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SUPPORTEDNESS AND TAMENESS DIFFERENTIALLESS GEOMETRY OF PLANE CURVES

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Abstract—We introduce a class of planar arcs and curves, called tame arcs, which is general enough to describe (parts of) the boundaries of planar real objects. A tame arc can have smooth parts as well as sharp (non-differentiable) corners; thus a polygonal arc is tame. On the other hand, this class of arcs is restrictive enough to rule out pathological arcs which have infinitely many inflections or which turn infinitely often: A tame arc can have only finitely many inflections, and its total absolute turn must be finite.

In order to relate boundary properties of discrete objects obtained by segmenting digital images to the corresponding properties of their continuous originals, the theory of tame arcs is based on concepts that can be directly transferred from the continuous to the discrete domain. A tame arc is composed of a finite number of supported arcs. We define supported digital arcs and motivate their definition by the fact that they can be obtained by digitizing continuous supported arcs. Every digital arc is tame, since it contains a finite number of points, and therefore it can be decomposed into a finite number of supported digital arcs.

Plane curves   Supporting lines   Supported arcs   Tame arcs   Digital arcs

1. INTRODUCTION

Any model for a class of real objects should, on the one hand, be able to reflect relevant shape properties of the objects as exactly as possible; on the other hand, it should be mathematically tractable. For example, it does not make much sense to model the class of boundaries of two-dimensional (2D) projections of real objects as all possible curves in the plane. This class is too general; it includes curves with very unnatural properties—e.g. plane-filling curves.

It should also be possible to relate shape properties of the 2D projections of real objects to properties of the discrete objects obtained by segmenting their digital images. In pattern recognition, the properties of discrete objects that are measured using digital algorithms are assumed to represent properties of the original objects.

For these reasons, in this paper, we will not use the classical tools of differential geometry to describe boundary curves of planar sets. Differential geometry is based on the concept of a derivative, which requires the calculation of limits of infinite sequences. This calculation cannot be transferred into a discrete space. Nevertheless, it is clear that many concepts of differential geometry, such as curvature, characterize important shape properties of boundary curves of planar sets. In this paper we will define these shape concepts using geometrical concepts that can be also applied to discrete spaces.

The class of continuous planar arcs and curves is very large; it includes many "pathological" examples such as the "space-filling" curves of Peano and the nowhere differentiable "snowflake" curve of Sierpinski. Requiring the arcs and curves to be differentiable is too restrictive, since this excludes polygons, which are not differentiable at their vertices. A somewhat better idea is to require differentiability at all but a finite number of points, but this is not restrictive enough, because it allows arcs that can oscillate infinitely many times [e.g. the graph of the function $x \sin(1/x)$, shown in Fig. 1(a)] or turn infinitely often [e.g. the inward-turning spiral illustrated in Fig. 1(b)].

This paper defines classes of continuous planar arcs and curves that include polygonal arcs and polygons, but exclude the pathological cases. Our definitions are based on the concept of a *supporting line* of a set S—a line l through a point of S such that S lies in one of the closed half-planes bounded by l. We call a set S *supported* if there is at least one supporting line through every point of S. It can be shown (Section 2) that a closed, bounded, connected S is supported iff S is (an arc of) the boundary of a convex set. The concept of a supporting hyperplane (in particular, a supporting line) is a fundamental tool in the theory of convex functions, which is a part of convex analysis. As

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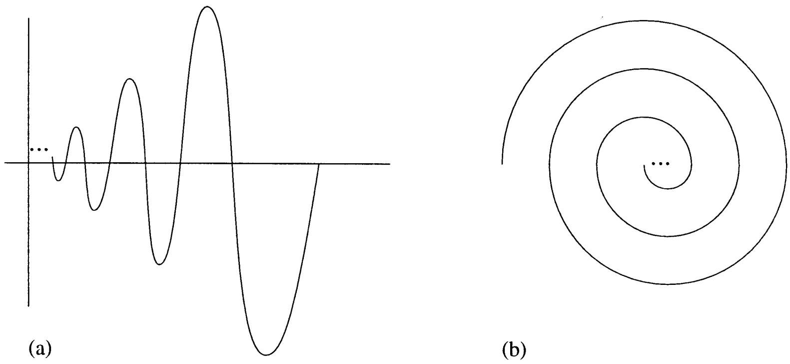

Fig. 1. (a) The graph of the function $x \sin(1/x)$. (b) An inward-turning spiral that turns infinitely often.

pointed out 25 years ago by Rockafellar[1] (p. XVI), "Supporting hyperplanes to convex sets can be employed in situations where tangent hyperplanes, in the sense of the classical theory of smooth surfaces, do not exist".

In Section 3, we define supporting sectors and rays, which correspond to directional tangent lines. In Section 4, we relate the properties of supported arcs to the standard properties of arcs in differential geometry. This relationship is based on the well-established connections between differential geometry and convex analysis. We show that a supported arc A has left and right derivatives at every point, and that A is differentiable at a point if it has a unique supporting line at that point. On the other hand, a supported arc A can have (even infinitely many) "cusps": non-endpoints at which A has more than one supporting line. (Here and in what follows, "arc" is short for "arc or simple closed curve".)

We call an arc A *uniquely supported* if it has a unique supporting line at every non-endpoint. Thus, a uniquely supported arc is everywhere differentiable and has no cusps. We will show that the curvature of a uniquely supported arc has the same sign at every point.

