$$\begin{aligned} \dot{V}(t) &= \dot{S}(t) + 2n\int_0^D e^x w(x,t) w_t(x,t)\mathrm{d}x \qquad (51) \\ &= \dot{S}(t) + n\int_0^D e^x \mathrm{d}w^2(x,t) \\ &= \dot{S}(t) + n e^D w^2(D,t) - n w^2(0,t) - n\int_0^D e^x w^2(x,t)\mathrm{d}x \\ &\leq -\frac{1}{4}\sum_{i=1}^{n} \varpi_i^2 \xi_i^2 - \frac{1}{2}\left(\sum_{i=1}^{n} \xi_i \varpi_i\right)^2 + n w^2(0,t) + n e^D w^2(D,t) - n w^2(0,t) - n\int_0^D e^x w^2(x,t)\mathrm{d}x \\ &= -\frac{1}{4}\sum_{i=1}^{n} \varpi_i^2 \xi_i^2 - \frac{1}{2}\left(\sum_{i=1}^{n} \xi_i \varpi_i\right)^2 + n e^D w^2(D,t) - n\int_0^D e^x w^2(x,t)\mathrm{d}x. \end{aligned}$$

With (31), we have

$$w^2(D,t) = \frac{1}{(c+1)^2}\alpha_1^2(P(t)). \tag{52}$$

Noting that $\alpha_1(Z(t)) = -\sum_{i=1}^{n} \varpi_i \xi_i$, we get

$$\dot{V}(t) \leq -\frac{1}{4n}\alpha_1^2(Z(t)) - \frac{1}{2}\alpha_1^2(Z(t)) + \frac{n e^D \alpha_1^2(P(t))}{(c+1)^2} - n\|w(t)\|^2. \tag{53}$$

With the help of (46), (47), it holds

$$\begin{aligned} \dot{V}(t) &\leq -\left(\frac{1}{4n} + \frac{1}{2}\right)\alpha_1^2(Z(t)) + \frac{n e^D(\varphi(2|Z(t)|) + \varphi(2\|w(t)\|))}{(c+1)^2} - n\|w(t)\|^2 \\ &\leq -\left(\left(\frac{1}{4n} + \frac{1}{2}\right) - \frac{n e^D M}{(c+1)^2}\right)\alpha_1^2(Z(t)) - \left(n - \frac{n e^D M}{(c+1)^2}\right)\|w(t)\|^2. \end{aligned} \tag{54}$$

Choosing

$$c_1^* = 2n\sqrt{2M}e^{D/2} \tag{55}$$

for all $c > c_1^*$, one has

$$\dot{V}(t) \leq -\left(\frac{1}{8n} + \frac{1}{4}\right)\alpha_1^2(Z(t)) - \frac{n}{2}\|w(t)\|^2, \tag{56}$$

so the target system (28), (29), (31) is asymptotically stable. Since the function $\alpha_1^2(Z(t))$ is positive definite in $Z(t)$, there exists a class $\mathcal{K}$ function γ_1 such that $\dot{V}(t) \leq -\gamma_1(V(t))$. Then, there exists a class $\mathcal{KL}$ function β_2 such that $V(t) \leq \beta_2(V(0), t)$ for all $t \geq 0$. With additional routine class $\mathcal{K}$ calculations, one finds β_1 that completes the proof of the lemma. □

Theorem 1. *Consider the closed-loop system consisting of (14)-(16) together with the control law (27). If there exists a $M > 0$ such that (46), (47) hold, then there exists $c_1^* > 0$ given by (55), for all $c > c_1^*$, the closed-loop system of (14)-(16), (27) is asymptotically stable, that is, there exists a class $\mathcal{KL}$ function β_3 such that*

$$|Z(t)| + \|u(t)\| \le \beta_3(|Z(0)| + \|u(0)\|, t) \tag{57}$$

for all $t \ge 0$.

Proof. Using Lemmas 2 , 3, and 4, we have

$$\begin{aligned} &|Z(t)| + \|u(t)\| \\ &\le \underline{\sigma}\left(|Z(t)| + \|w(t)\|\right) \\ &\le \underline{\sigma}\left(\beta_1\left(|Z(0)| + \|w(0)\|, t\right)\right) \\ &\le \underline{\sigma}\left(\beta_1\left(\overline{\sigma}\left(|Z(0)| + \|u(0)\|\right), t\right)\right) \end{aligned} \tag{58}$$

for all $t \ge 0$. Denote that $\beta_3(s,t) = \underline{\sigma}(\beta_1(\overline{\sigma}(s), t))$, (57) is drawn. Hence, the closed-loop system of (14)-(16), (27) is asymptotically stable.

Theorem 2. *Consider the closed-loop system consisting of (1)-(5) together with the control law (27). If there exists an $M > 0$ such that (46), (47) hold, then there exists $c_1^* > 0$ given by (55), for all $c > c_1^*$, the closed-loop system of (1)-(5), (27) is asymptotically stable, that is, there exists a class $\mathcal{KL}$ function β_4 such that*

$$|Z(t)| + \left(\int_{t-D}^{t} U^2(\theta)\mathrm{d}\theta\right)^{1/2} \le \beta_4\left(\|Z(0)\| + \left(\int_{-D}^{0} U^2(\theta)\mathrm{d}\theta\right)^{1/2}, t\right) \tag{59}$$

for all $t \ge 0$. □

Proof. Using Theorem 1, we get

$$\begin{aligned} &|Z(t)| + \left(\int_{t-D}^{t} U^2(\theta)\mathrm{d}\theta\right)^{1/2} \\ &= |Z(t)| + \|u(t)\| \\ &\le \beta_3\left(|Z(0)| + \|u(0)\|, t\right) \\ &= \beta_3\left(|Z(0)| + \left(\int_{-D}^{0} U^2(\theta)\mathrm{d}\theta\right)^{1/2}, t\right) \end{aligned} \tag{60}$$

for all $t \ge 0$. Choosing $\beta_4 = \beta_3$, (59) is obtained. Hence, the closed-loop system of (1)-(5), (27) is asymptotically stable. □

2.3 | Inverse optimal control for general strict-feedforward nonlinear systems

Theorem 3. *Consider the closed-loop system consisting of (14)-(16) together with the control law (27). If there exists an $M > 0$ such that (46), (47) hold, then there exists $c_1^{**} > c_1^* > 0$, for all $c > c_1^{**}$, the control law (27) minimizes the cost functional*

$$J = \lim_{t\to\infty}\left(\gamma V(t) + \int_0^t \left(L(\tau) + \frac{\gamma n e^D}{c} U^2(\tau)\right)\mathrm{d}\tau\right), \tag{61}$$

where $V(t)$ is given by (50), and L is a functional of $(Z(t), U(\theta))$ for all $t - D \le \theta \le t$ such that

$$L(t) \ge \gamma\left(\frac{\alpha_1^2(Z(t))}{8n} + \frac{n}{2}\|w(t)\|^2\right) \tag{62}$$

for an arbitrary $\gamma > 0$.

Proof. Let

$$L(t) = -\frac{\gamma n e^D}{c+1}\alpha_1^2\left(P(t)\right) + \gamma\left(\frac{1}{2}\sum_{i=1}^{n}\varpi_i^2\xi_i^2 + \frac{1}{2}\left(\sum_{i=1}^{n}\xi_i\varpi_i\right)^2 - w(0,t)\sum_{i=1}^{n}\varpi_i\xi_i + nw^2(0,t) + n\int_0^D e^x w^2(x,t)\mathrm{d}x\right). \tag{63}$$

It can be deduced that

$$L(t) \geq -\frac{\gamma n e^D}{c+1}\alpha_1^2(P(t)) + \frac{\gamma}{4}\sum_{i=1}^{n}\varpi_i^2\xi_i^2 + \frac{\gamma}{2}\left(\sum_{i=1}^{n}\xi_i\varpi_i\right)^2 + n\gamma\int_0^D e^x w^2(x,t)\mathrm{d}x. \tag{64}$$

With the help of (46), (47), there exists

$$c_1^{**} = 8n^2 M e^D \tag{65}$$

for all $c > c_1^{**}$, one has

$$L(t) \geq \frac{\gamma}{8n}\alpha_1^2(Z(t)) + \frac{n\gamma}{2}\|w(t)\|^2 \tag{66}$$

for any $t \geq 0$.

