Competitive Influence Maximization Model with Monetary Incentive

Nadia Niknami and Jie Wu
Department of Computer and Information Sciences
Temple University, Philadelphia, PA, USA

***Abstract*—The spreading of information in social networks can be modelled as a process of diffusing information with a probability from its source to its neighbors. There is a challenge in the real world where competing companies implement their strategies in order to gain influence in the same social network at the same time. To effectively control the spreading of processes within the network, the effective use of limited resources is of prime importance. When budgets are fixed, competitors will search for a set of seed members to diffuse influence and maximize the number of members that are affected. Each competitor seeks to maximize its influence by investing in the most influential members in the given social network. In this paper, we utilize the Colonel Blotto game to help competitors figure out how many resources should be allocated to influential nodes to increase the influences on nodes. This is done while also taking into account that competing campaigns are trying to do the same thing. We propose a Max-Influence-Independent-Set (MIIST) algorithm to determine the most influential independent set and find the optimal investment to gain maximum influence in the given social network. The effectiveness of this approach is evaluated under different parameter values, namely probability distributions, topologies, and density.**

Index Terms*—*Blotto game, budget allocation, competitive influence maximization, social network analysis, viral marketing.

I. INTRODUCTION

Online Social Networks (OSNs) have increasingly become an effective means of communication, thanks to the fast-spreading of information from user to user on the networks. In this regard, finding a set of the most influential nodes to generate the largest influence spread [1]–[3] is an ongoing research problem. The objective is to identify such sets of nodes that are most influential. A major idea is that it is possible to trigger a large cascade of information dissemination in a social network by targeting just a few nodes. These nodes are known as seed nodes. In general, a node with a significant propagation capability is regarded as influential because it can spread influence over a large number of nodes. Ideally, seed nodes in the graph $\mathcal{G}$ should have the greatest influence on the other nodes. For this reason, determining the propagation capabilities of nodes as well as identifying influential nodes are essential to ensure a successful and rapid spread within social networks. In an OSN, represented by a graph $\mathcal{G}(V, E, W)$, each individual node is either *active* or *inactive*. A node that is inactive is one that is not influenced by others. A node that is active is one that is either a source of influence or has been influenced by other nodes that are active or influenced. A node whose status has been changed from inactive to active cannot be further modified. The tendency for each node to become active increases monotonically as more of its neighbors become active. *Independent cascade* (IC) [1] and *linear threshold* (LT) [4] are the two most popularly studied models in information diffusion. With both models, we can distinguish between active nodes, which spread information, and inactive nodes. Nodes in the active state can potentially affect neighbors. The influence of an active node is determined by its relationships with its neighbors. Following a number of propagations, a large number of nodes become active in the network. According to the LT model, a node becomes active if a large number ($>$ threshold) of its neighbors are also active. In the IC model, the success of the seed node u in influencing one of its inactive neighbors v is only a function of the edge propagation probability between node u and node v. Both models terminate after a certain number of propagations when no further nodes are activated. Both the propagation of influence as well as the impact of the initial targeting are linear. The influence maximization problem is NP-complete; however, an approximate solution exists.

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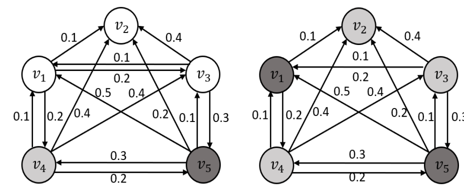


(a) Initial investment (b) Results of Influence

Fig. 1: Influence of seed member in a social network.

Fig. 1 illustrates a social network where two players (gray and dark gray) invested on some of the influential members. Part (a) shows the initial investment by these two players. Each one found two influential ones in his favor. Part (b) demonstrates the outcome of the influence propagation resulted from the gray player's seed nodes and the dark gray player's seed nodes. It is clear that the gray player player is the winner of this game, because he has more activated members in comparison to the dark gray player's activated persons. This result could be due to the influence probability on links and the importance of each seed node. In general, the result of the influence propagation depends on the importance of seed nodes in the case of influence, competitors' investment

TABLE I: Main Notations

Symbol	Meaning
$\mathcal{G}$	Social network
B_1/B_2	Total budget of player $1/2$
$B_1^{(v)}/B_2^{(v)}$	Allocated budget of player $1/2$ on node v
N_v	Neighbor set of node v
V	Set of nodes in the network
w_v	Weight of node v
w'_v	Effective weight of node v
p_{vu}	Influence probability of edge between v and u
$p_{vu}w_u$	Influence weight of node v on node u
ap_v	Activation probability node v
$\mathcal{S}$	Set of seed nodes in the network
$\mathcal{A}_1/\mathcal{A}_2$	Activated nodes by player $1/2$ in the network

strategies, the influence of nodes on each other, and the topology of the network.

Game theory provides a more realistic solution to the Influence Maximization problem in competitive networks competing sources of information. Such a scenario can be modeled as a multiple-person competition game where each of the players tries to make its influence spread maximally. An inactive node that receives influence from different parties at the same time will be activated by the one who sends the highest influence. An influenced node can increase its influence on its neighbors by allocating a large portion of the budget to those nodes. The Colonel Blotto Game (CBG) is one of the most widely adopted game-theoretic frameworks to model and analyze competitive resource allocation problems. Essentially, the CBG represents the competitive interactions between two players seeking to make investments across a network of nodes. The player who allocates the most resources to a given node wins it (influences it) and receives a corresponding valuation. In a CBG, the fundamental issue confronting players is how to allocate their resources in order to maximize the value of the nodes they win. The main difference between the competitive influence game and the influence game is the propagation that occurs in social networks. Nodes in CBG are independent, while influence maximization problems involve relationships between the nodes.