We call a (simple) arc *tame* if it can be subdivided into a finite number of supported subarcs (Section 5). Evidently, polygonal arcs are tame, but it can be verified that the pathological arcs of Fig. 1 are not tame. The subdivision points of a tame arc are called its *joints*. The points of a tame arc can be classified into *regular* and *inflection* points, according to whether they are interior to some supported subarc. Consequently, inflection points are mandatory joints, so that a tame arc can have only finitely many inflections, though it can have infinitely many cusps. As we shall see, both regular and inflection points can be cusps.

In Section 6 we show, using an associated polygonal arc, how the total curvature, which we will call the (total) turn, of a tame arc can be defined in terms of supporting lines. We prove that the total turn of a supported arc is at most 360°; this is well known for differentiable arcs. We also give alternative characterizations of cusps and inflections using an associated polygonal arc (Section 7).

In Section 8, we define supported digital arcs and motivate their definition by the fact that they can be obtained by digitizing continuous supported arcs. The extension of our theory of tame arcs to digital arcs is also based on the concepts of supporting lines and half-planes:

A digital set $S \subseteq \mathbb{Z}^2$ is *digitally supported* if every point $p \in S$ is a (4-) boundary point of a digital half-plane containing S (i.e. there exists a continuous half-plane containing S such that p has a 4-neighbor outside this half-plane).

We show that a digital set is supported iff it is contained in the boundary of the digitization of a convex set. This gives us a new definition of a convex-digital set as a digital set whose boundary is digitally supported. We also show that a digital set is supported iff it is contained in the digitization of a continuous supported arc.

2. SUPPORTING LINES; SUPPORTED SETS

We begin with a definition of fundamental tools used in convex analysis (e.g. see the book by Rockafellar,[1] Section 11, pp. 99–100):

Definition. Let S be a subset of the plane, and p a point of S. A straight line $l_S(p)$ through p is called a *supporting line* of S at p if S is contained in one of the closed half-planes into which $l_S(p)$ divides the plane. This closed half-plane is called a *supporting half-plane*.

We will deal here with supported sets, which are (not necessarily proper) arcs, as we will shortly see:

Definition. A subset S of the plane is *supported* if, for every $p \in S$, there exists at least one supporting line of S at p.

Note that if S has a supporting line at p, then p must be a boundary point of S (i.e. any neighborhood of

p contains points of the complement of S). Note also that supporting lines need not be unique. For example, if S is a single point, every line through that point is a supporting line of S; if S is a segment of a straight line l, the same is true for its endpoints, but at its interior points, l is the only supporting line of S.

Before stating some basic facts, we summarize some basic definitions: An *arc* is a subset of the plane which is a homeomorphic image of an interval of non-zero length: A: $[a, b] \rightarrow \mathbb{R}^2$, where $a < b$. The points $A(a)$ and $A(b)$ are called the *endpoints* of arc A. If the endpoints of an arc are the same [i.e. $A(a) = A(b)$], the arc becomes a *simple closed curve* (or a *Jordan curve*), and can be regarded as having no endpoints. A simple closed curve can also be defined as a homeomorphic image $C: S^1 \rightarrow \mathbb{R}^2$ of a circle of non-zero radius r. A simple closed curve and a single point are sometimes called non-proper arcs. If arc A is a subset of arc B or curve C, it is called a *subarc* of B or C. We recall[2,3] that the *closed convex hull conv*(S) of a set S is the intersection of all the closed half-planes that contain S.

We now state a few basic properties of supported sets. Since they can be derived from elementary concepts of convex analysis, their proofs will be omitted here; they are given in full in a report by the authors.[4]

Proposition 1. A set S is supported iff it is contained in the boundary of its closed convex hull.

Proposition 2. A closed, bounded, connected set S is supported iff S is a subarc (not necessarily proper) of the boundary of a convex set (the convex hull of S).

Let C be a simple closed curve. By the Jordan curve theorem, the complement of C has two non-empty connected components, one of which is bounded and surrounded by C. Let C^* be the closure of the bounded component. By the Jordan curve theorem, C is the boundary of C^*, and the interior of C^* is non-empty.

Proposition 3. A simple closed curve C is supported iff C^* is convex.

This result is a special case of a theorem which was proved by a number of prominent mathematicians, including Caratheodory,[5] Brunn,[6] and Minkowski,[7] and which can also be found in a more general form in the book by Valentine[2] (Theorem 4.1, p. 47).

The characterization of supported arcs can also be derived from a concept introduced by Latecki, Rosenfeld, and Silverman:[8]

Definition. A subset S of the plane is said to have property CP_3 if for every three collinear points of S, the line segment joining at least two of them is contained in S.

Property CP_3 appears to be the first simple intrinsic characterization of parts of the boundaries of convex sets. Considerably more complicated characterizations were given over 30 years ago by Menger[9] and Valentine.[10]

It can be shown[8] that an arc A is supported iff it has property CP_3 (Theorem 14), and that this in turn is equivalent to $A \cup A(a)\,A(b)$ being the boundary of a convex set (Theorem 12), where $A(a)\,A(b)$ is the line segment whose endpoints are $A(a)$ and $A(b)$. If $A \neq A(a)\,A(b)$, $A \cup A(a)\,A(b)$ is a simple closed curve; thus, either a supported arc is a line segment, or joining its endpoints yields a supported simple closed curve. We thus have

Proposition 4. An arc A is supported iff $A \cup A(a)\,A(b)$ is the boundary of a convex set.

