A Note on a Combinatorics Partition Problem

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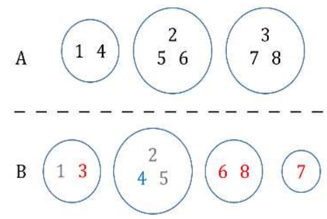


In recent years, I used the following partition problem to test analytical and logical skills of PhD candidates: A professor assigned his eight students to three groups in 2021 (partition A) and then to four groups in 2022 (partition B). Show that at least two students exist who are assigned to smaller groups in 2022 compared to the ones in 2021. Suppose the three groups in 2021 are {2, 5, 6}, {1, 4}, and {3, 7, 8} while the four groups in 2022 are {1, 3}, (2, 4, 5), {6, 8}, and {7}. Students 3, 6, 7, and 8 are in smaller groups, while student 4 is in a larger group in 2022.

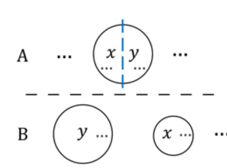
The above result applies to any number of students, if the difference between the number of groups of A and B remains 1. The solution to the partition problem is not straightforward. Most students used the pigeonhole principle directly or checking group sizes of each student under two partitions without success. Last week, one of my PhD students told me that a similar problem has been presented in a classic combinatorics textbook together with an elegant solution. I am disappointed as I was always under the impression that the problem is original. At the same time, I am excited because my non-conventional solution (Solution 4) has some merit. Furthermore, I came up two new solutions (Solutions 1 and 2).

The first two solutions rely on two intermediate steps, called "facts", to derive the conclusion.

Fact 1: When students are assigned to partition A and then to partition B with more groups, at least one student exists whose group size in B is smaller than the one in A.

Fact 2: When students are assigned to partition A and then to partition B with the same number of groups, if one student has a smaller group in B (compared to the group he was assigned to in A), there exists another student with a larger group in B.

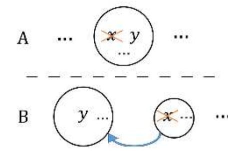
Both Solutions 1 and 2 use Fact 1 to find the first student x and then make group numbers of A and B the same to apply Fact 2 to find the second student. In Solution 1, x's group within A is split into two. In Solution 2, x's group within B is removed by deleting x and moving other elements to another group in B.



Solution 1: Let x_A and x_B denote the groups to which x is assigned in A and B, respectively, as well as their sizes. Based on Fact 1, there is a student x (the first student) such that $x_B < x_A$. Select another student y in x_A, if $y_B < y_A$, y is the second student. Therefore, we just check for $y_B \geq y_A$. Split group y_A into two groups, one which includes x and the other which includes y, to form a new partition A'. Both A' and B now have the same number of groups and $y_B > y_{A'}$. Based on Fact 2, there exists a student z in a smaller group within B for the partition pair (A', B). As $z_A \geq z_{A'}$, z is the second student for pair (A, B).

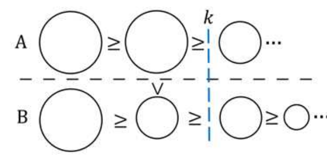
Notes on Fact 1 and Fact 2: k students assigned to a group contribute one group of size k. When all students maintain the same or larger group sizes in B than in A, B's average group size is at least A's average. We can show Fact 1 by contradiction, where B's average group size (with a higher number of groups) is less than A's average. When one student has a larger group size and all others have the same or larger groups sizes in B compared to in A, B's average group size is larger than A's average. We can show Fact 2 by contradiction as both A and B have the same average group size.

Solution 2: We find students x and y similarly to how we find them in Solution 1. Then we just need to check for $y_B \geq y_A$. Delete group x_B by first remove student x from x_B and x_A and then move all others students from x_B to y_B. The above process generates a new partition pair (A', B') with the same number of groups. Based on Fact 2, since $y_{B'} > y_{A'}$, there must exist a student z such that $z_{B'} < z_{A'}$ in pair (A', B'). Because $z_A \geq z_{A'}$ and $z_{B'} \geq z_B$, $z_B < z_A$. Therefore, z is the second student for pair (A, B).



Next, we present a direct solution listed in the textbook: *A Walk Through Combinatorics*. First, sort all groups within A (and B) in a non-increasing order of group size for both A and B. Compare A's prefix series with B's prefix series by gradually increasing prefix length until A's prefix is "larger" than B's prefix in terms of prefix sum. Then show that two students in the selected A's prefix must be placed groups in B with smaller group sizes.

Solution 3: Let a_i and b_i represent a group and it's size in partitions A and B, respectively. Sort all groups in A (and B) in a non-increasing order by the group size: a_1, a_2, … (and b_1, b_2, …). Find the smallest index k such that $a_1 + a_2 + \ldots + a_k > b_1 + b_2 + \ldots + b_k$. This k always exists as B has one more group than A for the same number of students. Clearly, $a_k > b_k$. One can then derive $(a_1 + a_2 + \ldots a_k) - (b_1 + b_2 + \ldots + b_{k-1}) \geq 2$. In other words, at least 2 students in $a_1, a_2, \ldots, a_k$ cannot all fit into $b_1, b_2, \ldots, b_{k-1}$. These students will be assigned to b_k, b_{k+1}, …. with smaller groups.



Our final solution uses a special auxiliary variable: \$1, assigns \$1 to each group of a partition, and draws a contradiction to using two "accounting" methods of all funds assigned to two partitions.

Solution 4: We assign \$1 to each group of A and B, which is equally distributed among group members. As B has one more group than A, the investment difference between B and A is \$1. On the other hand, each student's payment in a group is >\$0 but ≤\$1. In other word, the payment difference for each student in B and A is <\$1. As the investment difference between B and A equals the summation of payment differences of all students, fewer than two students is insufficient to add up to \$1.

The partition problem can be extended when the number of groups in A and B differ by k, then at least k+1 students are now in smaller groups in B than in A. Both Solutions 3 and 4 can be applied, while Solutions 1 and 2 need some work to be sufficient to solve the generalized problem, perhaps with revised facts and/or double induction. The merit of Solutions 1 and 2 is the use of an auxiliary partition pair to study partition pair (A, B).

The essence of this story is that it is important for researchers to conduct independent work to nurture their creativity, even if this work may produce suboptimal or unnatural solutions. Nonconventional solutions sometimes bring unexpected, pleasant surprises. More insights will be gained after comparing ones' solutions with others' work. It is an art to judiciously follow an iterative process of conducting independent research and following others' research. Our solutions echo G. H. Hardy's description of proof by contradiction as "one of a mathematician's finest weapons".