Active nodes can have inference on their inactive neighbors based on the influence probability. The higher the value of the weight, the higher the chance is to influence the given neighbor. Considering this fact, adding some extra investment on nodes just to increase the weight of influence would be helpful to raise the chance of influence propagation. The scenario described above can be modeled using a multi-person competitive one-shot game where both players try to maximize the number of influenced nodes by direct investment under a budget constraint. The budget allocation and influence propagation are done in one-shot, and the investment among nodes is based on the importance of the node in the case of influence propagation. In a one-shot game, players need to make a decision at beginning of the game without observing the result of the competition. The objective function of IM problem in the case of considering weights is submodular whenever the unweighted version is, so we can still use the greedy algorithm for obtaining a $(1-1/e-\epsilon)$-approximation. Contributions of this paper are summarized as follows:

- With two players competing simultaneously in a social network, we define a special one-shot game. Under budget constraints, players must select an optimal investment strategy to maximize their activation probabilities. This investment is done in one shot.
- We present a two-phase budget allocation strategy that integrates seed selection with budget allocation. By taking the effective weight of the nodes into account, this study improves the process of seed selection. Furthermore, it shifts the focus of the problem to convincing influential nodes to participate as seeds depending on the amount invested.
- We propose a competitive Max-Influence-Independent-Set (MIIS) in order to find the maximum influence independent sets in a social network. We introduce a new measurement called the effective weight, which is calculated for each independent maximum influence set. The node with the highest effective weight will be selected as the seed node.
- The model is validated by analyzing the effects of different parameters of influence distributions, densities, and network structure measurements.

This paper is organized as follows: Section II reviews the related works of competitive influence maximization and resource allocation approaches. The background and motivation are represented in the Section III. Section IV explains the MIIS approach based on the two different algorithms. The evaluation and performance of our scheme are discussed in Section V. We finally give a brief conclusion in Section VII.

II. Related Works

A. Competitive influence maximization problem

In a number of studies [5]–[8], maximization of competitive influence has been investigated in situations where multiple competing sources propagate simultaneously. Bharathi *et al.* [9] discussed the problem of competitive influence maximization for the first time in online social networks. They extended the single source IC model to the competitive setting and gave an approximation algorithm for computing the best response to an opponent's strategy. Competitive IM aims at finding strategies that maximize one's influence while minimizing his opponents' influence in a social network [10] [11]. In addition to the IC model and the LT model, there are other extensions that allow a variety of competing ideas to be spread in social networks [12] [9] rather than focusing on spreading a single idea. Li *et al.* [13] consider a model for competitive IM. According to a graph $\mathcal{G}$ and diffusion model, the strategy space comprises all IM algorithms that players can adopt. It is the aim of each player to find an optimal Nash equilibrium strategy which maximizes his influence over the game. Wu in [14] discussed the submodularity and approximation degree of the algorithm for competitive influence maximization based on the LT model with respect to viral marketing. Authors in [13], [15], [16] used the game-theoretic strategy to solve the competitive influence maximization problem. Authors in [17] modeled an attack-defender game where both parties want to maximize their territory. A two-party influence game is a special type of such a security game. Jafari and Navidi [18] introduced a game-theoretic approach for modeling competitive diffusion over social networks. They considered the topological structure of the graph, individual's initial tendency, and information content on the diffusion process. Authors in [19] proposed an optimization problem where there are multiple competitive and complementary products in the network. This approach is follower-based and aims to find the top-k influential nodes for the target product.

B. Resource allocation against opponents

Authors in [10] investigate competitive influence when players had to decide on resource allocation against their competitors. In such a game of Colonel Blotto, the price of competition appears to be unlimited. Authors in [20] consider the budget allocation scenario in terms of maximizing influence. When competitors allocate varying budgets to each node, nodes will favor the product of the company offering the highest value in the network. Competitors compete according to how much budget each of them allots to each node in the network. A Nash equilibrium based model is proposed by Masucci *et al.* [21] to compete for obtaining more customers in online social networks. In [22] the voter model is extended to include nodes with continuous states that reflect their opinion about the competitor. Each competitor has the objective of increasing their overall opinion within the network. Depending on the continuing action of the two competitors, the budget of the competitors may vary. According to their proposed model, there is a pure Nash equilibrium in the game. Additionally, they extended their analysis to repeated marketing campaigns. in [23].

III. PRELIMINARIES

A social network can be modeled with a weighted and directed graph $\mathcal{G}(V, E, P, W)$, where V and E are the set of nodes and edges, respectively. The network participants correspond to node set V and their relations are represented by arc set E. P represents the weight/influence probability of edges. Each edge $(u, v) \in E$ between node u and v has an influence probability $p_{uv} \in P$ which is in [0,1] and $\sum_{u \in N_v} p_{uv} \leq 1$. When $p_{uv} = 0$, it shows that edge $(u, v) \notin E$. In addition, we define W as a set of weights associated with each vertex in V. The importance of node v in the case of influence is shown by $w_v \in W$. Fig. 8 illustrates these parameters in a social network with five members. Assume there are players interested in promoting the ideas to the individuals in the social network. A member of the social network becomes active via one of the players if this member accepts the product or idea of the player. The term *active* indicates that a node v adopts a product or information and begins to exert influence by sharing that information with its neighbors u along edge $(u, v) \in E$

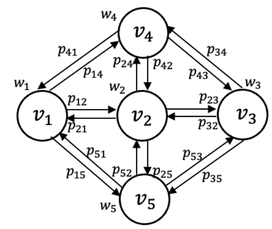


Fig. 2: Direct investment based on the weight of node.

with an influence probability of $p_{uv} \in P$. As a matter of fact, an attempt to trigger node v occurs with probability p_{uv}. It is important to note that activated nodes are permanently active and cannot be reactivated at a later date by another node. The propagation process comes to a halt if no node is activated within a certain time period.