3. SUPPORTING SECTORS

Definition. Let p be a point of a set S such that at least one supporting line of S at p exists. The *supporting sector* $\sigma_S(p)$ is defined as the intersection of all the closed supporting half-planes of S at p.

Clearly, $\sigma_S(p)$ is a closed and convex subset of the plane. We assume from now on that S is not a subset of a line (the contrary case was discussed in Section 2). If there is a unique supporting line l of S at p, then $\sigma_S(p)$ is the closed half-plane determined by l that contains S [see Fig. 2(a)]. If there is more than one supporting line of S at p, then $\sigma_S(p)$ is a closed angular sector with angle $\alpha_S(p)$ less than 180° [see Fig. 2(b)]. Note that a line through p is a supporting line of S at p iff it is contained in the closure of the complement of $\sigma_S(p)$.

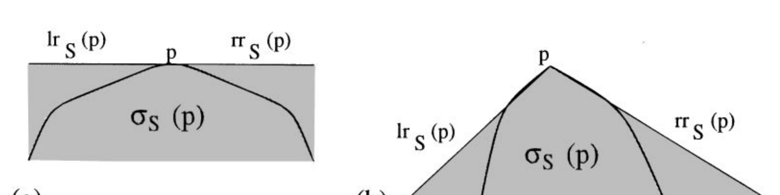

Fig. 2. $\sigma_S(p)$ is the supporting sector of S at point p.

If S has a unique supporting line at p, we define the *turn angle* of S at p as $0°$. If S has more than one supporting line at p, we define the *turn angle* of S at p as $180° - \alpha_S(p)$.

We now assume that the set S is an arc. The supporting sector $\sigma_S(p)$ of an arc S is bounded by two rays emanating from p that make an angle $\alpha_S(p) \leq 180°$. We will call these rays the left and right supporting rays of A at p depending on the direction in which we traverse A. The *left supporting ray* of S at p will be denoted by $lr_S(p)$ and the *right supporting ray* of S at p will be denoted by $rr_S(p)$ (see Fig. 2). These concepts are precisely defined in the Appendix.

4. RELATION TO DIFFERENTIAL GEOMETRY

Any arc is continuous, but we have not assumed that supported arcs are differentiable. In this section we show that a supported arc must have left and right derivatives at every point (Theorem 1).

Let $A: [a, b] \to \mathbb{R}^2$ be an arc and let $x, y \in (a, b)$. Consider the vector $(A(x) - A(y))/|x - y|$. As y approaches x from the left (right), this vector may approach a finite, non-zero limit; if so, the limit is called the left (right) derivative of A at x and is denoted by $A'_-(x)$ $[A'_+(x)]$. The right derivative of A at a, and the left derivative of A at b, are defined similarly.

The left (right) derivative exists at $x \in (a, b)$ and is a finite and non-zero vector $A'_-(x)$ $[A'_+(x)]$ iff the limit of the lines through $A(x)$ and $A(t)$ as t approaches x from below, i.e. $t < x$ (above, i.e. $x < t$).

If the directional derivatives $A'_-(x)$ and $A'_+(x)$ exist and are equal, then we say that the derivative of A at x exists and has value $A'(x) \equiv A'_-(x) = A'_+(x)$.

The following theorem describes a fundamental relation between supportedness of arcs and the existence of directional derivatives. It was noted as early as 1893 by Stoltz.[11] It follows, e.g. from Theorem 23.1, p. 213, in the book by Rockafellar.[1]*

Theorem 1. Let $A: [a, b] \to \mathbb{R}^2$ be a supported arc. Then the directional derivatives $A'_-(x)$ and $A'_+(x)$ exist and are finite and non-zero at every point $x \in (a, b)$, and the same for $A'_-(a)$ and $A'_+(b)$.

If the supporting line at a point x of A is unique, the left and right supporting rays at x are collinear, which implies that the left and right derivatives at x are equal. We thus have

Theorem 2. Let $A: [a, b] \to \mathbb{R}^2$ be a supported arc, and let $x \in (a, b)$.Then the derivative $A'(x)$ exists [i.e. $A'(x) = A'_-(x) = A'_+(x)$] iff the supporting line at x is unique.

*This theorem is stated for convex functions; but a supported arc can be locally treated as a graph of a convex function from a closed interal into $\mathbb{R}$.

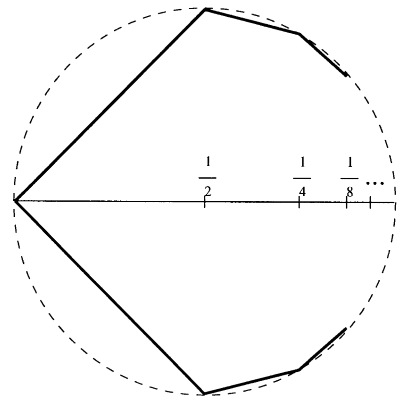

Fig. 3. A supported arc can have infinitely many cusps.

A proof of Theorem 2 can also be found in Rockafellar's book[1] (Theorem 25.1, p. 242).

Definition. We define a non-endpoint p of a supported arc A to be a *cusp* if there is more than one supporting line of A at p.

