Introduction to Probability, Statistics and Random Processes

Chapter 4: Continuous and Mixed Random Variables

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Introduction

- Discrete random variables can only take a countable number of possible values.
- Continuous random variables have a range in the form of
  - Interval on the real number line.
  - Union of non-overlapping intervals on real line.
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  - Union of non-overlapping intervals on real line.
- We also know that for any $k \in \mathbb{R}, P(X = k) = 0$.
- CDF works but PMF does not since $P(X = k) = 0$.

Probability density functions

- Random variables arise by a (never-ending) process of refinement from discrete random variables

  A discrete random variable associated with some experiment takes on the value 6.283 with probability p. If we refine, in the sense that we also get to know the fourth decimal, then the probability p is spread over the outcomes $6.2830, 6.2831, \cdots, 6.2839$.

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- Continuing the refinement process, the probabilities of the possible values approaches *zero*
- The probability that the possible values lie in some fixed interval $[a, b]$ will settle down.

Probability density functions

Definition

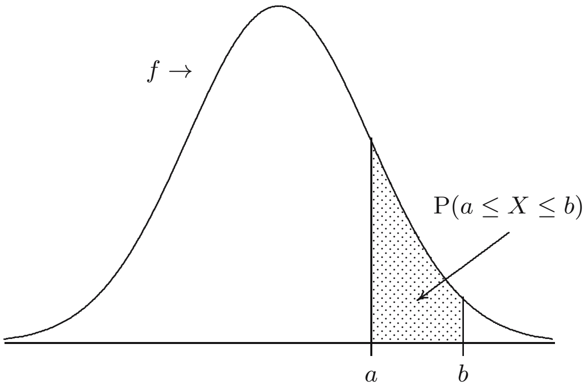
A random variable X is continuous if for some function $f : \mathbb{R} \to \mathbb{R}$ and for any numbers a and b with $a \leq b$

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

The function f has to satisfy $f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)dx = 1$. We call f the probability density function (or probability density) of X.

Probability density functions

- The probability that X lies in an interval $[a, b]$ is equal to the area under the probability density function f of X over the interval $[a, b]$



Area under a probability density function f on the interval $[a, b]$.

Probability density functions

- If the interval gets smaller and smaller, the probability will go to zero: for any positive ε

$$P(a-\varepsilon \leq X \leq a+\varepsilon) = \int_{a-\varepsilon}^{a+\varepsilon} f(x)dx$$

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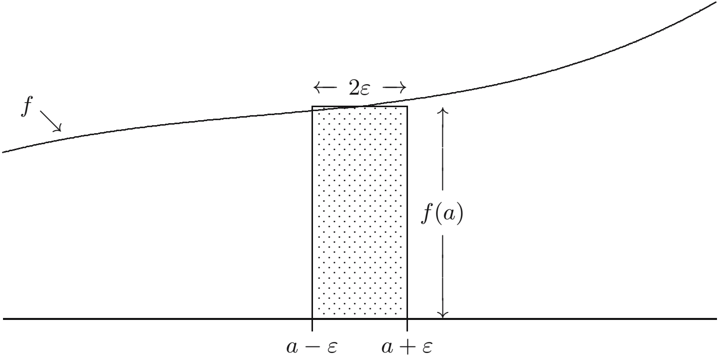
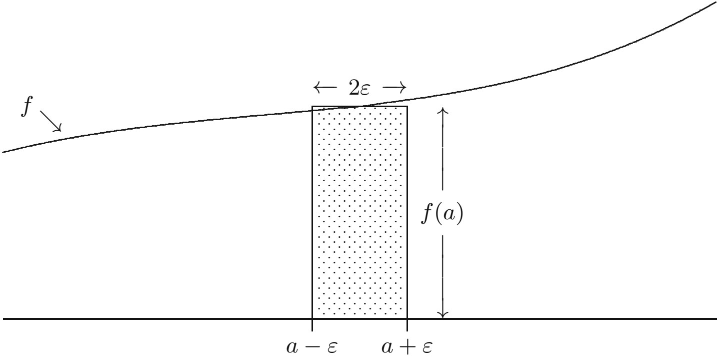
Probability density functions