A. Influence analysis

Social influence in a network occurs when an individual is influenced by their friends, without being forced to take those actions directly [24]. Activating a node u in $\mathcal{G}$ means accepting an idea from the player i. Once a node u accepts the idea of being occupied by a player i, it cannot change occupation to another party. If the given node does not accept any idea, it means that the state of the node u is inactive. Mathematically, we can define the influence diffusion as finding $\mathcal{S}_k^*$ such that $S_k^* = \text{argmax}_{\mathcal{S} \in V, |\mathcal{S}|=k} \, \sigma()$, where $\sigma(\mathcal{S})$ denotes an influence function that gives the expected number of influenced nodes resulted from the set of $\mathcal{S}$. An influence function $\sigma(.)$ is monotone iff $\sigma(\mathcal{S}) \leq \sigma(\mathcal{S}')$ for any $\mathcal{S} \subseteq \mathcal{S}' \subseteq V$. As well as, an influence function $\sigma(.)$ is submodular iff $\sigma(\mathcal{S} \cup \{v\}) - \sigma(\mathcal{S}) \geq \sigma(\mathcal{S}' \cup \{v\}) - \sigma(\mathcal{S}')$ for any $\mathcal{S} \subseteq \mathcal{S}' \subseteq V$.

One of the popular influence diffusion models in social networks is the Linear Threshold model [1] [4]. Linear Threshold model describes how influence is propagated throughout the network starting from the initial seed vertices. Each node v selects a threshold $\theta_v \in [0, 1]$ randomly. Thresholds intuitively represent the fraction of the neighbors of node v that must become active for node v to become active. Node v checks if the weighted sum of its active neighbors is greater than or equal to its threshold θ_v; if so, node v is activated. In a multi-round propagation, in step t, all nodes that were active in step $t-1$ remain active, and any node v for which the total weight of its active neighbors is at least θ_v ($\sum_{\text{u} \in \text{N(v)}} p_{uv} \geq \theta_v$) will be activated. The process runs until no more activation is possible.

The independent cascade is another model of influence propagation that uses a Greedy solution. With an initial set, the diffusion process of this model occurs in multiple steps. In the first step, only nodes in the initial set S are active, and all other nodes are inactive. The inactive nodes will be activated successively by each of the active nodes. During step $t + 1$, every newly activated node u in S_t is attempting to influence its inactivated neighbors v that do not belong to S_t with an independent probability p_{vu}. The process is repeated until an equilibrium state is reached and no further

Algorithm 1 Strict MIIS

Initialize $\mathcal{S}$ to an empty set
for each node $v \in V$ **do**
  Compute effective weight $w'_v = w_v + \sum_{u \in N_v} p_{vu} w_u$
while there is not eligible seed node in V **do**
  Choose $v \in v$ with the highest effective weight
  Add node v to the set $\mathcal{S}$
  Remove node v and all v's neighbours from V
  for each node $u \in V - \mathcal{S}$ **do**
    if $u \in N_v$ **then**
      Update $w_u = w_u(1 - p_{vu})$
return $\mathcal{S}$

Algorithm 2 Weak MIIS

Initialize $\mathcal{S}$ to an empty set
for each $v \in V$ **do**
  Compute effective weight $w'_v = w_v + \sum_{u \in N_v} p_{vu} w_u$
while V is not empty **do**
  Choose $v \in V$ with the highest effective weight
  Add node v to the set $\mathcal{S}$
  for each node $u \in N_v$ **do**
    Update $w_u = w_u(1 - p_{vu})$
  Add node v to $\mathcal{S}$
  Remove node v
  for each $u \in V - \mathcal{S}$ **do**
    if $u \in N_v$ **then**
      Update $w_u = w_u(1 - p_{vu})$
return $\mathcal{S}$

activation is possible [1]. The important point to note is that in an independent cascade model, each node will only be able to influence its neighbors once after it has been activated. The active nodes will remain active permanently. The number of activated nodes in the final step is the spread of influence of the seed set S, denoted by $\sigma()$.

B. Competitive influence maximization

When there are multiple competitors that attempt to make nodes active in their favor, we need to consider and compare different total weights of edges. Suppose that there is a CIM game with two players and n nodes in a social network $\mathcal{G}$. Player 1 has a budget of size B_1, and player 2 has a budget size of B_2. Each node u has a value, $w_u > 0$, which can be regarded as the reward of taking this node for players. The total value of n nodes in this social network is $W = \sum_{u \in V} w_u$. The winner of this game would be the player who can obtain the most reward by influencing the more important nodes. Players engage in a competition based on the amount of the budget they allocate toward seed nodes (the most influential nodes).

Competitive linear threshold model: The inactive node v would be activated by player 1 if for all of the neighbors of node v that are activated by player 1, there is $\sum_{u \in N_v^1} p_{uv} \geq \theta_v$. For all of the neighbors of u that are activated by player 2 there is $\sum_{u \in N_v^2} p_{uv} < \theta_v$. Similarly, the inactive node v would be activated by player 2 if $\sum_{u \in N_v^2} p_{uv} \geq \theta_v$ and $\sum_{u \in N_v^1} p_{uv} < \theta_v$. If both of players 1 and 2 have higher aggregated weights of edges than θ_v, then node v will be activated by 1 if:

$$\sum\nolimits_{u \in N_v^1} p_{uv} \geq \theta_v, \tag{1a}$$

$$\sum\nolimits_{u \in N_v^2} p_{uv} \geq \theta_v, \tag{1b}$$

$$\sum\nolimits_{u \in N_v^1} p_{uv} > \sum\nolimits_{u \in N_v^2} p_{uv}. \tag{1c}$$