By Theorem 2, the arc A is not differentiable at a cusp. Note that a supported arc can have infinitely many cusps, as illustrated in Fig. 3. It is not hard to see that at the accumulation point of the cusps in Fig. 3, the arc is differentiable.

Note that differentiability does not imply supportedness (i.e. the converse of Theorem 2 is not true); at a point of inflection of an arc (see Section 5), its derivative may exist, but it has no supporting line.

Definition. An arc A is *uniquely supported* if at every non-endpoint $p \in A$ there exists a unique supporting line $l_A(p)$.

Clearly, if an arc is uniquely supported, it is supported. However, the converse is not true; e.g. a convex simple polygonal arc is supported but does not have unique supporting lines at its vertices.

In this and the next two paragraphs we deal with arcs $A: [a, b] \to \mathbb{R}^2$ that belong to class C^2, i.e. the first and second derivatives of A exist and are non-zero vectors for every point t in the open interval (a, b)—in other words, a C^2 arc is an immersion [reference (12), p. 1–1]. For a simple closed curve $A: [a, b] \to \mathbb{R}^2$ to belong to class C^2, we additionally require that $A(a) = A(b)$, $A'(a) = A'(b)$, and $A''(a) = A''(b)$. Clearly, a supported C^2 arc (or curve) is uniquely supported.

We recall that the curvature of an arc can be defined as rate of change of slope (as a function of arc length). The magnitude of the curvature depends on how the arc is parameterized (which need not be by arc length), but its sign does not depend on the parameterization. We can now restate Theorem 8, pp. 1–26,

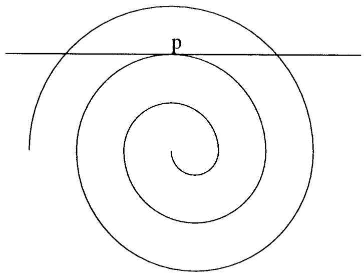

Fig. 4. A differentiable arc may not be supported.

from Spivak[12] (Spivak, p. 1–16, calls supported simple closed C^2 curves *convex*):

Theorem 3. A simple closed C^2 curve $C:[a, b] \to \mathbb{R}^2$ is supported iff, for every $p \in [a, b]$, the curvature $\kappa(p)$ satisfies $\kappa(p) \geq 0$ (or $\kappa(p) \leq 0$, depending on the direction in which C is traversed) for every $p \in [a, b]$.

The following theorem is a simple consequence of Theorem 3.

Theorem 4. If a C^2 arc A is supported, then for every non-endpoint $p \in A$, the curvature $\kappa(p)$ satisfies $\kappa(p) \geq 0$ (or $\kappa(p) \leq 0$, depending on the direction in which A is traversed).

Proof. We can extend arc A to a supported simple closed C^2 curve C, and then apply Theorem 3 to C. We need only construct a supported C^2 arc B, lying in the half-plane determined by the line segment $A(a)$ $A(b)$ that does not contain arc A, such that the endpoints and first and second derivatives at the endpoints of B coincide with those of A. □

The converse of Theorem 4 is not true. Consider a spiral S such that for all non-endpoints $p \in S$ the curvature $\kappa(p)$ exists and has the same sign; see Fig. 4. Evidently, for any point $p \in S$ such that the total turn of the part of the spiral from p to one of the endpoints is greater than 360°, there is no supporting line of S at p. (In Section 6 we will define the total turn of a supported arc and show that it is at most 360°.)

5. TAME ARCS

The class of supported arcs is quite restricted; they cannot turn by more than 360°, and they can only turn in one direction, so they cannot have inflections. In this section we study a class of arcs which we call "tame"; a tame arc consists of a finite number of subarcs each of which is supported.

Definition. An arc $T:[a, b] \to \mathbb{R}^2$ will be called *tame* if there exist points $x_1, \ldots, x_n \in [a, b]$ with $x_1 = a$ and $x_n = b$ such that $T([x_i, x_{i+1}])$ is supported for $i = 0, \ldots, n-1$. The points $x_2, \ldots, x_{n-1}$ will be called the *joints* of T.

Note that there are many choices for the joints; as we shall see below, only inflections are mandatory joints, but it may also be necessary to introduce additional "optional" joints to ensure that the subarcs are supported (e.g. see Fig. 4).

Trivially, a supported arc is tame. In general, a tame arc T can be described as "piecewise"-supported. Note that even at its joints, a tame arc has one-sided derivatives. If they are unequal, so that the union of the left and right supporting rays is not a straight line, we shall call the joint a *cusp*.

The class of tame arcs seems to be general enough to describe (arcs of) the boundaries of planar objects. On the other hand, this class is restrictive enough to rule out the pathological examples of arcs and curves discussed in the introduction. For example, the spiral shown in Fig. 1, which turns inward infinitely often, is an arc, but is not tame, since no matter how we divide it into a finite number of subarcs, the first or last arc still turns infinitely often, and so does not have a supporting line at every point. Note also that a tame arc can only have finitely many inflections, since its curvature cannot change sign except possibly at the x_i's. Thus, e.g. the graph of $x \sin(1/x)$ is not tame, since it oscillates infinitely often as x approaches 0. Sierpinski's "snowflake" curve and Peano's "space-filling" curves are not tame for the same reason.