With the help of (49), (51), after some calculations, and noting $U^*(t) = \frac{c}{c+1}\alpha_1(P(t))$, we have

$$\begin{aligned}
L(t) &= -\frac{\gamma n e^D}{c+1}\alpha_1^2(P(t)) + \gamma\left(\frac{1}{2}\sum_{i=1}^{n}\varpi_i^2\xi_i^2 + \frac{1}{2}\left(\sum_{i=1}^{n}\xi_i\varpi_i\right)^2 - w(0,t)\sum_{i=1}^{n}\varpi_i\xi_i + nw^2(0,t) + n\int_0^D e^{gx}w^2(x,t)\mathrm{d}x\right)\\
&= -\frac{\gamma n e^D}{c+1}\alpha_1^2\left(P(t)\right) + \gamma n e^D w^2(D,t) - \gamma\dot{V}(t)\\
&= -\frac{\gamma n e^D}{c+1}\alpha_1^2\left(P(t)\right) + \gamma n e^D(U(t) - \alpha_1\left(P(t)\right))^2 - \gamma\dot{V}(t)\\
&= -\frac{\gamma n e^D(c+1)}{c^2}(U^*(t))^2 + \gamma n e^D\left(U(t) - \frac{c+1}{c}U^*(t)\right)^2 - \gamma\dot{V}(t)\\
&= \frac{\gamma n e^D}{c}(U^*(t))^2 + \gamma n e^D\left((U(t) - U^*(t))^2 - \frac{2}{c}U(t)U^*(t)\right) - \gamma\dot{V}(t),
\end{aligned} \tag{67}$$

and hence, it can be deduced that

$$\gamma V(t) + \int_0^t\left(L(\tau) + \frac{\gamma n e^D}{c}U^2(\tau)\right)\mathrm{d}\tau = \gamma V(0) + \gamma\int_0^t n e^D\left(1 + \frac{1}{c}\right)(U(t) - U^*(t))^2\mathrm{d}\tau \tag{68}$$

so the minimum of (61) is reached with

$$U(t) = U^*(t) \tag{69}$$

such that

$$J = \gamma V(0). \tag{70}$$

□

Remark 1. c_1^{**} given by (65) is bigger than c_1^* defined by (55).

3 | LINEARIZABLE STRICT-FEEDFORWARD SYSTEMS

From the work of Krstic,[6] it was shown that a strict-feedforward system (1)-(5) for $D = 0$ is linearizable provided the following assumption is satisfied.

Assumption 1. The functions $\varphi_i(\underline{Z}_{i+1})$ and $\phi_i(\underline{Z}_{i+1})$ can be written as $\phi_{n-1}(Z_n) = \theta_n'(Z_n)$ and $\varphi_{n-1}(Z_n) = 0$, and

$$\phi_i\left(\underline{Z}_{i+1}\right) = \sum_{j=i+1}^{n-1}\frac{\partial\theta_{i+1}\left(\underline{Z}_{i+1}\right)}{\partial Z_j}\phi_j\left(\underline{Z}_{j+1}\right) + \frac{\partial\theta_{i+1}\left(\underline{Z}_{i+1}\right)}{\partial Z_n} \tag{71}$$

$$\varphi_i\left(\underline{Z}_{i+1}\right) = \sum_{j=i+1}^{n-1}\frac{\partial\theta_{i+1}\left(\underline{Z}_{i+1}\right)}{\partial Z_j}\left(Z_{j+1} + \varphi_j\left(\underline{Z}_{j+1}\right)\right) - \theta_{i+2}\left(\underline{Z}_{i+2}\right) \tag{72}$$

for $i = n-2, \ldots, 1$, using some C^1 scalar-valued functions $\theta_i(\underline{Z}_i)$ satisfying $\theta_i(0) = (\partial\theta_i(0)/\partial Z_j) = 0$, for $i = 2, \ldots, n, j = i, \ldots, n$.

The nominal control design ($D = 0$) for linearizable strict-feedforward (1)-(5) is given by Krstic[6] as

$$U(t) = \alpha_1\left(Z(t)\right), \tag{73}$$

where $\vartheta_{n+1} = 0$, $\alpha_{n+1} = 0$, and, for $i = n, n-1, \dots, 2, 1$,

$$\alpha_i\left(\underline{Z}_i\right) = -\sum_{j=i}^{n}\left(Z_j - \vartheta_{j+1}\left(\underline{Z}_{j+1}\right)\right), \tag{74}$$

$$\zeta_n^{[i]}\left(\tau, \underline{Z}_i\right) = e^{-\tau}\sum_{k=0}^{n-i}\frac{(-\tau)^k}{k!}\left(Z_{n-k} - \vartheta_{n-k+1}\left(\underline{Z}_{n-k+1}\right)\right) \tag{75}$$

$$\zeta_j^{[i]}\left(\tau, \underline{Z}_i\right) = e^{-\tau}\sum_{k=0}^{j-i}\frac{(-\tau)^k}{k!}\left(Z_{j-k} - \vartheta_{j-k+1}\left(\underline{Z}_{j-k+1}\right)\right) + \vartheta_{j+1}\left(\zeta_{j+1}^{[i]}\left(\tau, \underline{Z}_i\right)\right) \tag{76}$$

$$\vartheta_i\left(\underline{Z}_i\right) = -\int_0^{\infty}\left[\zeta_i^{[i]}\left(\tau, \underline{Z}_i\right) + \varphi_{i-1}\left(\underline{\zeta}_i^{[i]}\left(\tau, \underline{Z}_i\right)\right) + \phi_{i-1}\left(\underline{\zeta}_i^{[i]}\left(\tau, \underline{Z}_i\right)\right)\alpha_i\left(\underline{\zeta}_i^{[i]}\left(\tau, \underline{Z}_i\right)\right)\right]\mathrm{d}\tau. \tag{77}$$

3.1 | Predictor control for linearizable strict-feedforward systems

Consider the linearizable strict-feedforward system with actuator delay

$$\dot{Z}_i(t) = Z_{i+1}(t) + \varphi_i\left(\underline{Z}_{i+1}(t)\right) + \phi_i\left(\underline{Z}_{i+1}(t)\right)u(0,t) \tag{78}$$

$$u_t(x,t) = u_x(x,t) \tag{79}$$

$$u(D,t) = U(t), \tag{80}$$

where $i = 1, 2, \dots, n$.

With the diffeomorphic transformation $h = G(Z)$ defined by

$$h_n = Z_n \tag{81}$$

$$h_i = \sum_{j=i}^{n}\begin{pmatrix} n-i \\ j-i \end{pmatrix}(-1)^{j-i}\left(Z_j - \vartheta_{j+1}\left(\underline{Z}_{j+1}\right)\right), i = n-1, n-2, \dots, 1 \tag{82}$$

and $\vartheta_j, j = 1, 2, \dots, n$ given by (74)-(77), system (78)-(80) is transferred to the following system:

$$\dot{h}_i(t) = h_{i+1}(t),\ i = 1, 2, \dots, n-1, \tag{83}$$

$$\dot{h}_n(t) = u(0,t) \tag{84}$$

$$u_t(x,t) = u_x(x,t) \tag{85}$$

$$u(D,t) = U(t). \tag{86}$$

The predictor feedback for system (83)-(86) is

$$U(t) = \frac{c}{c+1}\alpha_1\left(G^{-1}\left(\eta(D,t)\right)\right) = -\frac{c}{c+1}\sum_{i=1}^{n}\begin{pmatrix} n \\ i-1 \end{pmatrix}\eta_i(D,t) = U^*(t), \tag{87}$$

where $c > 0$ is sufficiently large, and $\eta(D,t) = [\eta_1(D,t), \dots, \eta_n(D,t)]^T$ is given by

$$\frac{\partial}{\partial x}\eta_i(x,t) = \eta_{i+1}(x,t),\ i = 1, 2, \dots, n-1, \tag{88}$$

$$\frac{\partial}{\partial x}\eta_n(x,t) = u(x,t) \tag{89}$$

with initial condition $\eta(0,t) = h(t)$ for $x = D$.

