Introduction to Probability, Statistics and Random Processes

Chapter 4: Continuous and Mixed Random Variables

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https://cis-linux1.temple.edu/∼tug29203/25fall-2033/index.html

The normal distribution

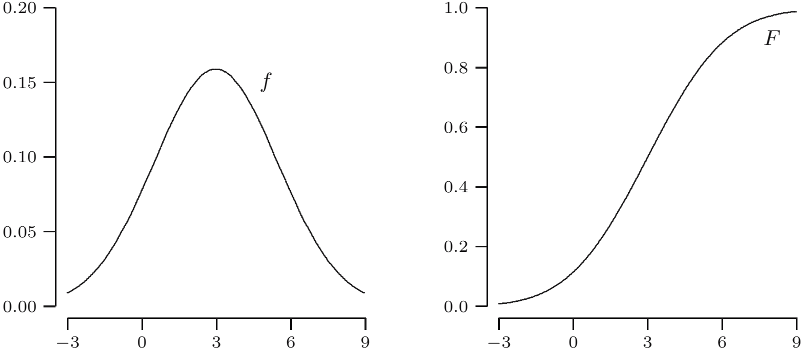

Figure: The probability density and the distribution function of the N(3,6.25) distribution.

The normal distribution

Definition

If X has an $N(\mu, \sigma^2)$ distribution, then its distribution function is given by

$$F(a) = \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \text{ for } -\infty < a < \infty$$

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- Unfortunately there is no explicit expression for F; f has no antiderivative
- Any $N(\mu, \sigma^2)$ distributed random variable can be turned into an $N(0, 1)$ distributed random variable by a simple transformation
- A table of the $N(0, 1)$ distribution suffices

Right tail probabilities table

- Right tail probabilities $1 - F(a) = P(Z \geq a)$ for $Z \sim N(0, 1)$.

a	0	1	2	3	4	5	6	7	8	9
0.0	5000	4960	4920	4880	4840	4801	4761	4721	4681	4641
0.1	4602	4562	4522	4483	4443	4404	4364	4325	4286	4247
0.2	4207	4168	4129	4090	4052	4013	3974	3936	3897	3859
0.3	3821	3783	3745	3707	3669	3632	3594	3557	3520	3483
0.4	3446	3409	3372	3336	3300	3264	3228	3192	3156	3121
0.5	3085	3050	3015	2981	2946	2912	2877	2843	2810	2776
0.6	2743	2709	2676	2643	2611	2578	2546	2514	2483	2451
0.7	2420	2389	2358	2327	2296	2266	2236	2206	2177	2148
0.8	2119	2090	2061	2033	2005	1977	1949	1922	1894	1867
0.9	1841	1814	1788	1762	1736	1711	1685	1660	1635	1611
1.0	1587	1562	1539	1515	1492	1469	1446	1423	1401	1379
1.1	1357	1335	1314	1292	1271	1251	1230	1210	1190	1170
1.2	1151	1131	1112	1093	1075	1056	1038	1020	1003	0985
1.3	0968	0951	0934	0918	0901	0885	0869	0853	0838	0823
1.4	0808	0793	0778	0764	0749	0735	0721	0708	0694	0681
1.5	0668	0655	0643	0630	0618	0606	0594	0582	0571	0559
1.6	0548	0537	0526	0516	0505	0495	0485	0475	0465	0455
1.7	0446	0436	0427	0418	0409	0401	0392	0384	0375	0367
1.8	0359	0351	0344	0336	0329	0322	0314	0307	0301	0294
1.9	0287	0281	0274	0268	0262	0256	0250	0244	0239	0233
2.0	0228	0222	0217	0212	0207	0202	0197	0192	0188	0183
2.1	0179	0174	0170	0166	0162	0158	0154	0150	0146	0143
2.2	0139	0136	0132	0129	0125	0122	0119	0116	0113	0110
2.3	0107	0104	0102	0099	0096	0094	0091	0089	0087	0084
2.4	0082	0080	0078	0075	0073	0071	0069	0068	0066	0064

Examples: Using the right tail probabilities table

- Use Φ to denote the distribution function (F) for standard normal distribution random variable Z
- Find the probability that a standard normal random variable Z is smaller than or equal to 1
  - Ans: $P(Z \leq 1) = 1 - P(Z \geq 1)$, in which $P(Z \geq 1) = 1 - \Phi(1) = 0.1587$
- Find $P(Z > 1.07)$
  - Ans: stay in the same row in the table but move to the seventh column to find that $P(Z > 1.07) = 0.1423$.

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- Find $P(Z > 1.07)$
  - Ans: stay in the same row in the table but move to the seventh column to find that $P(Z > 1.07) = 0.1423$.
- Find $P(Z \leq 0.75)$, how do you know — without doing any calculations — that the answer should be larger than 0.5?

The chemical reactor example, revisited

Example

Recall the chemical reactor example, where the residence time T, measured in minutes, has an exponential distribution with parameter $\lambda = v/V = 0.25$. The mean time the particle stays in the vessel is 4 minutes. However, from the viewpoint of process control this is not the quantity of interest. Often, there will be some minimal amount of time the particle has to stay in the vessel to participate in the chemical reaction, and we would want that at least 90% of the particles stay in the vessel this minimal amount of time.

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- We are interested in the number q with the property that $P(T > q) = 0.9$

$$P(T \leq q) = 0.1.$$

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- The number q is called the 0.1th quantile or 10th percentile of the distribution.

$$P(T \leq q) = 1 - e^{-0.25q} = 0.1.$$

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$$P(T \leq q) = 1 - e^{-0.25q} = 0.1.$$

- This holds exactly when $e^{-0.25q} = 0.9$. So $q = 0.42$. (10% of the particles stays less than 25 seconds!)

Quantile

Definition

Let X be a continuous random variable and let p be a number between 0 and 1. The *pth* quantile or 100*pth* percentile of the distribution of X is the smallest number qp such that

$$F(q_p) = P(X \leq q_p) = p.$$

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- If F (of X) is strictly increasing from 0 to 1 on some interval, then

$$q_p = F^{inv}(p)$$

Quantiles for an exponential distribution

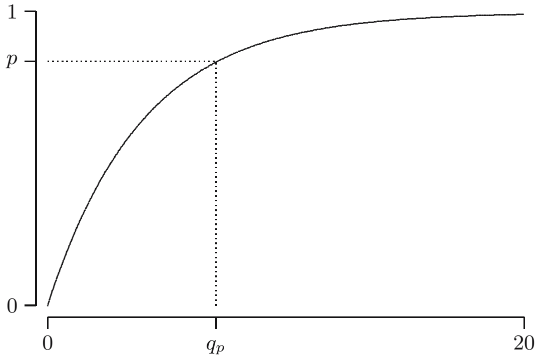

Figure: The pth quantile q_p of the $Exp(0.25)$ distribution.

Using the right tail probabilities table

- Find the 90th percentile of a standard normal
    - $\Phi(q_{0.9}) = 0.9$, that is $1 - \Phi(q_{0.9}) = 0.1$
    - The table gives us $q_{0.9} = 1.28$.

Using the right tail probabilities table

- Find the 90th percentile of a standard normal
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