- What does $f(a)$ represent?

$$P(a - \varepsilon \leq X \leq a + \varepsilon) \approx 2\varepsilon f(a)$$

Probability density functions



- What does $f(a)$ represent?

$$P(a-\varepsilon \leq X \leq a+\varepsilon) \approx 2\varepsilon f(a)$$

- $f(a)$: a (relative) measure of how likely it is that X will be near a. (can be arbitrarily large)

Discrete VS. Continuous

- discrete random variables: no probability density function f
- continuous random variables: no probability mass function p
- both have a distribution function $F(a) = P(X \leq a)$.

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a).$$

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- for a continuous variable X

$$F(b) = \int_{-\infty}^{b} f(x)dx \qquad f(x) = \frac{d}{dx}F(x).$$

Probability Density Function(PDF)

- We define the PDF of random variable X as

Definition: Consider a continuous random variable X with CDF $F(x)$. The function $f(x)$ is the probability density function (PDF) of X, defined by

$$f(x) = \frac{dF(x)}{dx} = F'(x),$$

if $F(x)$ is differentiable at x.

The darts example

- Model an experiment: "an object hits a disc of radius r in a completely arbitrary way"
  - we are interested in the distance X between the hitting point and the center of the disc.
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- Find out F, f
  - $F(b) = P(X \leq b) = \frac{\pi b^2}{\pi r^2} = \frac{b^2}{r^2}$ for $0 \leq b \leq r$.
  - $f(x) = \frac{d}{dx}F(x) = \frac{1}{r^2}\frac{d}{dx}x^2 = \frac{2x}{r^2}$ for $0 \leq b \leq r$.

Properties of PDF

Consider a continuous random variable X with PDF $f(x)$. We have

1. $f(x) \geq 0$ for all $x \in \mathbb{R}$.
2. $\int_{-\infty}^{\infty} f(u)du = 1$.
3. $P(a < X \leq b) = F(b) - F(a) = \int_a^b f(u)du$.
4. More generally, for a set A, $P(X \in A) = \int_A f(u)du$.

Range

- Range of a random variable X is the set of all possible values of the random variable.
- For a continuous random variable, we can define it as the set of all real numbers with non-zero PDF.

$$R_X = \{x | f(x) > 0\}$$

- R_X defined here might not show all possible values of X but the difference is unimportant.

Continuous Random Variables and their Distributions: the uniform distribution

- Example: Choose a real number uniformly at random in the interval $[a, b]$ and call it X.
- By uniformly at random, we mean all intervals in $[a, b]$ that have the same length have the same probability.
- Find the CDF of X.

Continuous Random Variables and their Distributions: the uniform distribution

- Uniformity implies that probability of an interval in $[a, b]$ is proportional to its length.

$$P(X \in [x_1, x_2]) \propto (x_2 - x_1)$$

- Since $P(X \in [a, b]) = 1$, we have

$$P(X \in [x_1, x_2]) = \frac{x_2 - x_1}{b - a}, \text{ where } a \leq x_1 \leq x_2 \leq b.$$

- From the definition of CDF, $F(x) = P(X \leq x)$ we get

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b \end{cases}$$

Continuous Random Variables and their Distributions: the uniform distribution

- CDF for a continuous random variable uniformly distributed over $[a, b]$.

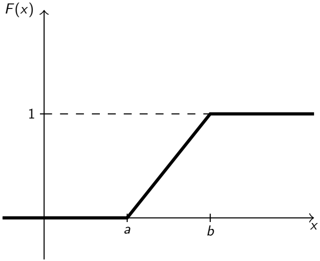

Figure: CDF for a continuous random variable uniformly distributed over $[a, b]$.