It can be deduced that

$$\eta_i(x,t) = \sum_{j=i}^{n}\frac{x^{j-i}}{(j-i)!}h_j(t) + \int_0^x \frac{(x-y)^{n+1-i}}{(n+1-i)!}u(y,t)\mathrm{d}y, \tag{90}$$

for $i = 1, 2, \dots, n$. By (81)-(82), we have

$$\eta_i(x,t) = \sum_{j=i}^{n}\frac{x^{j-i}}{(j-i)!}\sum_{l=j}^{n}\begin{pmatrix} n-j \\ l-j \end{pmatrix}(-1)^{l-j}\left(Z_l - \vartheta_{l+1}\left(\underline{Z}_{l+1}\right)\right) + \int_0^x \frac{(x-y)^{n+1-i}}{(n+1-i)!}u(y,t)\mathrm{d}y, \tag{91}$$

for $i = 1, 2, \ldots, n$. Hence, the feedback law for system (83)-(86) can be rewritten as

$$U(t) = -\frac{c}{c+1}\sum_{i=1}^{n}\binom{n}{i-1}\left(\sum_{j=i}^{n}\frac{D^{j-i}}{(j-i)!}\sum_{l=j}^{n}\binom{n-j}{l-j}(-1)^{l-j}\left(Z_l - \vartheta_{l+1}\left(\underline{Z}_{l+1}\right)\right) + \int_0^D \frac{(D-y)^{n+1-i}}{(n+1-i)!}u(y,t)\mathrm{d}y\right). \tag{92}$$

Noting that $u(x,t) = U(x+t-D)$, the predictor control law for system (78)-(80) is

$$U(t) = -\frac{c}{c+1}\sum_{i=1}^{n}\binom{n}{i-1}\left(\sum_{j=i}^{n}\frac{D^{j-i}}{(j-i)!}\sum_{l=j}^{n}\binom{n-j}{l-j}(-1)^{l-j}\left(Z_l - \vartheta_{l+1}\left(\underline{Z}_{l+1}\right)\right)\int_{t-D}^{t} \frac{(D-y)^{n+1-i}}{(n+1-i)!}U(\sigma)\mathrm{d}\sigma\right), \tag{93}$$

where $c > 0$ is sufficiently large.

Next, we will prove that the closed-loop system consisting of (78)-(80) together with the control law (93) is asymptotically stable.

With a diffeomorphic transformation

$$\xi_{n-i} = \sum_{j=0}^{i}\binom{i}{j}h_{n-j}, \quad i = 0, 1, 2, \ldots, n-1, \tag{94}$$

system (83)-(86) is transferred to

$$\dot{\xi}_i(t) = \sum_{j=i+1}^{n}\xi_j(t) + u(0,t),\ i = 1, 2, \ldots, n-1 \tag{95}$$

$$\dot{\xi}_n(t) = u(0,t) \tag{96}$$

$$u_t(x,t) = u_x(x,t) \tag{97}$$

$$u(D,t) = U(t), \tag{98}$$

and it can be deduced that

$$\sum_{i=1}^{n}\binom{n}{i-1}h_i(t) = \sum_{i=1}^{n}\xi_i(t). \tag{99}$$

The infinite-dimensional backstepping transformation is defined as follows:

$$w(x,t) = u(x,t) + \sum_{i=1}^{n}\binom{n}{i-1}\eta_i(x,t), \tag{100}$$

where $\eta_i(x,t), i = 1, 2, \ldots, n$ are given by (90).

Noting that $\eta_i(0,t) = h_i(t)$, with the help of (87), (98)-(100), system (95)-(98) is transferred to the target system

$$\dot{\xi}_i(t) = -\sum_{j=1}^{i}\xi_j(t) + w(0,t),\ i = 1, 2, \ldots, n \tag{101}$$

$$w_t(x,t) = w_x(x,t) \tag{102}$$

$$w(D,t) = \frac{1}{c}\sum_{i=1}^{n}\binom{n}{i-1}\eta_i(D,t). \tag{103}$$

The inverse backstepping transformation of (100) is defined as follows:

$$u(x,t) = w(x,t) - \sum_{i=1}^{n}\varpi_i(x,t), \tag{104}$$

where

$$\frac{\partial}{\partial x}\varpi_i(x,t) = -\sum_{j=1}^{i}\varpi_j(x,t) + w(x,t),\ i = 1, 2, \ldots, n \tag{105}$$

with initial condition $\varpi_i(0,t) = \xi(t)$.

Under the inverse backstepping transformation (104), the target system (101)-(103) is transferred to system (95)-(98).

Lemma 5. *Consider the target system (101)-(103), there exists $c^* > 0$ such that system (101)-(103) is asymptotically stable for all $c > c^*$, that is, there exist $R > 0$, $\overline{\lambda} > 0$, such that for all $c > c^*$,*

$$|\xi(t)|^2 + \|w(t)\|^2 \leq Re^{-\overline{\lambda} t}(|\xi(0)|^2 + \|w(0)\|^2) \tag{106}$$

for all $t \geq 0$.

Proof. Denote

$$A = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 0 \\ -1 & -1 & -1 & \cdots & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}. \tag{107}$$

Since A is a Hurwitz matrix, for any a positive matrix Q, there exists a positive matrix P such that $AP + PA^T = -Q$.

Considering system (101)-(103), an overall Lyapunov function is given as follows:

$$V(t) = \xi^T P\xi + l \int_0^D e^x w^2(x,t)\mathrm{d}x, \tag{108}$$

where $l > \frac{2\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q)}$. We have that

$$\begin{aligned} \dot{V}(t) &= \xi^T \left(AP + PA^T\right) \xi + 2\xi^T PBw(0,t) + 2l \int_0^D e^x w(x,t)w_t(x,t)\mathrm{d}x \\ &= -\xi^T Q\xi + 2\xi^T PBw(0,t) + 2l \int_0^D e^x w(x,t)w_t(x,t)\mathrm{d}x \\ &\leq -\lambda_{\min}(Q)\xi^T\xi + \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(PBB^TP)}\xi^T PBB^T P\xi + \frac{2\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q)}w^2(0,t) + l\int_0^D e^x \mathrm{d}w^2(x,t) \\ &\leq -\frac{\lambda_{\min}(Q)}{2}\xi^T\xi + \frac{2\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q)}w^2(0,t) + le^D w^2(D,t) - lw^2(0,t) - l\int_0^D e^x w^2(x,t)\mathrm{d}x \\ &\leq -\frac{\lambda_{\min}(Q)}{2}\xi^T\xi + le^D w^2(D,t) - l\int_0^D e^x w^2(x,t)\mathrm{d}x \\ &\leq -\frac{\lambda_{\min}(Q)}{2}|\xi|^2 + le^D w^2(D,t) - l\|w(t)\|^2. \end{aligned} \tag{109}$$

From (103), we have

$$w^2(D,t) = \frac{1}{c^2}\left(\sum_{i=1}^n \begin{pmatrix} n \\ i-1 \end{pmatrix} \eta_i(D,t)\right)^2. \tag{110}$$

Using (90), we get

$$\begin{aligned} \eta_i(D,t) &\leq \left|\sum_{j=i}^n \frac{D^{j-i}}{(j-i)!}h_j(t)\right| + \left|\int_0^D \frac{(D-y)^{n+1-i}}{(n+1-i)!}u(y,t)\mathrm{d}y\right| \\ &\leq e^D|h(t)| + \frac{D^{n+1-i}}{(n+1-i)!}\sqrt{D}\|u(t)\| \\ &\leq e^D|h(t)| + \max\left\{D, \frac{D^2}{2!}, \ldots, \frac{D^n}{n!}\right\}\sqrt{D}\|u(t)\|, \end{aligned} \tag{111}$$

so

$$w^2(D,t) \leq \frac{1}{c^2}(2^n-1)^2\left(2e^{2D}|h(t)|^2 + 2\varsigma D\|u(t)\|^2\right), \tag{112}$$

where

$$\varsigma = \left(\max\left\{D, \frac{D^2}{2!}, \ldots, \frac{D^n}{n!}\right\}\right)^2. \tag{113}$$

