Introduction to Probability, Statistics and Random Processes

Chapter 2: Counting

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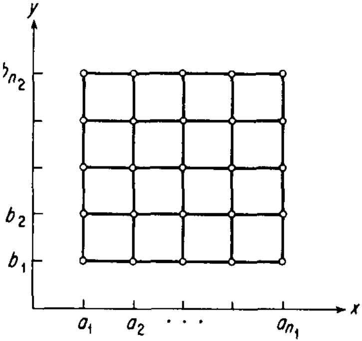
Finding Probabilities Using Counting Methods

- Counting methods that can be used for discrete sample spaces with equally likely outcomes.
- For such a finite sample space S, the probability of an event A is
$$P(A) = \frac{|A|}{|S|}$$
- Thus finding the probability in this case reduces to a counting problem.

Rudiments of Combinatorial Analysis

Theorem

Given n_1 elements $a_1, a_2, \cdots, a_{n_1}$ and n_2 elements $b_1, b_2, \cdots, b_{n_2}$, there are precisely $n_1 \times n_2$ distinct ordered pairs (a_i, b_j) containing one element of each kind.



Rudiments of Combinatorial Analysis

Theorem

Given n_1 *elements* $a_1, a_2, \cdots, a_{n_1}$ *and* n_2 *elements* $b_1, b_2, \cdots, b_{n_2}$*, etc., up to* n_r *elements* $x_1, x_2, \cdots, x_{n_r}$*, there are precisely* $n_1 \times n_2 \times \cdots \times n_r$ *distinct ordered pairs* $(a_{i_1}, b_{i_2}, \cdots, x_{i_r})$ *containing one element of each kind.*

Multiplication Principle

- Everything we need to know about counting comes from the **multiplication principle**.
- The formal statement of the principle is as follows-

> Suppose that we perform r experiments such that the kth experiment has n_k possible outcomes, for $k = 1, 2, \cdots, r$. Then there are a total of $n_1 \times n_2 \times n_3 \times \cdots \times n_r$ possible outcomes for the sequence of r experiments.

Example

- What is the probability of getting three sixes in a throw of three dice?
- Hint: $N = 6^3 = 216$ equiprobable outcomes of throwing three dice.

Example

- Suppose that I want to purchase a tablet computer. I can choose either a large or a small screen; a 64GB, 128GB, or 256GB storage capacity; and a black or white cover. How many different options do I have?
- **Answer:** There are 12 possible options. The multiplication principle states that we can simply multiply the number of options in each category (screen size, memory, color) to get the total number of possibilities, i.e., the answer is $2 \times 3 \times 2 = 12$.

Some Terminology

- **Sampling** — Choosing an element from a set. We **draw** a sample at random from a given set in which each element of set has equal chance of being chosen.
- **With replacement**- While drawing multiple samples from a set, if we put each element back after each draw, we call it sampling with replacement. It also means *repetition allowed*.
- **Without replacement** — While drawing multiple samples, we do not put each element back after every draw, i.e *repetitions not allowed*.
- **Ordered** — Sampling in which ordering matters.
- **Unordered** — Sampling in which ordering does not matter.

Ordered Sampling with Replacement

- We need to make r draws from a set of n-elements in which ordering matters and there is repetition.
  - We choose r objects in succession from a population (set) of n distinct objects $\{a_1, a_1, \cdots, a_n\}$, in such a way that after choosing each object and recording the choice, we return the object to the set before making the next choice. This gives an "ordered sample" of the form $(a_{i_1}, a_{i_2}, \cdots, a_{i_r})$.
- Setting $n_1 = n_2 = \cdots = n_r = n$ in theorem 2, we have $n \times n \times \ldots \times n = n^r$.

Example Problem

- For example if $A = \{1, 2, 3\}$ and $k = 2$, how many different possibilities are there for ordered sampling with replacement?
- If $A = \{1, 3, 5, 7\}$ and $k = 3$, how many different possibilities are there for ordered sampling with replacement?

Ordered Sampling without Replacement: Permutations

- When ordering matters and repetitions are not allowed:
  - Choose r objects in succession from a population of n distinct objects $\{a_1, a_1, \cdots, a_n\}$, in such a way that an object once chosen is removed from the population Then we again get an ordered sample, but now there are n - 1 objects left after the first choice, n - 2 objects left after the second choice, and so on.

$$n \times (n-1) \times ... \times (n-k+1).$$

- This is referred to as *k-permutations of an n-element set*:

$$P_k^n = n \times (n-1) \times ... \times (n-k+1).$$

- Note that for $k > n$, we have $P_k^n = 0$ and if $k = n$,

$$P_n^n = n \times (n-1) \times ... \times 1.$$

- P_n^n is equal to $n!$ and is pronounced as *n factorial*.

Example

- Suppose we place r distinguishable objects into n different "cells" (r $\leq$ n), with no cell allowed to contain more than one object. Find the total number of distinct arrangements of the objects in the cells?

Example

- A subway train made up of n cars is boarded by r passengers (r $\leq$ n), each entering a car completely at random. What is the probability of the passengers all ending up in different cars?
- **Answer:** there are $N = n^r$ distinct equiprobable arrangements of passengers in the cars. Let A be the event that "no more than one passenger enters any car.", $N(A) = n \times (n-1) \times \cdots \times (n-r+1)$. Thus $P(A) = \frac{n \times (n-1) \cdots (n-r+1)}{n^r}$.