The uniform distribution

- The PDF of a random variable with $Uniform(a, b)$ distribution is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$

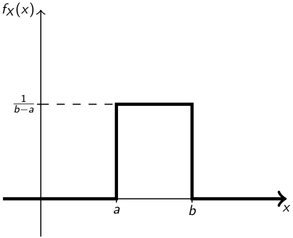

Figure: PDF for a continuous random variable uniformly distributed over $[a, b]$.

Continuous Random Variables and their Distributions: Recap

- We have the definition of a continuous random variable

> **Definition:** A random variable X with CDF $F(x)$ is said to be continuous if $F(x)$ is a continuous function for all $x \in \mathbb{R}$.

- The CDF is a continuous function with no jumps.
- No jumps is consistent with the fact that $P(X = x) = 0$ for all x.
- CDF of a continuous random variable is differentiable almost everywhere in $\mathbb{R}$.

Expected Value and Variance

- Remember the definition of expected value for a *discrete* random variable

$$E[X] = \sum_{k \in R_X} kp(k) = \sum_{k \in R_X} kP(X = k).$$

Expected Value and Variance

- Remember the definition of expected value for a *discrete* random variable

$$E[X] = \sum_{k \in R_X} kp(k) = \sum_{k \in R_X} kP(X = k).$$

- We can write the definition of expected value of a *continuous* random variable as

$$\boxed{E[X] = \int_{-\infty}^{\infty} xf(x)dx}$$

Expectation: the center of gravity

- $E[X]$ is indeed the center of gravity of the mass distribution described by the function f

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \frac{\int_{-\infty}^{\infty} xf(x)dx}{\int_{-\infty}^{\infty} f(x)dx}$$

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- $E[X]$ is indeed the center of gravity of the mass distribution described by the function f

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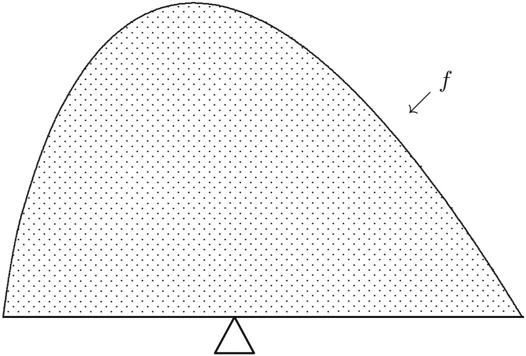

Expected Value of a Function of a Continuous Random Variable

Law of the unconscious statistician (LOTUS) for continuous random variables:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

- Expectation is a linear operation.
  - $E[aX + b] = aE[X] + b$ for all $a, b \in \mathbb{R}$, Prove
  - $E[X_1 + X_2 + ... + X_n] = EX_1 + EX_2 + ... + EX_n$ for any set of random variables $X_1, X_2, .., X_n$.

Variance

- Variance of a random variable is defined as

$$Var(X) = E[(X - \mu_X)^2] = EX^2 - (EX)^2$$

- For a continuous random variable we can write

$$\boxed{\begin{aligned} \mathrm{Var}(X) &= E\left[(X - \mu_X)^2\right] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x)dx \\ &= EX^2 - (EX)^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - {\mu_X}^2 \end{aligned}}$$

- For $a, b \in \mathbb{R}$, we have

$$Var(aX + b) = a^2 Var(X)$$

Discrete vs Continuous Random Variables

Discrete	Continuous
PMF $p(x) = P(X = x)$	PDF $f(x) = \frac{dF(x)}{dx}$
$\sum$	$\int$
$E[X] = \sum_{k \in R_X} kp(k)$	$E[X] = \int_{-\infty}^{\infty} xf(x)dx$
LOTUS $E[g(x)] = \sum_{k \in R_X} g(k)p(k)$	LOTUS $E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$