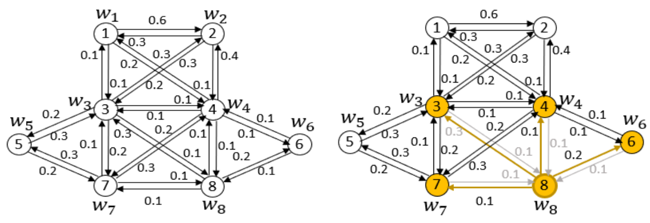
Competitive independent cascade model: If a node v is activated at a certain time step t, it immediately tries to activate all its neighbors u along edge $(v, u) \in E$. In the event that a node v activation attempt is successful (which occurs with probability p_{vu}), node v becomes active at time $t + 1$ and subsequently tries to influence its neighbors. The nodes can be activated either by an influence cascade initiated by player 1 or by an influence cascade initiated by player 2. In the event of a tie, each player has a 50% chance. The active node is permitted to attempt to activate each neighbor only once during propagation, and each activation attempt is presumed to occur independently. The propagation process stops if no nodes are activated within a certain period of time.

Algorithm 3 Monetary-Incentive-CIM

Require: $\mathcal{G}(V, E, W, P)$, total budget B_1 and B_2
Step 1 (Finding Seed Sets):
  $\mathcal{S} \leftarrow$ Call $MIIS(\mathcal{G})$
Step 2 (Investments):
  for each set $s \in S$ **do**
    Investment on s based on w'_s
Step 3 (Propagation):
  for each node $u \in V$ **do**
    if node u is invested by both players directly **then**
      Calculate $ap_u = B_1^u / (B_1^u + B_2^u)$
    else if node u is invested by player 1 directly **then**
      $\mathcal{A}_1 = \mathcal{A}_1 \cup u$
    else if node u is invested by player 2 directly **then**
      $\mathcal{A}_2 = \mathcal{A}_2 \cup u$
    else if node u is under different influence v and w **then**
      $ap_u = (p_{vu})(1 - p_{vu} p_{wu}) / (p_{vu} + p_{wu})$
return list of activated nodes $\mathcal{A}_1$ and $\mathcal{A}_2$

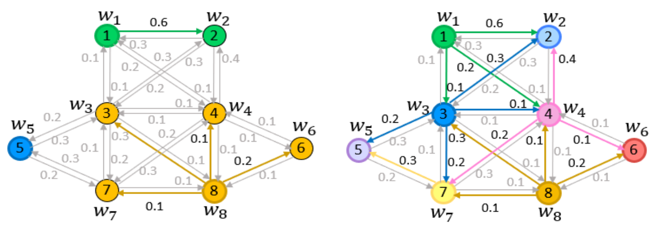
C. Blotto game

The Colonel Blotto problem is a zero-sum game about how to best position resources. In the Colonel Blotto game, two players concurrently allocate resources across n battlefields. The player with the greatest resources in each battlefield wins that battle and the player with the most overall wins is the victor. The Colonel Blotto game involves two colonels simultaneously distributing their troops across different battlefields. Colonels receive their ultimate reward based on the number of battles they win. Colonel Blotto is a game which is commonly used for the analysis of a wide variety of applications. The search for the optimal strategy for the Colonel Blotto game



(a) Original Network (b) First Set

Fig. 3: Influence Independent Sets



(a) Sets with Strict MIIS (b) Sets with Weak MIIS

Fig. 4: Strict and Weak Influence Independent Sets

has been ongoing for some time. Colonel Blotto is a zero-sum game, but since the number of pure strategies of the agents is exponential in terms of the number of troops and the number of battlefields, it is quite difficult to determine which strategy is best. Several attempts have been made to solve variants of this problem. After almost a century Ahmadinejad *et al.* in [25] provided an algorithm for finding the optimal strategies in polynomial time.

IV. Maximum Influence Independent Set (MIIS)

The information diffusion game models the diffusion process of information in the social network $\mathcal{G}$ for the competitive players where players want to spread their information as much as possible. The scenario we are interested in is one in which players identify the most influential nodes in the network, and then they compete over them by directly investing in the nodes in order to activate the node according to their individual preference. This investment is personalized. The weight of edges between node v and u in the $\mathcal{G}$ shows the amount of influence that node v gives node u. The competitors in a CIM problem select the seed nodes and allocate portions of the budget depending on the importance of the selected node. A ranked list of nodes based on the importance of nodes in the given social network is advantageous for the players to have an optimal investment. Suppose that there is a CIM game with two players, 1 and 2, and n nodes in a social network $\mathcal{G}$. Player 1 has a budget of size B_1, and player 2 has a budget size of B_2.

Each node v has a value, $w_v > 0$, which can be regarded as the reward that a given player would achieve for taking this node. The total value of n nodes in this social network is $W = \sum_{v \in V} w_v$. The winner of this game would be the player who can obtain the most reward by influencing the more important nodes. Players engage in the competition based on the amount of the budget they allocate toward seed nodes (the most influential nodes). In this game, three types of competition can occur. The first type of competition is players' competition on seed nodes by the amount of allocated budget, which can be called Node-Node competition. The second type is Link-Link, which is the competition of influence when two different links with different influences try to activate the given node in their favor. The last one is Node-Link. This will happen when one of the competitors allocates some portion of the budget to the given node, and the influence of another competitor reaches this node by the influence of the link.