Unordered Sampling without Replacement: Combinations

- We want to make k draws from a set of n-elements in which ordering does not matter and repetition is not allowed.
- This means that we have to chose a k-element subset of A, and is also called k-**combination** of the set A.
- The number of k-element subsets of A is given by $\binom{n}{k}$ and is read as *n choose k*.

Unordered Sampling without Replacement: Combinations

- The difference between $\binom{n}{k}$ and P_k^n is in the ordering.
- For any k-element subset of A, we can order the elements in $k!$ ways. Thus

$$P_k^n = \binom{n}{k} \times k!$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- If $k > n$, then $\binom{n}{k} = 0$.

Binomial combination

Theorem

A population of n elements has precisely

$$C_k^n = \frac{n!}{r!(n-k)!}$$

subpopulations of size $r \leq n$.

- $\binom{n}{k}$ is also called the **binomial coefficient**; as the coefficients in the binomial theorem are given by $\binom{n}{k}$.
- n choose k is also denoted as $C_{n,k}$, $C(n,k)$, nCk, etc.

Example

- What is the probability that two playing cards picked at random from a full deck are both aces?
- **Answer:** A full deck consists of 52 cards, of which 4 are aces. There are $C_2^{52} = 1326$ ways of selecting a pair of cards from the deck. Of these 1326 pairs, there are $C_2^4 = 6$ consisting of two aces. Thus ...

Example

- A batch of 100 manufactured items is checked by an inspector, who examines 10 items selected at random. If none of the 10 items is defective, he accepts the whole batch. Otherwise, the batch is subjected to further inspection. What is the probability that a batch containing 10 defective items will be accepted?
- **Solution:** The number of ways of selecting 10 items out of a batch of 100 items equals $N = C_{10}^{100}$. Let A be the event that "the batch of items is accepted by the inspector." $N(A) = C_{10}^{90}$. Thus ...

Interpretation of $\binom{n}{k}$ and Some Useful Identities

- We can interpret $\binom{n}{k}$ as

  The total number of ways to divide n distinct objects into two groups A and B such that group A consists of k objects and group B consists of $n-k$ objects is $\binom{n}{k}$.

- For non-negative integers k, m and n, we have
  1. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
  2. For $0 \leq k < n$, $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$.
  3. Vandermonde's identity- $\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$

Extending Binomial Coefficients

Theorem

Given a population of n elements,let $n_1, n_2, \cdots, n_k$ be positive integers such that $n_1 + n_2 + \cdots n_k = n$. Then there are precisely $N = \frac{n!}{n_1! \times n_2! \times \cdots \times n_k!}$ ways of partitioning the population into k subpopulations, of sizes $n_2, n_2, \cdots, n_k$, respectively.

Bernoulli Trials and Binomial Distributions

- A **Bernoulli Trial** is a random experiment that has 2 possible outcomes usually labeled as *success* and *failure*.
- Example: A coin toss with possible outcomes *heads* and *tails*.
- We denote the probability of success as p and failure as $q = 1 - p$.
- A **Binomial experiment** is an experiment in which we perform n independent Bernoulli trials and count the total number of successes.

Bernoulli Trials and Binomial Distributions

- In general we have

Binomial Formula:

For n independent Bernoulli trials where each trial has success probability p, the probability of k successes is given by

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Example Problem

- I have a coin for which $P(H) = p$ and $P(T) = 1 - p$. I toss the coin 5 times.
  a. What is the probability of outcome $THHHH$?
  b. What is the probability of outcome $HTHHH$?
  c. What is the probability of outcome $HHTHH$?
  d. What is the probability that I will observe exactly 4 heads and 1 tail?
  e. What is the probability that I will observe exactly 3 heads and 2 tail?
  f. If I toss the coin n times, what is the probability that I observe exactly k heads and $n - k$ tails?

Unordered Sampling with Replacement

- This is the most challenging type of sampling.
- **Lemma:** The total number of distinct k samples from an n-element set such that repetition is allowed and ordering does not matter is the same as the number of distinct solutions to the equation

$$x_1 + x_2 + ... + x_n = k, \text{ where } x_i \in \{0, 1, 2, 3, ...\}.$$

- The number of distinct solutions to this equation is given by the following theorem.

Unordered Sampling with Replacement: Intuition

- **Lemma:** The total number of distinct k samples from an n-element set such that repetition is allowed and ordering does not matter is the same as the number of distinct solutions to the equation

$$x_1 + x_2 + \dots + x_n = k, \text{ where } x_i \in \{0, 1, 2, 3, \dots\}.$$

- Intuition: Consider a set $A = \{a_1, a_2, a_3\} = \{1, 2, 3\}$ and $k = 2$. There are 6 ways: $\{1,1\}, \{1,2\}, \{1,3\}, \{2,2\}, \{2,3\}, \{3,3\}$. Encode each by a vector $x_1 + x_2 + x_3 = 2$, where x_i denotes the number of occurrences of a_i.

Unordered Sampling with Replacement

Theorem: The number of distinct solutions to the equation

$$x_1 + x_2 + \dots + x_n = k, \text{ where } x_i \in \{0, 1, 2, 3, \dots\} \tag{1}$$

is equal to

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

- Intuition: out of $n + k - 1$ positions (k '1' and $n - 1$ '+'), pick k position (for '1') ...

Example

- Ten passengers get on an airport shuttle at the airport. The shuttle has a route that include 5 hotels, and each passenger gets off the shuttle at his/her hotel. The driver records how many passengers leave the shuttle at each hotel.
- How many different possibilities exist?
- **Answer:** Let x_i be the number of passengers that get off the shuttle at Hotel i. $x_1 + \cdots + x_5 = 10$. Thus C_{10}^{5+10-1}...