It can be deduced that the inverse of (94) is

$$h_{n-i}(t) = \sum_{j=0}^i (-1)^{i+j} \begin{pmatrix} i \\ j \end{pmatrix} \xi_{n-j}(t), \quad i = 0,1,2,\ldots,n-1, \tag{114}$$

and after some calculation, we have

$$|h(t)| \leq \frac{\sqrt{4^n - 1}}{\sqrt{3}} |\xi(t)|. \tag{115}$$

It is easy to get from (105) that

$$\varpi(x,t) = e^{Ax}\xi(t) + \int_0^x e^{A(x-s)} B w(s,t) ds, \tag{116}$$

where A and B are given by (107). Furthermore, we get

$$\begin{aligned} |\varpi(x,t)|^2 &\leq 2e^{2|A|x}|\xi(t)|^2 + 2\left|\int_0^x e^{A(x-s)} B w(s,t) ds\right|^2 \\ &\leq 2e^{2|A|x}|\xi(t)|^2 + 2\int_0^x \left|e^{A(x-s)}B\right|^2 ds \int_0^x w^2(s,t) ds \\ &\leq 2e^{2|A|x}|\xi(t)|^2 + 2|B|^2 \int_0^x e^{2|A|(x-s)} ds \int_0^x w^2(s,t) ds \\ &= 2e^{2|A|x}|\xi(t)|^2 + |B|^2 \frac{e^{2|A|x} - 1}{|A|} \int_0^x w^2(s,t) ds. \end{aligned} \tag{117}$$

Using (104), we have

$$\begin{aligned} u^2(x,t) &\leq 2w^2(x,t) + 2\left(\sum_{i=1}^n \varpi_i(x,t)\right)^2 \\ &\leq 2w^2(x,t) + 2n\sum_{i=1}^n \varpi_i^2(x,t) \\ &= 2w^2(x,t) + 2n|\varpi(x,t)|^2. \end{aligned} \tag{118}$$

By (117), (118), it can be deduced that

$$\|u(t)\|^2 \leq 2\|w(t)\|^2 + \frac{2n\left(e^{2|A|D} - 1\right)}{|A|}|\xi(t)|^2 + \frac{2n|B|^2}{|A|}\left(\frac{e^{2|A|D} - 1}{2|A|} - D\right)\|w(t)\|^2. \tag{119}$$

With the help of (112), (115), (119), we arrive at

$$\begin{aligned} w^2(D,t) &\leq \frac{1}{c^2}(2^n - 1)^2\left(\frac{2e^{2D}(4^n - 1)}{3}|\xi(t)|^2 + 2\varsigma D\|u(t)\|^2\right) \\ &\leq \frac{2(2^n - 1)^2}{c^2}\frac{e^{2D}(4^n - 1)}{3}|\xi(t)|^2 \\ &\quad + \frac{2(2^n - 1)^2}{c^2}\varsigma D\left(2|w(t)|^2 + \frac{2n\left(e^{2|A|D} - 1\right)}{|A|}|\xi(t)|^2 + \frac{2n|B|^2}{|A|}\left(\frac{e^{2|A|D} - 1}{2|A|} - D\right)|w(t)|^2\right) \\ &= \frac{2(2^n - 1)^2}{c^2}\left(\frac{e^{2D}(4^n - 1)}{3} + \frac{2n\varsigma D\left(e^{2|A|D} - 1\right)}{|A|}\right)|\xi(t)|^2 \\ &\quad + \frac{4(2^n - 1)^2}{c^2}\varsigma D\left(1 + \frac{n|B|^2}{|A|}\left(\frac{e^{2|A|D} - 1}{2|A|} - D\right)\right)|w(t)|^2, \end{aligned} \tag{120}$$

where ς is given by (113). Using (109), (120), we get

$$\begin{aligned} \dot{V}(t) &\leq -\frac{\lambda_{\min}(Q)}{2}|\xi(t)|^2 + \frac{2(2^n - 1)^2 l e^D}{c^2}\left(\frac{e^{2D}(4^n - 1)}{3} + \frac{2n\varsigma D\left(e^{2|A|D} - 1\right)}{|A|}\right)|\xi(t)|^2 \\ &\quad + \frac{4le^D(2^n - 1)^2}{c^2}\varsigma D\left(1 + \frac{n|B|^2}{|A|}\left(\frac{e^{2|A|D} - 1}{2|A|} - D\right)\right)|w(t)|^2 - l\|w(t)\|^2. \end{aligned} \tag{121}$$

Choosing

$$c^* = 2\sqrt{2}(2^n-1)e^{\frac{D}{2}} \max\left\{\sqrt{\frac{le^{2D}(4^n-1)}{3\lambda_{\min}(Q)} + \frac{2nl\varsigma D\left(e^{2|A|D}-1\right)}{|A|\lambda_{\min}(Q)}}, \sqrt{\varsigma D\left(1+\frac{n|B|^2}{|A|}\left(\frac{e^{2|A|D}-1}{2|A|}-D\right)\right)}\right\} \tag{122}$$

for all $c > c^*$, we get

$$\begin{aligned}\dot{V}(t) &\leq -\frac{\lambda_{\min}(Q)}{4}|\xi(t)|^2 - \frac{l}{2}\|w(t)\|^2 \\ &\leq -\min\left\{\frac{\lambda_{\min}(Q)}{4}, \frac{l}{2}\right\}\left\{|\xi(t)|^2 + \|w(t)\|^2\right\}.\end{aligned} \tag{123}$$

With (108), we have

$$\begin{aligned}&\min\{\lambda_{\min}(P), l\}\left(|\xi(t)|^2 + \|w(t)\|^2\right) \\ &\leq V(t) \\ &\leq \max\left\{\lambda_{\max}(P), le^D\right\}\left(|\xi(t)|^2 + \|w(t)\|^2\right).\end{aligned} \tag{124}$$

Thus, from (123), (124), it holds that

$$\dot{V}(t) \leq -\overline{\lambda}V(t) \tag{125}$$

with

$$\overline{\lambda} = \frac{\min\left\{\frac{\lambda_{\min}(Q)}{4}, \frac{l}{2}\right\}}{\max\{\lambda_{\max}(P), le^D\}}. \tag{126}$$

We arrive at

$$\begin{aligned}V(t) &\leq e^{-\overline{\lambda}t}V(0) \\ &\leq e^{-\overline{\lambda}t}\max\left\{\lambda_{\max}(P), le^D\right\}\left(|\xi(0)|^2 + \|w(0)\|^2\right).\end{aligned} \tag{127}$$

With the help of (124), we have

$$\begin{aligned}|\xi(t)|^2 + \|w(t)\|^2 &\leq \frac{V(t)}{\min\{\lambda_{\min}(P), l\}} \\ &\leq \frac{\max\left\{\lambda_{\max}(P), le^D\right\}}{\min\{\lambda_{\min}(P), l\}}e^{-\overline{\lambda}t}\left(|\xi(0)|^2 + \|w(0)\|^2\right).\end{aligned} \tag{128}$$

Thus, for all $c > c^*$, we get (106) where $c^*, \overline{\lambda}$ are given by (122) and (126), respectively, and $R = \frac{\max\{\lambda_{\max}(P), le^D\}}{\min\{\lambda_{\min}(P), l\}}$. The proof is completed. □

Lemma 6. *Considering system (83)-(86), there exists $c^* > 0$ such that system (83)-(86) is asymptotically stable for all $c > c^*$, that is, there exist $\overline{R} > 0, \overline{\lambda} > 0$, such that for all $c > c^*$,*

$$|h(t)|^2 + \|u(t)\|^2 \leq \overline{R}e^{-\overline{\lambda}t}\left(|h(0)|^2 + \|u(0)\|^2\right) \tag{129}$$

for all $t \geq 0$.