A polygonal arc (or polygon) is piecewise straight, so that in particular it is tame. The term "tame" is used in knot theory to describe knots that are equivalent to polygonal knots;[13] in our case too, tameness is a generalization of polygonality.

Definition. A point x of a tame arc T will be called a *regular point* if there exists a supported subarc T' of T such that x is interior to T', i.e. $x \in T'$ and x is not an endpoint of T'. A non-endpoint that is not a regular point will be called an *inflection point*.

Evidently, an inflection point of a tame arc T must be a joint of T. The other joints of T, if any, are "optional" joints.

Classically, an inflection point of a differentiable arc A [e.g. Fig. 5(a)] is a non-endpoint at which the curvature of A changes sign. By Theorem 4, such a point cannot be an interior point of a supported subarc of A; hence, if A is differentiable and tame, each of its subarcs between consecutive joints is uniquely supported, and its curvature has a constant sign on each of these subarcs. However, a tame arc need not be differentiable; it can have both regular and inflection points that are cusps. For example, the cusps in Figs 5(b)–(d) are inflection points. In Section 7 we will use total turn concepts to classify inflection cusps.

As we have already seen, a tame arc that contains only regular non-endpoints is not necessarily

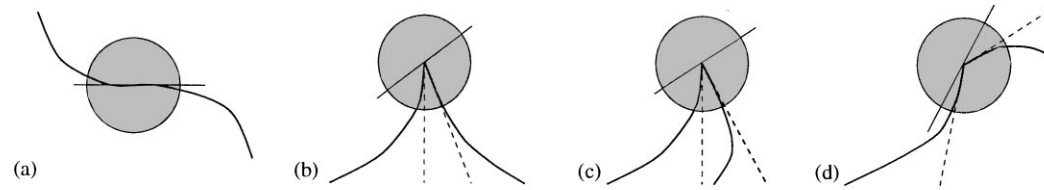

Fig. 5. Inflection points.

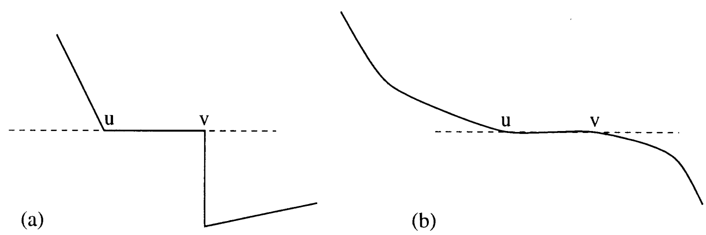

Fig. 6. Inflection segments.

supported, e.g. the spiral in Fig. 4. A polygonal arc also has no inflection points, but as shown in Fig. 6(a), it need not be supported. More generally, an arc that has a straight subarc [Fig. 6(b)] may have no inflection points, but may not be supported. In both of the cases in Fig. 6, at least one of the points in the subarc uv must be a joint.

Definition. A maximal straight subarc S of a tame arc T will be called an *inflection segment* if there does not exist a supported subarc T' of T such that S is interior to T', i.e. $S \subseteq T'$ and S does not contain an endpoint of T'. Evidently, at least one point of an inflection segment must be a joint.

6. THE TOTAL TURN OF A TAME ARC

In differential geometry, the total curvature of an arc is defined by integrating the curvature. Similarly, the "total turn" of a polygonal arc is defined by summing the turns of its vertices. In Section 3, we defined the turn angle at a point of a supported arc. In this section we will define the (total) turn of a tame arc. Our definition will be based on associating a polygonal arc with the tame arc. The associated polygonal arc is not unique, but as we shall see, the turn of every polygonal arc associated with a given tame arc is the same. Therefore, we can define the total turn of a tame arc as the total turn of any associated polygonal arc.

6.1. Simple polygonal arcs

Let $\{x_i : 0 \le i \le n-1\}$ be distinct points in the plane. The ordered sequence of vectors $(x_i\, x_{i+1})_{i=0,\ldots,n-1}$ defines a polygonal arc *polyarc*(x_0, x_n) joining x_0 to x_n. If $x_i\, x_{i+1} \cap x_j\, x_{j+1} = \{x_i, x_{i+1}\} \cap \{x_j, x_{j+1}\}$ for $0 \le i \neq j \le n$, we

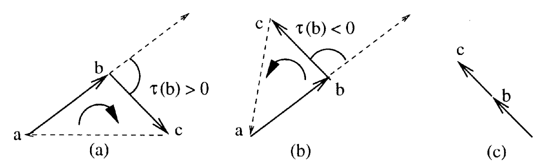

Fig. 7. The turn angle $\tau(b)$ at a vertex b with respect to two vectors ab and bc: (a) $\tau(b) > 0$, (b) $\tau(b) < 0$, and (c) $\tau(b) = 0$.

call *polyarc*(x_0, x_n) a *simple polygonal arc.* If $x_0 = x_n$, it is called a (*simple*) *polygon.*

Let ab and bc be non-collinear vectors; then the *turn angle* $\tau(b)$ is defined as $sign * \alpha$, where α is the angle between ab and bc and $sign = +1$ or -1 depending on whether the triangle abc is oriented clockwise or counterclockwise [see Figs 7(a) and (b)]. If ab and bc are collinear, then $\tau(b) = 0$ if ab and bc point in the same direction [see Fig. 7(c)]. (The case where ab and bc are collinear but point in opposite directions will be discussed later.)