TABLE II: Weight and Effective Weight

Node (v)	**Weight** (w_v)	**Effective Weight** $(w_v + \sum_{u \in N_v} p_{vu} w_u)$
1	6	6+(2)(0.6)+(4)(0.1)+(3)(0.2)=8.2
2	2	2+(6)(0.3)+(3)(0.1)+(4)(0.3)=5.3
3	4	4+(6)(0.1)+(3)(0.1)+(2)(0.3)+(2)(0.2)+(4)(0.1)+(8)(0.1)=7.1
4	3	3+(6)(0.2)+(2)(0.4)+(4)(0.1)+(1)(0.1)+(4)(0.3)+(8)(0.1)=7.5
5	2	2+(4)(0.3)+(4)(0.3)=4.4
6	1	1+(3)(0.1)+(8)(0.1)=2.1
7	4	4+(4)(0.2)+(3)(0.3)+(2)(0.2)+(8)(0.1)=6.9
8	8	8+(4)(0.3)+(3)(0.1)+(1)(0.2)+(4)(0.1)=10.1

1) Node-Node influence competition: A Node-Node competition on the node u is the competition of two competitors with the amount of allocated budget. Suppose that $B_1^{(u)}$ and $B_2^{(u)}$ are the amount of the budget that players 1 and 2 have allocated to node u. The winning probability for player 1 for this competition is as follows:

$$ap_u = B_1^{(u)} / (B_1^{(u)} + B_2^{(u)}) \tag{2}$$

2) Link-Link influence competition: Each active node propagates its influence on its neighbors following the budget allocation process. Consider the case where there is a node u with two active neighbors with different influences. This node receives the influence from player 1 via node v with influence probability p_{vu}. In addition, it receives the influence from player 2 via another neighbor, node w, with p_{wu}. Here, link-link influence competition will happen. The probability that node u would be activated by player 1 is as follows:

$$ap_u = \frac{p_{vu}}{(p_{vu} + p_{wu})} \times (1 - p_{vu} p_{wu}), \tag{3}$$

where $(1 - p_{vu} p_{wu})$ considers the probability of activation of node u by at least one of the players. The probability that node u would be activated by player 2 is as follows:

$$ap_u = \frac{p_{wu}}{(p_{vu} + p_{wu})} \times (1 - p_{vu} p_{wu}) \tag{4}$$

3) Node-Link influence competition: Based on our assumptions, we assumed that direct allocation has a higher priority in comparison with the influence of links. Consider the scenario in which player 1 has a direct investment in node v. If player 2 is influencing node v via its neighbors, node v will be activated by player 1.

A. Activation probability and utility function

Definition 1. (Influence Weight) When a node v is selected as a seed node, each $u \in N_v$ will have the adjusted weight

TABLE III: Updated Weights with Strict MIIS (second step)

Node (v)	**Weight** (w_v)	**Effective Weight** ($w_v + \sum_{u \in N_v} p_{vu} w_u$)
1	6(0.9)(0.8)=4.32	4.32+(0.84)(0.6)=4.82
2	2(0.6)(0.7)=0.84	0.84+(4.32)(0.3)=2.32
3	4	Ineligible
4	3	Ineligible
5	2(0.7)(0.7)=0.98	0.98+(0)+(0)=0.98
6	1	Ineligible
7	4	Ineligible
8	8	Selected

TABLE IV: Updated Weights with Weak MIIS

Node (v)	**Weight** (w_v)	**Effective Weight** ($w_v + \sum_{u \in N_v} p_{vu} w_u$)
1	6	6+(2)(0.6)+(2.8)(0.1)+(2.1)(0.2)=7.9
2	2	2+(6+0.3)(2.8*0.3)+(2.1)(0.1)=4.85
3	4(1-0.3)=2.8	2.8+(6)(0.3)+(2)(0.3)+(2.1)(0.1)+(2)(0.2)+(2.4)(0.1)=6.05
4	3(1-0.1)=2.1	2.1+(6)(0.2)+(2)(0.4)+(2.8)(0.1)+(0.8)(0.1)+(2.4)(0.3)=5.18
5	2	2+(2.8)(0.3)+(2.4)(0.3)=3.56
6	1(1-0.2)=0.8	0.8+(2.1)(0.1)=1.01
7	4(1-0.1)=2.4	2.4+(2.8)(0.2)+(2.1)(0.3)+(2)(0.2)=14.7
8	8	Selected

$(1 - p_{vu})w_u$, where $p_{vu}w_u$ is called the influence weight of node v on node u.

Definition 2. (Effective Weight) The effective weight of i is the summation of its own weight plus influence weights of its neighbors. It can be calculated by

$$w'_v = w_v + \sum_{u \in N_v} p_{vu} w_u. \quad (5)$$

Theorem 1. In $\mathcal{G}(V, E, P, W)$, for each node v and its neighbors $N_v = \{u_1, u_2, ..., u_k\}$, which are not in seed set $\mathcal{S}$, the probability that node v being activated is in proportion to this relationship:

$$\begin{aligned} ap_v = &\sum_{u_i \in N_v} ap_{u_i} \cdot p_{u_i v} \\ &- \sum_{u_i, u_j \in N_v, i<j} (ap_{u_i} \cdot p_{u_i v})(ap_{u_j} \cdot p_{u_j v}) + ... \\ &+ (-1)^k (ap_{u_1} \cdot p_{u_1 v})...(ap_{u_k} \cdot p_{u_k v}). \end{aligned}$$

Proof: The probability that node v is activated by the influence of its neighbor u_1 is $ap_v = ap_{u_1} \cdot p_{u_1 v}$. The failure probability for u_1 to activate node v in step 1 is $1 - ap_v$. There are same attempts from all the neighbors to influence node v; the probability that u_2 succeeded is $ap_{u_2} \cdot p_{u_2 v}$; To simplify the notations, we define $\sigma_{u_i} = ap_{u_i} \cdot p_{u_i v}$. Node v cannot be activated by both of these neighbors. The conditional probability that u_1 failed but u_2 succeeded in activating node v is $(1 - ap_v)\sigma_{u_2}$. Total probability that node v becomes activated includes activation by u_1 or by u_2. Thus:

$$ap_v = ap_v + (1 - ap_v)\sigma_{u_2} = \sigma_{u_1} + \sigma_{u_2} - \sigma_{u_1}\sigma_{u_2}.$$