Proof. With the help of (94), we get

$$|\xi(t)| \leq \frac{\sqrt{4^n-1}}{\sqrt{3}}|h(t)|. \tag{130}$$

Using (90), we have

$$\begin{aligned}\eta_i(x,t) &\leq \left|\sum_{j=i}^{n}\frac{x^{j-i}}{(j-i)!}h_j(t)\right| + \left|\int_0^x \frac{(x-y)^{n+1-i}}{(n+1-i)!}u(y,t)\mathrm{d}y\right| \\ &\leq e^x|h(t)| + \frac{x^{n+1-i}}{(n+1-i)!}\sqrt{x}\|u(t)\| \\ &\leq e^x|h(t)| + \max\left\{x, \frac{x^2}{2!}, \dots, \frac{x^n}{n!}\right\}\sqrt{x}\|u(t)\| \\ &\leq e^x|h(t)| + e^x\sqrt{x}\|u(t)\|,\end{aligned} \tag{131}$$

with $i = 1, 2, \dots, n$. By (100), it can be deduced that

$$
\begin{aligned}
\|w(t)\|^2 &= \int_0^D w^2(x,t)\mathrm{d}x \\
&\le 2\int_0^D u^2(x,t)\mathrm{d}x + 2\int_0^D \left(\sum_{i=1}^n \binom{n}{i-1}\eta_i(x,t)\right)^2 \mathrm{d}x \\
&\le 2\|u(t)\|^2 + 4(2^n-1)^2\int_0^D \left(e^{2x}|h(t)|^2 + e^{2x}x\|u(t)\|^2\right)\mathrm{d}x \\
&= \left(2 + 2(2^n-1)^2\left(De^{2D} - e^{2D} + 1\right)\right)\|u(t)\|^2 + 2(2^n-1)^2(e^{2D}-1)|h(t)|^2.
\end{aligned}
\tag{132}
$$

We deduce from (115), (119) that

$$
\begin{aligned}
|h(t)|^2 + \|u(t)\|^2 &\le \left(\frac{4^n-1}{3} + \frac{2n\left(e^{2|A|D}-1\right)}{|A|}\right)|\xi(t)|^2 \\
&\quad + \left(2 + \frac{2n|B|^2}{|A|}\left(\frac{e^{2|A|D}-1}{2|A|} - D\right)\right)\|w(t)\|^2 \\
&\le \Lambda_1\left(|\xi(t)|^2 + \|w(t)\|^2\right),
\end{aligned}
\tag{133}
$$

where

$$
\Lambda_1 = \max\left\{\frac{4^n-1}{3} + \frac{2n\left(e^{2|A|D}-1\right)}{|A|}, 2 + \frac{2n|B|^2}{|A|}\left(\frac{e^{2|A|D}-1}{2|A|} - D\right)\right\}. \tag{134}
$$

Using Lemma 5, there exist $R > 0$, $\overline{\lambda} > 0$, such that for all $c > c^*$,

$$
\begin{aligned}
|h(t)|^2 + \|u(t)\|^2 &\le \Lambda_1\left(|\xi(t)|^2 + \|w(t)\|^2\right) \\
&\le \Lambda_1 R e^{-\overline{\lambda}t}\left(|\xi(0)|^2 + \|w(0)\|^2\right)
\end{aligned}
\tag{135}
$$

for all $t \ge 0$. With the help of (115), (132), we get

$$
\begin{aligned}
|h(t)|^2 + \|u(t)\|^2 &\le \Lambda_1 R e^{-\overline{\lambda}t}\left(|\xi(0)|^2 + \|w(0)\||^2\right) \\
&\le \Lambda_1 R e^{-\overline{\lambda}t}\left(\frac{4^n-1}{3} + 2(2^n-1)^2(e^{2D}-1)\right)|h(0)|^2 \\
&\quad + \Lambda_1 R e^{-\overline{\lambda}t}\left(2 + 2(2^n-1)^2(De^{2D} - e^{2D} + 1)\right)\|u(0)\|^2 \\
&\le \Lambda_1\Lambda_2 R e^{-\overline{\lambda}t}\left(|h(0)|^2 + \|u(0)\|^2\right)
\end{aligned}
\tag{136}
$$

for all $t \ge 0$, with

$$
\Lambda_2 = \max\left\{\frac{4^n-1}{3} + 2(2^n-1)^2\left(e^{2D}-1\right), 2 + 2(2^n-1)^2\left(De^{2D} - e^{2D} + 1\right)\right\}. \tag{137}
$$

Denote

$$
\overline{R} = R\Lambda_1\Lambda_2, \tag{138}
$$

(129) is drawn. The proof is completed. □

Theorem 4. *Consider the closed-loop system consisting of (78)-(80) together with the control law (87). Under Assumption 1, there exist $c^* > 0$ and a $\mathcal{KL}$ function β_5, such that for all $c > c^*$,*

$$
|Z(t)|^2 + \int_{t-D}^t U^2(\sigma)\mathrm{d}\sigma \le \beta_5\left(|Z(0)|^2 + \int_{-D}^0 U^2(\sigma)\mathrm{d}\sigma, t\right) \tag{139}
$$

for all $t \ge 0$.

Proof. With the diffeomorphic transformation $h(t) = G(Z(t))$ defined by (81)-(82), there exist $\mathcal{K}$ functions γ_1, γ_2 such that

$$
|Z(t)|^2 = |G^{-1}(h(t))|^2 \le \gamma_1\left(|h(t)|^2\right), \tag{140}
$$

$$
|h(t)|^2 = |G(Z(t))|^2 \le \gamma_2\left(|Z(t)|^2\right). \tag{141}
$$

Using Lemma 6, there exists $c^* > 0$ such that the closed-loop system (78)-(80) together with the control law (87) holds that

$$\begin{aligned}|Z(t)|^2 + \int_{t-D}^{t} U^2(\sigma)\mathrm{d}\sigma &= |Z(t)|^2 + \|u(t)\|^2 \\ &\leq \gamma_1\left(|h(t)|^2\right) + \|u(t)\|^2 \\ &\leq \gamma_3\left(|h(t)|^2 + \|u(t)\|^2\right) \\ &\leq \gamma_3\left(\overline{R}e^{-\overline{\lambda}t}\left(|h(0)|^2 + \|u(0)\|^2\right)\right) \\ &\leq \gamma_3\left(\overline{R}e^{-\overline{\lambda}t}\left(\gamma_2(|Z(0)|^2) + \|u(0)\|^2\right)\right) \\ &\leq \gamma_3\left(\overline{R}e^{-\overline{\lambda}t}\left(\gamma_4\left(|Z(0)|^2 + \|u(0)\|^2\right)\right)\right) \\ &\leq \gamma_3\left(\overline{R}e^{-\overline{\lambda}t}\left(\gamma_4\left(|Z(0)|^2 + \int_{-D}^{0} U^2(\sigma)\mathrm{d}\sigma\right)\right)\right)\end{aligned} \tag{142}$$

for all $t \geq 0$, with $\gamma_3(s) = \gamma_1(s) + s, \gamma_4(s) = \gamma_2(s) + s$. Choosing $\beta_5(s,t) = \gamma_3(\overline{R}e^{-\overline{\lambda}t}(\gamma_4(s)))$, where $\overline{\lambda}$, $\overline{R}$ are given by (126), (138), respectively, (139) is obtained. The proof is completed. □

3.2 | Inverse optimal control for linearizable strict-feedforward systems

Theorem 5. *Consider the closed-loop system consisting of (78)-(80) together with the control law (87). Under Assumption 1, there exists a sufficiently large $c^{**} > c^* > 0$, for all $c > c^{**}$, the control law (87) minimizes the cost functional*

$$J = \lim_{t\to\infty}\left(\gamma V(t) + \int_0^t \left(L(\tau) + \frac{\gamma l e^D}{c}U^2(\tau)\right)\mathrm{d}\tau\right) \tag{143}$$

where $l > \frac{2\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q)}$, and L is a functional of $(Z(t), U(\theta))$, for all $t - D \leq \theta \leq t$, such that

$$L(t) \geq l\gamma\left(\frac{\lambda_{\min}(Q)}{4}|\xi(t)|^2 + \frac{1}{2}\|w(t)\|^2\right) \tag{144}$$

for an arbitrary $\gamma > 0$.