Definition. If $x_0 \neq x_n$, we define the *turn* of a simple polygonal arc *polyarc*(x_0, x_n) as

$$\tau(polyarc(x_0, x_n)) = \sum_{i=1}^{n-1} \tau(x_i),$$

where $\tau(x_i)$ is the turn angle at vertex x_i with respect to vectors $x_{i-1}\, x_i$ and $x_i\, x_{i+1}$ for $i = 1, \ldots, n-1$. If $x_0 = x_n$, i.e. the arc *polyarc*(x_0, x_n) is a polygon, then we define its turn as

$$\tau(polyarc(x_0, x_n)) = \sum_{i=0}^{n-1} \tau(x_i),$$

where the turn angle at vertex x_0 is defined with respect to vectors $x_{n-1}x_0$ and x_0x_1.

For example, the turn of *polyarc*(x_0, x_5) in Fig. 8(a) is given by $\tau(polyarc(x_0, x_5)) = \tau(x_1) + \cdots + \tau(x_4)$,

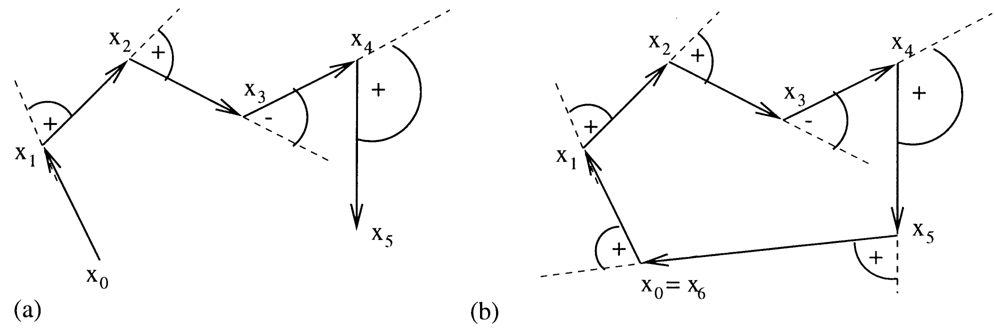

Fig. 8. The turn of the polygonal arc in (a) is $\tau(x_1) + \cdots + \tau(x_4)$. The turn of the closed polygonal arc in (b) is $\tau(x_0) + \cdots + \tau(x_5)$.

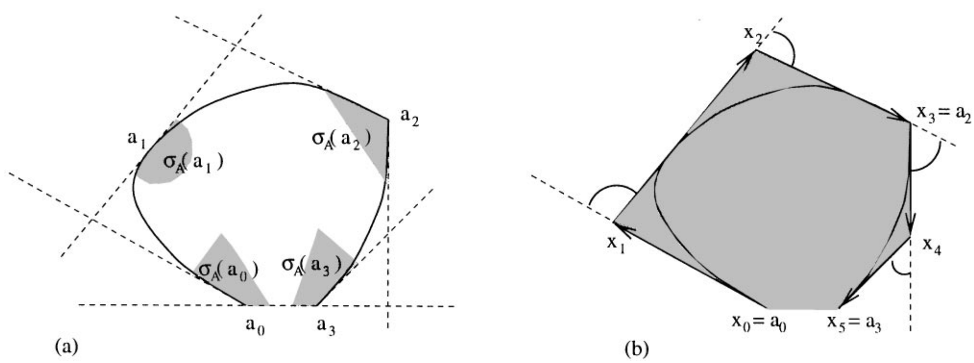

Fig. 9. (b) Shows an associated polygonal arc of the supported arc in (a).

and the turn of *polyarc* (x_0, x_6) with $x_0 = x_6$ in Fig. 8(b) is given by $\tau(polyarc(x_0, x_5)) = \tau(x_0) + \cdots + \tau(x_5)$.

If $polyarc(x_0, x_n) = (x_i\, x_{i+1})_{i=0,\ldots,n-1}$ is a simple polygon, the bounded region surrounded by *polyarc*(x_0, x_n) is called its *interior*. If the interior of the polygon is to the right of each vector $x_i x_{i+1}$, then the turn angle $\tau(x_i)$ at each vertex x_i is equal to 180° minus the interior angle of the simple polygon at x_i. A positive value of the turn at a vertex x_i indicates that x_i is a convex vertex of the polygon [e.g. $\tau(x_2) > 0$ in Fig. 8(b)], and a negative value of the turn at a vertex x_i indicates that x_i is a concave vertex of the polygon [e.g. $\tau(x_3) < 0$ in Fig. 8(b)].

It is well known that if $polyarc(x_0, x_n)$ is a simple polygon, then $|\tau(polyarc(x_0, x_n))| = 360°$ [see, e.g. reference (14), Lemma 4.16, p. 182]. Note that the sign of the turn depends on the direction in which we traverse the polygonal arc, i.e. $\tau(polyarc(x_0, x_n)) = -\tau(polyarc(x_n, x_0))$, where $polyarc(x_n, x_0) = (x_{k+1}\, x_k)_{i=n-1,\ldots,0}$. We thus have

Proposition 5. The turn of a simple polygon C is $\tau(C) = \pm 360°$.

6.2. *Supported arcs*

We will now show how to associate a polygonal arc $P(A)$ with any supported arc A. We will then define the turn of A as the turn of $P(A)$, and show that this turn is the same for any $P(A)$ associated with A.