If neither u_1 nor u_2 activated node v, u_3 attempts to activate node v. The probability that u_3 can activate v is $ap_{u_3} \cdot p_{u_3 v} = \sigma_{u_3}$. Therefore, the conditional probability that u_1, u_2 failed but u_3 succeeded in activating node v is $(1 - ap_v)\sigma_{u_3}$. Thus, the total probability that node v becomes activated.

$$\begin{aligned} ap_v =& ap_v + (1 - ap_v)\sigma_{u_3} \\ =& \sigma_{u_1} + \sigma_{u_2} + \sigma_{u_3} - \sigma_{u_1}\sigma_{u_2} - \sigma_{u_1}\sigma_{u_3} - \sigma_{u_2}\sigma_{u_3} \\ &+ \sigma_{u_1}\sigma_{u_2}\sigma_{u_3}. \end{aligned}$$

Under the condition that none of node v's previous neighbors activated, u_k attempts to activate node v. The total probability that node v becomes activated is the following:

$$\begin{aligned} ap_v =& ap_v + (1 - ap_v)\sigma_{u_k} \\ =& \sum_{u_i \in N_v} \sigma_{u_i} + \sum_{u_i, u_j \in N_v, i<j} \sigma_{u_i}\sigma_{u_j} \\ &+ ... + (-1)^k + \sigma_{u_1}\sigma_{u_2}...\sigma_{u_k}. \end{aligned}$$

This completes the proof. ■

In the proposed game, the utility for the players is a function of the total number of activated nodes at the end of the propagation process. In addition, it is a function of the weight of activated nodes. In this influence maximization game, the player with the highest number of weighted activated nodes will be deemed the winner. Several parameters affect the gain of players, including the total budget of players, seed selection strategy, and the influence distribution between nodes.

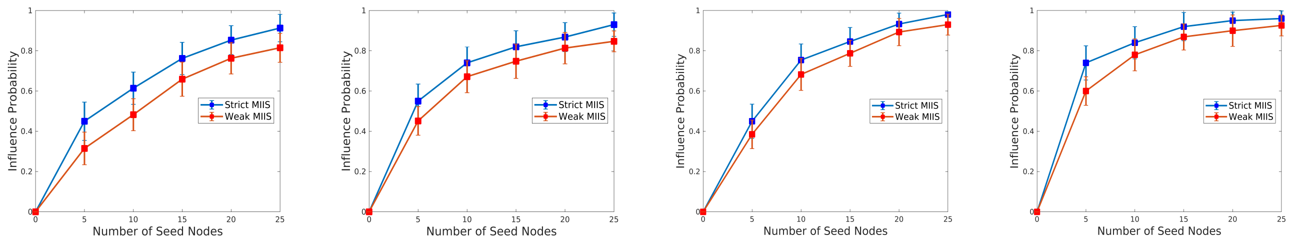
B. *Selecting seed nodes and propagation model*

Consider a static social network $\mathcal{G}$ and B_1 and B_2 for players as their fixed budget. Each player attempts to reach the maximum gain by activating as many nodes as possible within a given budget. Based on the effective weights of each node, players select seed nodes. Investing in nodes with higher effective weights will have a greater likelihood of propagating the influence to more nodes across the social network. The influence propagation between nodes is based on the influence probability of the links associated with the relationship. When two players select the same seed node for investment, the winner will be determined with the help of a budget proportion.

C. *Algorithms description*

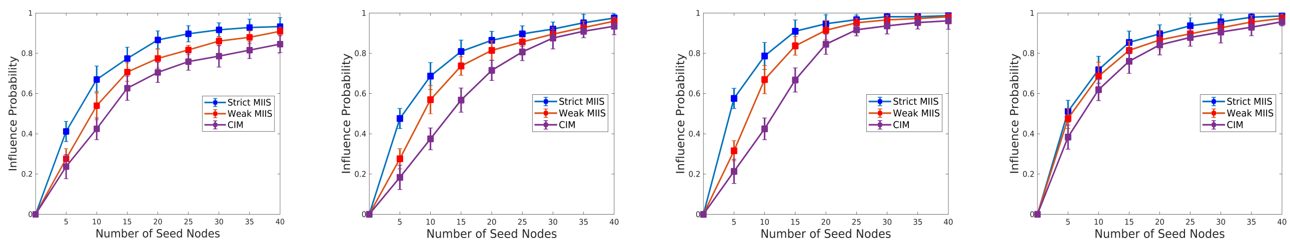
As mentioned before, each node v has a weight w_v, and its influence to its neighbor u is based on probability p_{vu}. When a node v is selected, its neighbor's weight (u's weight) should be adjusted based on $(1 - p_{vu})w_u$, where $p_{uv}w_u$ is called the influence weight of node v on u. The effective weight of node v, w'_v, is the summation of its own weight plus influence weights of its neighbors. Algorithm 1 shows the Strict MIIS algorithm. An eligible node v in $\mathcal{G}$ will be selected. It is a node with the maximum effective weight w'. Then, it is required to adjust weights of node v's neighbors and label them ineligible. Node v should be removed from network $\mathcal{G}$. This process continues until there are no eligible nodes in $\mathcal{G}$. Algorithm 2 presents Weak MIIS. The steps are similar to Algorithm 1. We select an eligible node v in $\mathcal{G}$ with the maximum effective weight w'. The only difference between Strict MIIS and Weak MIIS is that in the Weak MIIS after adjusting the weights of node v's neighbors, v's neighbors still will be eligible. We remove node v from $\mathcal{G}$ but keep its neighbors in the social network $\mathcal{G}$.