Proof. Let

$$L(t) = -\frac{\gamma l e^D}{c+1}\alpha_1^2\left(G^{-1}(\eta(D,t))\right) - \gamma\left(-\xi^T(t)Q\xi(t) + 2\xi^T(t)PBw(0,t) - lw^2(0,t) - l\int_0^D e^x w^2(x,t)\mathrm{d}x\right), \tag{145}$$

where $l > \frac{2\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q)}$. It can be deduced that

$$\begin{aligned}L(t) \geq & -\frac{\gamma l e^D}{c+1}\alpha_1^2\left(G^{-1}(\eta(D,t))\right) - \gamma\left(-\lambda_{\min}(Q)\xi^T(t)\xi(t) + \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(PBB^TP)}\xi^T(t)PBB^TP\xi(t)\right. \\ & \left. + \frac{2\lambda_{\max}\left(PBB^TP\right)}{\lambda_{\min}(Q)}w^2(0,t) - lw^2(0,t) - l\int_0^D e^x w^2(x,t)\mathrm{d}x\right) \\ \geq & -\frac{\gamma l e^D}{c+1}\alpha_1^2\left(G^{-1}(\eta(D,t))\right) + \frac{\gamma\lambda_{\min}(Q)}{2}|\xi(t)|^2 + l\gamma\|w(t)\|^2.\end{aligned} \tag{146}$$

From (87), (111), we know

$$\begin{aligned}\alpha_1^2\left(G^{-1}(\eta(D,t))\right) &= \left(-\sum_{i=1}^{n}\binom{n}{i-1}\eta_i(D,t)\right)^2 \\ &\leq (2^n-1)^2\left(e^D|h(t)| + \max\left\{D, \frac{D^2}{2!}, \dots, \frac{D^n}{n!}\right\}\sqrt{D}\|u(t)\|\right)^2 \\ &\leq (2^n-1)^2\left(2e^{2D}|h(t)|^2 + 2e^{2D}D\|u(t)\|^2\right).\end{aligned} \tag{147}$$

With the help of (115), (119), we arrive at

$$\begin{aligned}\alpha_1^2\left(G^{-1}(\eta(D,t))\right) &\leq 2e^{2D}(2^n-1)^2\left(|h(t)|^2+D\|u(t)\|^2\right)\\ &\leq 2e^{2D}(2^n-1)^2\left(\frac{4^n-1}{3}+\frac{2nD\left(e^{2|A|D}-1\right)}{|A|}\right)|\xi(t)|^2\\ &\quad+4De^{2D}(2^n-1)^2\left(1+\frac{n|B|^2}{|A|}\left(\frac{e^{2|A|D}-1}{2|A|}-D\right)\right)\|w(t)\|^2.\end{aligned} \tag{148}$$

Choosing

$$c^{**}=\max\left\{\frac{8le^{3D}(2^n-1)^2\left(\frac{4^n-1}{3}+\frac{2nD(e^{2|A|D}-1)}{|A|}\right)}{\lambda_{\min}Q},8De^{3D}(2^n-1)^2\left(1+\frac{n|B|^2}{|A|}\left(\frac{e^{2|A|D}-1}{2|A|}-D\right)\right),c^*\right\}, \tag{149}$$

where c^* is given by (122), for all $c>c^{**}$, by (146), (148), it holds

$$L(t)\geq\frac{l\gamma\lambda_{\min}(Q)}{4}|\xi(t)|^2+\frac{l\gamma}{2}\|w(t)\|^2. \tag{150}$$

Noting $U^*(t)=\frac{c}{c+1}\alpha_1(G^{-1}(\eta(D,t)))$, after some calculations, we have

$$\begin{aligned}L(t)&=-\frac{\gamma le^D}{c+1}\alpha_1^2\left(G^{-1}(\eta(D,t))\right)-\gamma\left(-\xi^T(t)Q\xi(t)+2\xi^T(t)PBw(0,t)-lw^2(0,t)-l\int_0^D e^xw^2(x,t)\mathrm{d}x\right)\\ &=-\frac{\gamma le^D}{c+1}\alpha_1^2\left(G^{-1}(\eta(D,t))\right)+\gamma le^Dw^2(D,t)-\gamma\dot V(t)\\ &=-\frac{\gamma le^D}{c+1}\alpha_1^2\left(G^{-1}(\eta(D,t))\right)+\gamma le^D\left(U(t)-\alpha_1\left(G^{-1}(\eta(D,t))\right)\right)^2-\gamma\dot V(t)\\ &=-\frac{\gamma le^D(c+1)}{c^2}(U^*(t))^2+\gamma le^D\left(U(t)-\frac{c+1}{c}U^*(t)\right)^2-\gamma\dot V(t)\\ &=\frac{\gamma le^D}{c}(U^*(t))^2+\gamma le^D\left((U(t)-U^*(t))^2-\frac{2}{c}U(t)U^*(t)\right)-\gamma\dot V(t),\end{aligned} \tag{151}$$

and hence, it can be deduced that

$$\gamma V(t)+\int_0^t\left(L(\tau)+\frac{\gamma le^D}{c}U^2(\tau)\right)\mathrm{d}\tau=\gamma V(0)+\gamma le^D\int_0^t\left(1+\frac{1}{c}\right)(U(t)-U^*(t))^2\mathrm{d}\tau \tag{152}$$

so the minimum of (143) is reached with

$$U(t)=U^*(t) \tag{153}$$

such that

$$J=\gamma V(0). \tag{154}$$

The proof is completed. □

4 | EXAMPLE

Example 1. Consider a strict-feedforward nonlinear system given by Krstic[6] as

$$\dot Z_1(t)=Z_2(t)+Z_3^2(t) \tag{155}$$

$$\dot Z_2(t)=Z_3(t)+Z_3(t)U(t-D) \tag{156}$$

$$\dot Z_3(t)=U(t-D), \tag{157}$$

where $Z_1,Z_2,Z_3\in R$ are the states, U is a scalar control input, and $D\in R^+$ is an actuator delay. It is illustrated in the aforementioned work[6] that the overall system (155)-(157) is not linearizable.

The nominal control design ($D = 0$) for system (155)-(157) is obtained by Krstic[6] as

$$U(t) = -Z_1(t) - 3Z_2(t) - 3Z_3(t) - \frac{3}{8}Z_2^2(t) + \frac{3}{4}Z_3(t)\left(-Z_1(t) - 2Z_2(t) + \frac{1}{2}Z_3(t) + \frac{1}{2}Z_2(t)Z_3(t) + \frac{5}{8}Z_3^2(t) - \frac{1}{8}Z_3^3(t) - \frac{3}{8}\left(Z_2 - \frac{Z_3^2}{2}\right)^2\right). \tag{158}$$

By Theorem 2, the predictor control for system (155)-(157) is designed as

$$U(t) = \frac{c}{c+1}U_1(t) = U^*(t), \tag{159}$$

where $c > 0$ is sufficiently large and

$$U_1(t) = -P_1(t) - 3P_2(t) - 3P_3(t) - \frac{3}{8}P_2^2(t) + \frac{3}{4}P_3(t)\left(-P_1(t) - 2P_2(t) + \frac{1}{2}P_3(t) + \frac{1}{2}P_2(t)P_3(t) + \frac{5}{8}P_3^2(t) - \frac{1}{8}P_3^3(t) - \frac{3}{8}\left(P_2 - \frac{P_3^2}{2}\right)^2\right), \tag{160}$$

and $P_1(t) = p_1(D,t), P_2(t) = p_2(D,t)$, and $P_3(t) = p_3(D,t)$ are provided for $x = D$ by

$$p_1(x,t) = Z_1(t) + \int_0^x \left(p_2(y,t) + p_3^2(y,t)\right) \mathrm{d}y \tag{161}$$

$$p_2(x,t) = Z_2(t) + \int_0^x \left(p_3(y,t) + p_3^2(y,t)u(y,t)\right) \mathrm{d}y \tag{162}$$

$$p_3(x,t) = Z_3(t) + \int_0^x u(y,t)\mathrm{d}y. \tag{163}$$

Responses of the states of system (155)-(157) under the control law (159) are shown for $c = 100$ in Figure 1. One can observe that the closed-loop system is asymptotically stable. By Theorem 3, the control law (159) is inverse optimal.