Definition. Let $A : [a, b] \to \mathbb{R}^2$ be a supported arc. We will show below that there exists a set of points $\{a_i \in A : 0 \le i \le k\}$ such that $A(a) = a_0$, $A(b) = a_k$, and

$$\bigcap \{\sigma_A(a_i) : 0 \le i \le k\}$$

is a bounded region whose boundary is a simple polygon [see Fig. 9(a)]. Let $\{x_i : 0 \le i \le n\}$ be the set of vertices of this polygon ordered such that $a_0 = x_0$ and $a_k = x_n$, and such that the interior of the polygon is either to the right of each vector $x_i x_{i+1}$ or to the left of each vector $x_i x_{i+1}$ [see Fig. 9(b)]. We associate the simple polygonal arc $P(A) = (x_i\, x_{i+1})_{i=0,\ldots,n-1}$ with the supported arc A, and define the *turn* of A as

$$\tau(A) = \tau(P(A)).$$

We now show that for any supported arc A, there always exists such a set of points $\{a_i \in A : 0 \le i \le k\}$. In fact, in addition to the endpoints $A(a) = a_0$ and $A(b) = a_k$, it is sufficient to take a point of A with maximal positive x-coordinate, one with maximal negative x-coordinate, one with maximal positive y-coordinate, and one with maximal negative y-coordinate; thus four points (besides the endpoints) always

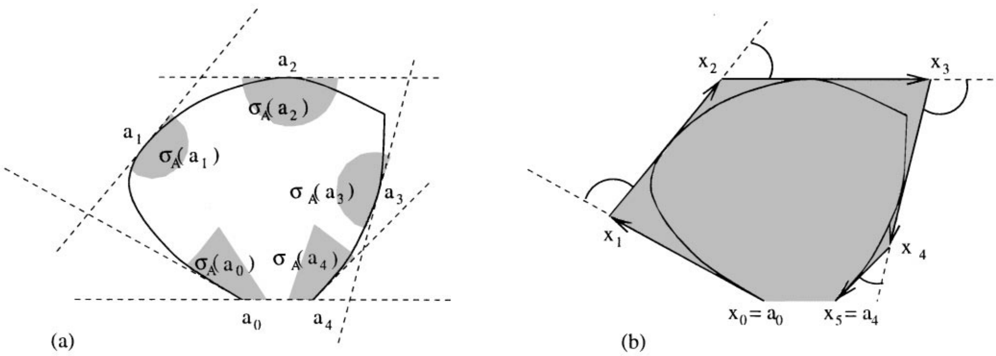

Fig. 10. (b) Shows a different associated polygonal arc of the supported arc in (a).

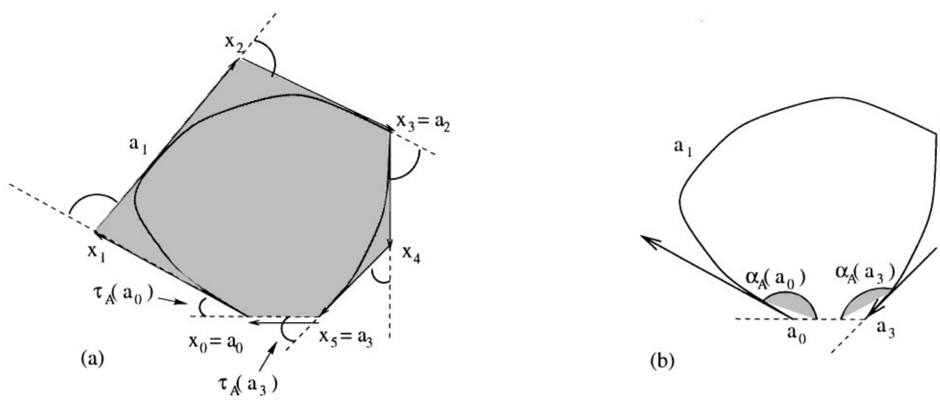

Fig. 11. The absolute turn of the supported arc A with endpoints a_0 and a_3 is equal to $\alpha_A(a_0) + \alpha_A(a_3)$.

suffice. We emphasize that the $a's$ are not unique; e.g. a different set of $a's$ for the supported arc in Fig. 9 is shown in Fig. 10.

Note that if a point v of the supported arc A coincides with a vertex of an associated $P(A)$, which is the case for $a_2 = x_3$ in Fig. 9, then $|\tau(v)| = 180^\circ - \alpha_A(v)$, where $\alpha_A(v)$ is the angle of the supporting sector $\sigma_A(v)$ of A at $v \in A$, i.e. $|\tau(v)|$ [with respect to $P(A)$] is the turn angle of A at v.

We now show that the turn of arc A defined in this way does not depend on the choice of the points $\{a_i \in A: 0 \le i \le k\}$. We show this for the absolute value of the turn of arc A in order to abstract from the particular orientation of arc A. Observe first that $\bigcap \{\sigma_A(a_i): 0 \le i \le k\}$ is convex (it is a finite intersection of supporting half-planes of A). Consequently, the polygonal arc $P(A)$ is part of the boundary of a convex set.