This process continues until there is not any node to select and network $\mathcal{G}$ is empty. Algorithm 3 represents the steps of the proposed algorithm in detail. This algorithm includes three



(a) $\sigma = 0.25$ (b) $\sigma = 0.5$ (c) $\sigma = 0.75$ (d) $\sigma = 1$

Fig. 5: Different influence distributions.



(a) Density=50% (b) Density=65% (c) Density=75% (d) Density=85%

Fig. 6: Effect of network sparsity

steps: 1) Finding the seed set, 2) Investment, 3) Propagation. Step 1 ends when k nodes which are the nodes with high effective weight w' are selected. These seed nodes are selected through the MIIS in Algorithm 1 or Algorithm 2. In step 2, players invest on the seed node in $\mathcal{S}$ based on the total budget and weight of each seed node. Step 3 shows the results of the competition for seed nodes as well as the results of diffusion process for each of the active nodes. As a result, each node v has a probability of activation, and this probability is determined by the ratio of the sum of the budgets invested by the competitors or by a probability ratio based on the influence of the neighbors. A number of the nodes will be activated by direct investments. A number of the nodes will be activated by the influence of their neighbors. Basically, what this algorithm determines is the list of nodes that each competitor has activated so far. Figs 3 and 4 show the difference between Strict and Weak MIIS for a toy example. We tried to display the steps with the help of different colors. Tables III and IV present the details of calculating weights and effective weights of nodes. Ineligible nodes are the neighbor of selected nodes in the Strict MIIS algorithm.

V. EVALUATION OF THE ONE-SHOT DIFFUSION GAME

In this section, we conduct several experiments to show the efficacy of our proposed framework. We used the Python library to represent the the nodes and their relations in the graphs. We used IC as the diffusion model. In order to check the impact of influence propagation, we consider normal distribution, with the same μ and different σ^2. In the case of budget, we mainly focus on the symmetric case where $B_1 = B_2 = B$. The steps in this game are as follows: First, the players have to determine the independent sets from the initial graph. After selecting seed nodes from the obtained maximum set, players begin direct investments in seed nodes based on the importance of nodes (effective weight w'). In the end, we use a spread model to measure the number of nodes that were influenced by the two players in a one-shot model.

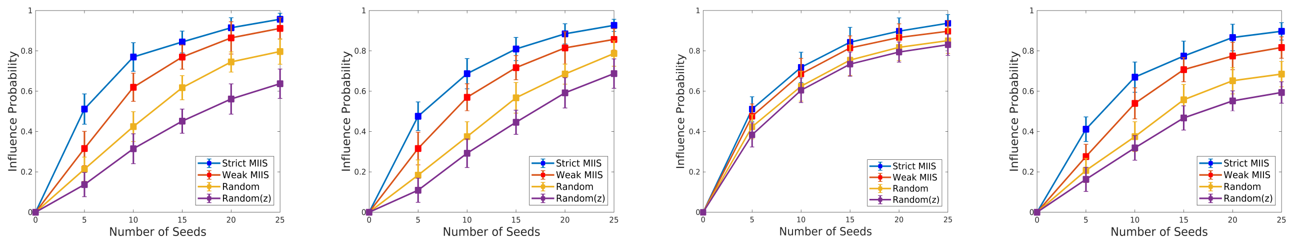
A. Comparison methods

In order to evaluate the performance of proposed method, we have used the following strategies in our experiments:

- Strict MIIS: After selecting a node with the maximum effective weight w', weights of the given node's neighbors will be adjusted and neighbors will be labeled ineligible.
- Weak MIIS: After selecting a node with the maximum effective weight w', weights of the given node's neighbors will be adjusted and neighbors will be labeled eligible.
- CIM: The traditional version of the competitive influence maximization method [9].
- Random: With this strategy, we start with the first seed node and allocate a random value (greater than zero) to each node while considering $\sum_v B^{(v)} = B$.
- Random(z): z is the maximum amount of the budget allocated to a node. It starts with the first seed node and allocate a random value (greater than zero and less than z) to each node while considering budget constraint.

B. Effect of influence distribution

To evaluate final activation probability in the multi-person competition game we need to consider the effect of influence distribution. This is because influence probability has a significant effect on the influence propagation and the total number of active nodes among the neighbors of seed nodes. Influence distribution determines the probability of inference of activated nodes on their neighbors. We analyze the effect



(a) Mesh network (b) Tree network (c) Star network (d) Real social network

Fig. 7: Influence probability under different methods for different topologies.

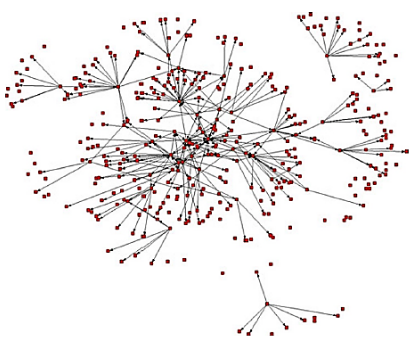


Fig. 8: Shape of the real social network in Gelsenkirchen.

of different influence distribution settings on the proposed models. We consider the weight of the edges in the range $\sigma = \{0.25, 0.5, 0.75, 1\}$. Fig. 5 depicts the impact of different influence distributions on influence probability for Strict MIIS and Weak MIIS. The result in this figure presents that Strict MIIS has a higher influence probability in comparison with Weak MIIS. In the different parts of Fig. 5, we considered a normal distribution with different variances. With smaller sigmas, there are larger numbers of edges with high influence probability. As a result, the influence propagation would be high in comparison to a network where there is a large sigma in the influence distribution formula. This shows that when there are more edges with a higher influence probability, there would be a higher activation probability, even with a small number of seed nodes. In addition, it shows that investment in more seed nodes leads to a higher activation probability.

