Introduction to Probability, Statistics and Random Processes

Chapter 2: Counting

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Recap: The Basic Rules of Probability Theory

Probability summation rule

- Probability summation rule: The probability that one of two mutually exclusive events (it does not matter which of them) occurs is equal to the sum o f the probabilities of these events.

$$P(A\ or\ B) = P(A \cup B) = P(A) + P(B) = \frac{n(A) + n(B)}{n}$$

- generalized for any number of events

$$P(A_1\ or A_2\ or\ \cdots or\ A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

Recap: The Basic Rules of Probability Theory

Probability multiplication rule

- The probability of the combination of two events (that is, of the simultaneous occurrence o f both o f them) is equal to the probability of one of them multiplied by the probability of the other provided that the first event has occurred.

$$P(A \text{ and } B) = P(A) \times P(B|A) = P(B) \times P(A|B)$$

- When A and B are independent

$$P(A \text{ and } B) = P(A) \times P(B)$$

- generalized for any number of events $\cdots$

The urn problem (1)

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- Solution: $P(A \text{ and } B) = \frac{3}{7} \times \frac{1}{3} = \frac{1}{7}$

The urn problem (2)

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- Solution: According to the multiplication rule, $P(1,2,3,4,5) = (1/5) \times (1/4) \times (1/3) \times (1/2) \times 1 = 1/120.$

The urn problem (3)

An urn contains 3 white balls and 4 black balls. One ball is drawn from the urn. Find the probability that the ball is white (event A). Now consider a new experiment with the same initial conditions: we draw one ball from the urn and place it in a drawer without looking at it. Then we draw a second ball from the urn. Does the probability that this second ball is white (event B) change?

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- Hint: in the second experiment, the fact that we have taken the ball of unknown colour out of the urn in no way affects the probability of the appearance of a white ball in the second draw.
- $P(A) = P(B) = 3/7 \cdots$

The cut-up alphabet problem

In a boy's schoolbag let there be 8 cut-up alphabet cards with the letters: two a's, three c's and three t's. We draw out three cards one after the other and put them on the table in the order that they appeared. Find the probability that the word "cat" will appear.

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- Solution: According to the multiplication rule, $P(\text{"}cat\text{"}) = (3/8) \times (2/7) \times (3/6) = 3/56$.

The cut-up alphabet problem — modified

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- $P(A) = (3/56) \times 6 = 9/28$

Three riflemen problem

Three riflemen take one shot each at the same target. The probability of the first rifleman hitting the target is $P_1 = 0.4$, of the second $P_2 = 0.5$, and of the third $P_3 = 0.7$. Find the probability that two riflemen hit the target.

- Event A (two hits at the target) breaks down into $C(3, 2) = C(3, 1) = 3$ variants:

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$$
\begin{aligned}
A &= (++- \text{ or } +-+ \text{ or } -++) \\
P(+\ +\ -) &= 0.4 \times 0.5 \times 0.7 = 0.140 \\
P(-\ +\ +) &= 0.6 \times 0.5 \times 0.7 = 0.210 \\
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&\ldots \\
P(A) &= 0.410
\end{aligned}
$$

The two hunters problem

Two hunters — Sam and George — went hunting, saw a bear and simultaneously shot at it. The bear was killed. There was only one hole in its hide. Which of the hunters was responsible was unknown. Though it is more likely that it was Sam- he was older and more experienced, and it was with the probability of 0.8 that he hit the target of such a size and from that distance. George, the younger and less experienced hunter, hit the same target with the probability of 0.4. The hide was sold for \$100. How to divide fairly this sum of money between Sam and George?

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- Solution (Hint): A_1 — Sam hit and George missed, A_2 — George hit and Sam missed.
  - $P(A_1) = 0.8 \times 0.6 \cdots$