Example 2. Consider a cart with an inverted pendulum system given by Wei[21] as follows:

$$(m_1 + m_2)\ddot{q}_1 + m_2 l\cos(q_2)\ddot{q}_2 = m_2 l\sin(q_2)\dot{q}_2^2 + F \tag{164}$$

$$\cos(q_2)\ddot{q}_1 + l\ddot{q}_2 = g\sin(q_2), \tag{165}$$

where m_1 and q_1 are the mass and position of the cart; m_2, l, and $q_2 \in (-\pi/2, \pi/2)$ are the mass, length of the link,

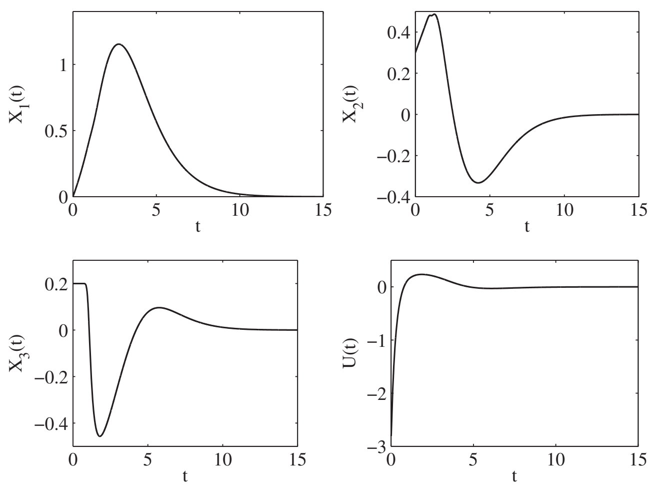

FIGURE 1 Responses of the states $X_1(t), X_2(t), X_3(t)$ of system (155)-(157) with the control law (159) for initial conditions as $X_1(0) = 0, X_2(0) = 0.3, X_3(0) = 0.2$ and $U(\theta) = 0$, for $\theta \in [0, 1]$

and angle of the pole, respectively, and $g = 9.8$ is the acceleration of gravity. Let $\dot{q}_2 = p_2$, $\dot{p}_2 = u$. Applying the feedback law (see the work of Wei[21])

$$F = -ul\left(m_1 + m_2 \sin^2(q_2)\right) / \cos(q_2) + (m_1 + m_2) g \tan(q_2) - m_2 l \sin(q_2)\dot{q}_2^2 \tag{166}$$

and with the following global change of coordinates

$$x_1 = \lambda \left(q_1 + l \ln \left(\frac{1 + \tan(q_2/2)}{1 - \tan(q_2/2)} \right) \right) \tag{167}$$

$$x_2 = \dot{q}_1 + (l / \cos(q_2))\, p_2, \tag{168}$$

we get

$$\dot{x}_1 = \lambda x_2 \tag{169}$$

$$\dot{x}_2 = \tan(q_2) \left(g + \frac{l}{\cos(q_2)} p_2^2 \right) \tag{170}$$

$$\dot{q}_2 = p_2 \tag{171}$$

$$\dot{p}_2 = u, \tag{172}$$

where $\lambda > 0$. To map the upper half-plane to R, we use another global change of coordinates and control as follows:

$$x_3 = \tan(q_2) \tag{173}$$

$$x_4 = \left(1 + \tan^2\left(q_2\right)\right) p_2 \tag{174}$$

$$v = \left(1 + x_3^2\right) u + \frac{2 x_3 x_4^2}{\left(1 + x_3^2\right)} + \left(g x_3 + \frac{g}{2} x_4 \right) \sqrt{1 + x_3^2}. \tag{175}$$

Finally, the dynamics of the cart-pole system is transformed into the following (assuming $l = 1$):

$$\dot{x}_1 = \lambda x_2 \tag{176}$$

$$\dot{x}_2 = x_3 \left(g + \frac{x_4^2}{\left(1 + x_3^2\right)^{3/2}} \right) \tag{177}$$

$$\dot{x}_3 = x_4 \tag{178}$$

$$\dot{x}_4 = -(g x_3 + (g/2) x_4) \sqrt{1 + x_3^2} + v. \tag{179}$$

From the aforementioned work,[21] the control law

$$v = v_1 + v_2 \tag{180}$$

$$v_1 = -2x_4 - x_3 - \left(1 / \sqrt{1 + x_3^2} \right) z_1 \tag{181}$$

$$z_1 = x_2 + \left(x_4 / \sqrt{1 + x_3^2} \right) + (g/2) x_3 \tag{182}$$

$$v_2 = \mu_2^{-1} \left(\frac{1}{2} x_3 \sqrt{1 + x_3^2} - x_4 \sqrt{1 + x_3^2} - \frac{1}{2} x_2 \right) - \mu_2 z_2 \tag{183}$$

$$z_2 = x_1 - N_2 \tag{184}$$

$$N_2 = -x_2 - \frac{g}{2} x_3 - \frac{1}{2g} x_4 - \frac{x_4}{\sqrt{1 + x_3^2}} - \frac{5}{4} \left(\frac{x_3 \sqrt{1 + x_3^2}}{2} + \frac{1}{2} \ln \left(x_3 + \sqrt{1 + x_3^2} \right) \right) \tag{185}$$

$$\mu_2 = \frac{1}{2g} + \frac{1}{\sqrt{1 + x_3^2}} \tag{186}$$

globally asymptotically stabilizes system (176)-(179).

We consider system (176)-(179) with input delay as follows:

$$\dot{x}_1 = \lambda x_2 \tag{187}$$

$$\dot{x}_2 = x_3\left(g + \frac{x_4^2}{\left(1+x_3^2\right)^{3/2}}\right) \tag{188}$$

$$\dot{x}_3 = x_4 \tag{189}$$

$$\dot{x}_4 = -\left(gx_3 + (g/2)x_4\right)\sqrt{1+x_3^2} + U(t-D), \tag{190}$$

where $D \in R^+$ is an actuator delay.

By Theorem 2, the control law for system (187)-(190) is given by

$$U(t) = \frac{c}{c+1}U_1(t) = U^*(t), \tag{191}$$

where $c > 0$ is sufficiently large, and $U_1(t) = v(t)$ is given as (180)-(186) by replacing $x_i(t), i = 1,2,3,4$, with $P_i(t), i = 1,2,3,4$, and $P_1(t) = p_1(D,t), P_2(t) = p_2(D,t), P_3(t) = p_3(D,t)$, and $P_4(t) = p_4(D,t)$ are provided for $x = D$ by

$$p_1(x,t) = x_1(t) + \int_0^x \lambda p_2(y,t)\mathrm{d}y \tag{192}$$

$$p_2(x,t) = x_2(t) + \int_0^x \left(p_3(y,t)(g + \frac{p_4^2(y,t)}{\left(1+p_3^2(y,t)\right)^{3/2}}\right)\mathrm{d}y \tag{193}$$

$$p_3(x,t) = x_3(t) + \int_0^x p_4(y,t)\mathrm{d}y \tag{194}$$

$$p_4(x,t) = x_4(t) + \int_0^x -\left(gp_3(y,t) + (g/2)p_4(y,t)\right)\sqrt{1+p_3^2(y,t)} + u(y,t)\mathrm{d}y. \tag{195}$$

Figures 2 and 3 show the simulation results for the cart-pole system with the initial state $(q_1, p_1, q_2, p_2) = (5, 0, \pi/3, 0)$ (ie, $(x_1, x_2, x_3, x_4) = (2.5 + 0.5\ln(\frac{\sqrt{3}+1}{\sqrt{3}-1}), 0, \sqrt{3}, 0)$), and $c = 100$. In Figure 4, clearly, the control law (191) stabilizes the inverted pendulum in its upright position after a rather short time. The parameters are chosen as $m_1 = m_2 = l = 1$.