C. *Effect of sparsity*

The sparsity of the network inhibits a significant extent of spreads for spreading models like the independent cascade even at high probabilities of influence. Fig. 6 shows the impact of different densities for varying numbers of seed nodes. In a social network with 85% density, there is higher influence propagation than a social network with 50% density. The higher the density, the higher the chance for the player to find an active node in his favor. In addition, there are more relations between nodes in a network with higher density. As a result, there will be a greater likelihood for an activated node to propagate its influence throughout the network. There will therefore be a greater probability of influence propagation. The reason for this can be attributed to the fact that edges help players to propagate their influence without the need to

TABLE V: Proposed Methods vs. Optimal One

Budget	**Influence Maximization Methods**		
	Weak MIIS	**Strict MIIS**	**Optimal**
10$	20%	23.2%	25%
20$	25%	27.3%	31.5%
30$	34%	37.5%	41%
40$	39.3%	42%	47.5%
50$	45.5%	49%	52%

consider additional seed nodes. The result in Fig. 6 presents that Strict MIIS, under different density, has a higher influence probability in comparison with Weak MIIS.

D. *Comparison with baseline approaches*

The topology of a social network is a significant parameter in analysing the influence maximization problem. That is because the number of links and the structure of relations between nodes depend on topology of network.In this section, we compare Strict MIIS and Weak MIIS with Random and Random(z) methods in the case of the different topologies and different number of seed nodes. The Random method considers the k nodes with high weight w and starts from the first node and allocates a random value greater than zero to each node with the considering budget constraint. The Random(z) method allocates budget with a limitation on maximum value. The maximum amount of the allocation to a node should be z. We compare the proposed method on social networks with mesh, tree, and star topology. In addition, we consider a real social network in which the relations between the members are based on the cooperation in the development of new ideas related to education that works very well in Gelsenkirchen [26]. Fig. 7 illustrates the influence probability of these methods taking into account different topologies of social networks. It is clear that Strict MIIS improves the result of the competitive influence maximization problem for all of the different topologies. In the case of the star network, the results of different methods seem to be close to each other.

E. *Comparison of proposed methods and the optimal one*

Table V shows the results of the competitive influence maximization problem for a simple network of 50 nodes under a variety of budget allocations. We observe that the results of Strict MIIS are similar to the optimal values. Nevertheless, the Weak MIIS method is insufficient to maximize the likelihood of influence for the case of maximum influence probability.

VI. Discussion

In summary, the experiments indicate that Strict MIIS has a higher influence probability than Weak MIIS. Moreover, it is clear that Strict MIIS increases the result of the competitive influence maximization problem in all topologies. In the case of the star network, the results of the different approaches are close to each other. In the case of considering networks with different densities, the result shows that Strict MIIS has a higher influence probability compared to Weak MIIS. In addition, the higher the density, the more often the player will find an active node in his favor. In a network with a higher density, there are more connections between nodes. Therefore, an activated node has a greater likelihood of propagating its influence across the network. Therefore, there will be a greater level of influence propagation. In comparison to a network where there is a large sigma in the influence distribution formula, the propagation of influence would be greater. The results of scenarios with different influence probability distributions show that when there are more edges with a higher influence probability, the activation probability would be higher. This is even with a small number of seed nodes. Furthermore, it shows that investing in more seed nodes results in a higher probability of activation. When there are enough seed nodes, a different influence distribution does not substantially affect the outcome of the competition.

VII. Conclusion

In this paper, we investigated the strategic investment of players in one-shot competitive influence game where two competing players need to simultaneously decide what proportion of budget should be invested directly on nodes in order to have the maximum inference on neighbors. We utilized the Colonel Blotto game to characterize the optimal strategic allocation of resources for the competitive maximization influence game. We proposed a novel approach based on the independent set for selecting seed nodes. Strict MIIS improves the influence probability under scenarios with different influence probability distributions, different network topologies, and different density. Even with a small number of seed nodes, the experiment shows a higher activation probability when there are more edges with a higher influence probability.

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Nadia Niknami received her B.S. degree in Computer Science from University of Isfahan, Iran, in 2011, and MSc degrees From Tarbiat Modares University, Tehran, Iran in 2015. She is currently pursuing the Ph.D. degree in the Department of Computer and Information Sciences, Temple University, Philadelphia. Her current research focuses on security, privacy-preserving, attack-defense scenarios, Markov chain, social network, game theory, Software Defined Network.

Jie Wu is the Director of the Center for Networked Computing and Laura H. Carnell professor at Temple University. He also serves as the Director of International Affairs at College of Science and Technology. He served as Chair of Department of Computer and Information Sciences from the summer of 2009 to the summer of 2016 and Associate Vice Provost for International Affairs from the fall of 2015 to the summer of 2017. Prior to joining Temple University, he was a program director at the National Science Foundation and was a distinguished professor at Florida Atlantic University. His current research interests include mobile computing and wireless networks, routing protocols, cloud and green computing, network trust and security, and social network applications. Dr. Wu regularly publishes in scholarly journals, conference proceedings, and books. He serves on several editorial boards, including IEEE Transactions on Mobile Computing, IEEE Transactions on Service Computing, Journal of Parallel and Distributed Computing, and Journal of Computer Science and Technology. Dr. Wu was general cochair for IEEE MASS 2006, IEEE IPDPS 2008, IEEE ICDCS 2013, ACM MobiHoc 2014, ICPP 2016, and IEEE CNS 2016, as well as program chair/cochair for IEEE INFOCOM 2011, CCF CNCC 2013, and ICCCN 2021. He was an IEEE Computer Society Distinguished Visitor, ACM Distinguished Speaker, and chair for the IEEE Technical Committee on Distributed Processing (TCDP). Dr. Wu is a Fellow of the AAAS and a Fellow of the IEEE. He is the recipient of the 2011 China Computer Federation (CCF) Overseas Outstanding Achievement Award.