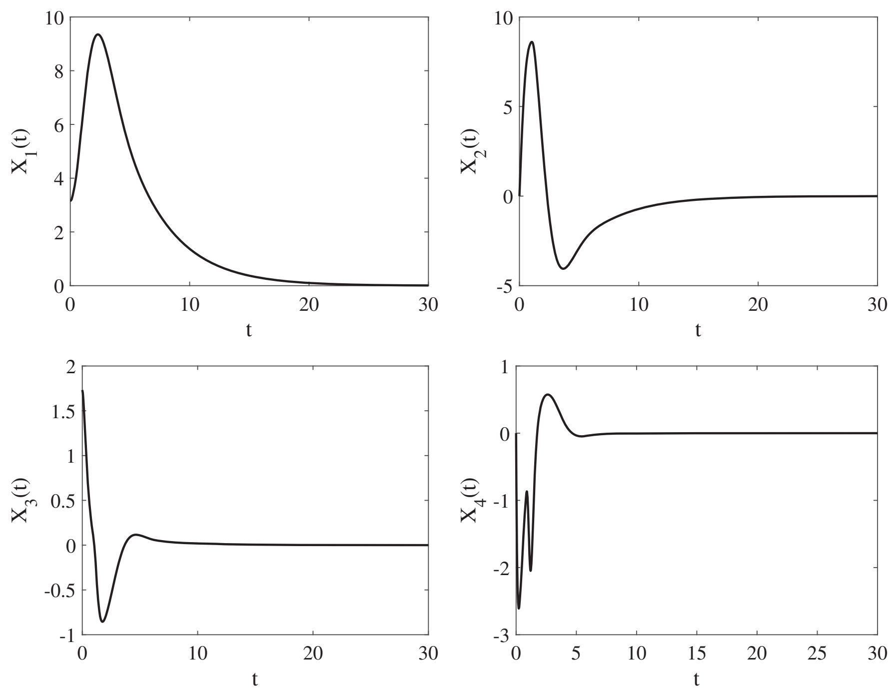

FIGURE 2 State trajectory of system (187)-(190)

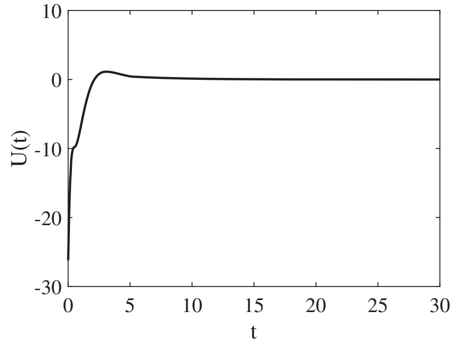

FIGURE 3 Control law (191)

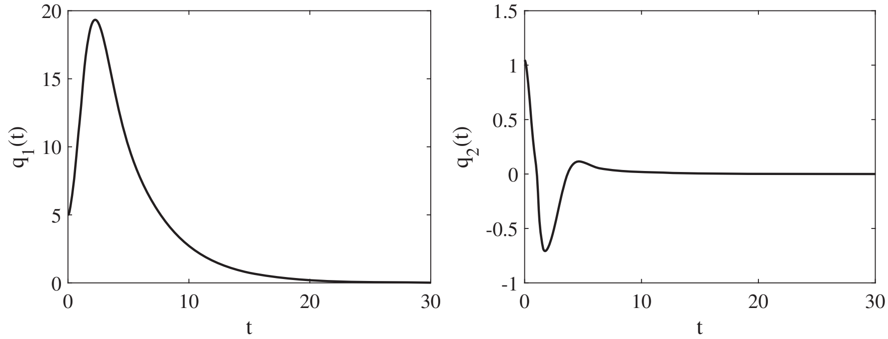

FIGURE 4 Position of the cart-pole system (164)-(165)

5 | CONCLUSIONS

Inverse optimal control for strict-feedforward systems with input delays is studied in this paper. A basic predictor control is designed for compensation for this class of nonlinear systems. Furthermore, it is shown that it is inverse optimal with respect to a meaningful differential game problem. For a class of linearizable strict-feedforward system, an explicit formula for compensation for input delay, which is also inverse optimal with respect to a meaningful differential game problem, is also obtained. A cart with an inverted pendulum system is given to illustrate the validity of the proposed method.

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REFERENCES

1. Mazenc F, Praly L. Adding integrations, saturated controls, and stabilization of feedforward systems. *IEEE Trans Autom Control*. 1996;41(11):1559-1578.
2. Sepulchre R, Jankovic M, Kokotovic PV. Integrator forwarding: a new recursive nonlinear robust design. *Automatica*. 1997;33(5):979-984.
3. Arcak M, Teel A, Kokotovic PV. Robust nonlinear control of feedforward systems with unmodeled dynamics. *Automatica*. 2001;37(2):265-272.
4. Xudong Y. Universal stabilization of feedforward nonlinear systems. *Automatica*. 2003;39(10):141-147.
5. Krstic M. Feedback linearizability and explicit integrator forwarding controllers for classes of feedforward systems. *IEEE Trans Autom Control*. 2004;49(10):1668-1682.
6. Krstic M. Input delay compensation for forward complete and strict-feedforward nonlinear systems. *IEEE Trans Autom Control*. 2010;55(2):287-303.
7. Krstic M. Lyapunov tools for predictor feedbacks for delay systems: inverse optimality and robustness to delay mismatch. *Automatica*. 2008;44(11):2930-2935.
8. Artstein Z. Linear systems with delayed controls: a reduction. *IEEE Trans Autom Control*. 1982;27(4):869-879.
9. Manitius AZ, Olbrot AW. Finite spectrum assignment problem for systems with delays. *IEEE Trans Autom Control*. 1979;24(4):541-553.

10. Mondie S, Michiels W. Finite spectrum assignment of unstable time-delay systems with a safe implementation. *IEEE Trans Autom Control*. 2003;48(12):2207-2212.
11. Bresch-Pietri D, Petit N. Robust compensation of a chattering time-varying input delay. Paper presented at: 53rd IEEE Conference on Decision and Control; 2014; Los Angeles, CA.
12. Mazenc F, Malisoff M, Niculescu SI. Stability and control design for time-varying systems with time-varying delays using a trajectory based approach. *SIAM J Control Optim*. 2017;55(1):533-556.
13. Bekiaris-Liberis N, Krstic M. Compensation of time-varying input and state delays for nonlinear systems. *J Dyn Syst Meas Control*. 2012;134(1):1-14.
14. Bekiaris-Liberis N, Krstic M. Robustness of nonlinear predictor feedback laws to time- and state-dependent delay perturbations. *Automatica*. 2013;49(6):1576-1590.
15. Cai X, Lin Y, Liu L. Universal stabilisation design for a class of non-linear systems with time-varying input delays. *IET Control Theory Appl*. 2015;9(10):1481-1490.
16. Bekiaris-Liberis N, Krstic M. Compensation of wave actuator dynamics for nonlinear systems. *IEEE Trans Autom Control*. 2014;59(6):1555-1570.
17. Cai X, Krstic M. Nonlinear control under wave actuator dynamics with time-and state-dependent moving boundary. *Int J Robust Nonlinear Control*. 2015;25(2):222-253.
18. Cai X, Krstic M. Nonlinear stabilization through wave PDE dynamics with a moving uncontrolled boundary. *Automatica*. 2016;68:27-38.
19. Karafyllis I, Krstic M. *Predictor Feedback for Delay Systems: Implementations and Approximations*. Cham, Switzerland: Springer; 2016.
20. Krstic M, Li Z. Inverse optimal design of input-to-state stabilizing nonlinear controllers. *IEEE Trans Autom Control*. 1998;43(3):336-350.
21. Wei X. The optimal stabilization of cart-pole system: A modified forwarding control method. Paper presented at: Proceedings of the IITA International Conference on Services Science, Management and Engineering; 2009; Zhangjiajie, China.

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