Suppose first that $a_0 = a_k$ (i.e. A is a simple closed curve). Then $P(A)$ is a simple polygon, and $\tau(P(A)) = \pm 360^\circ$ by Proposition 5. Since $\tau(P(A)) = \pm 360^\circ$ for every simple polygon, this result does not depend on the choice of the $a's$. Thus, we have

Proposition 6. The absolute turn of a supported simple closed curve A is $|\tau(A)| = |360^\circ|$.

We now assume that $a_0 \ne a_k$. Consider the polygonal arc $P'(A) = P(A) \circ x_n x_0$, where "$\circ$" represents concatenation, so that $P'(A)$ is $P(A)$ followed by $x_n x_0$. Evidently, $P'(A)$ is a simple polygon; thus $|\tau(P'(A))| = 360^\circ$ (Proposition 5). The absolute turn of $P(A)$ is equal to the turn of $P'(A)$ minus the turn angles at vertices $a_0 = x_0$ and $a_k = x_n$ [see Fig. 11(a)]. These turn angles are $|\tau(a_0)| = 180^\circ - \alpha_A(a_0)$ and $|\tau(a_k)| = 180^\circ - \alpha_A(a_k)$ [see Fig. 11(b)]. Therefore,

$$\begin{aligned} |\tau(A)| = |\tau(P(A))| &= ||\tau(P'(A))| - (\tau(a_0) + \tau(a_k))| \\ &= |360^\circ - (180^\circ - \alpha_A(a_0) + 180^\circ - \alpha_A(a_k))| \\ &= \alpha_A(a_0) + \alpha_A(a_k). \end{aligned}$$

Consequently, the absolute value of the turn of a supported arc A does not depend on the choice of $a_0, \ldots, a_k$. We also have [see Fig. 11(b)]

Proposition 7. The absolute turn of a supported arc A with endpoints $A(a) \ne A(b)$ is given by

$$|\tau(A)| = \alpha_A(A(a)) + \alpha_A(A(b)).$$

Since a supported arc A is contained in one of the closed half-planes determined by its endpoints

$A(a) \neq A(b)$, the angle of the supporting sector $\alpha_A(p)$ is less than or equal to 180°, where $p = A(a)$ or $p = A(b)$. Consequently, we obtain:

Proposition 8. *The absolute turn of a supported arc A is* $|\tau(A)| \leq 360°$.

Now we prove two technical propositions which we will need in Section 7.

Proposition 9. *Let* $A:[a,b] \to \mathbb{R}^2$ be a supported arc. Then A is contained in the convex hull of an associated polygonal arc $P(A)$.

Proof. We showed above that there exists a set of points $\{a_i \in A: 0 \leq i \leq k\}$ such that $A(a) = a_0$, $A(b) = a_k$, and

$$C(A) = \bigcap \{\sigma_A(a_i): 0 \leq i \leq k\}$$

is a bounded region whose boundary is a simple polygon which contains $P(A)$. The statement of the proposition follows from the facts that $C(A)$ is the convex hull of $P(A)$ and A is contained in $C(A)$. □

Proposition 10. Let $A:[a,b] \to \mathbb{R}^2$ be a supported arc. For every $\varepsilon > 0$, there exists a subarc A' of A such that $A(a) \in A'$ and $|\tau(A')| < \varepsilon$.

Proof. Let $a_o, \ldots, a_k$ be as in the proof of Proposition 9. Since the supporting sectors at the a's intersect in a bounded region, in particular $lr_A(a_0)$ intersects $rr_A(a_1)$ [see Fig. 12(a)]. Let B be the subarc of A between a_0 and a_1. The angle α between $lr_A(a_0)$ and $rr_A(a_1)$ shown in Fig. 12(a) is equal to $|\tau(B)|$, and $|\tau(B)| < 180°$.

It is not hard to see that there exists a point $p_1 \in B$ distinct from a_0 and a_1 such that the line l parallel to line segment a_0a_1 is a supporting line of B at p_1 [see Fig. 12(b)]. (Proof. Consider the function d which associates with each point of B its perpendicular distance to line segment a_0a_1. Since d is a continuous function, and B is a compact set, there is a point of B at which d attains its maximum value. Evidently the line through any such point p_1 parallel to a_0a_1 must be a supporting line of B.)

The angle β between $lr_A(a_0)$ and l is equal to $\alpha_B(a_0)$, the angle of the supporting sector $\sigma_B(a_0)$. Let B_1 be the subarc of A with endpoints a_0 and p_1. Then β is the turn of B_1.

Let $a_1 = b_1, b_2, \ldots$ be a sequence of points of B that converge to a_0 (in the standard Euclidean distance on the plane). Let p_i be determined for each b_i, $i > 1$, in the same way that p_1 was determined for $b_1 = a_1$, and let B_i be the subarc of B with endpoints a_0 and p_i. Then $\tau(B_i) = \alpha_{B_i}(a_0)$.

Since the left derivative of A at a_0 exists (Theorem 1), rays a_0b_i approach $lr_A(a_0)$ if b_i approaches a_0. Consequently, the angle $\alpha_{B_i}(a_0)$ goes to 0 if b_i approaches a_0. Therefore, for every $\varepsilon > 0$, there exists an i such that $|\tau(B_i)| < \varepsilon$. □

6.3. *Tame arcs*

Let $T:[a,b] \to \mathbb{R}^2$ be a tame arc (or in particular a tame simple closed curve). Then T can be divided into a finite number of supported arcs, i.e., there exist points $t_0 < \cdots < t_m \in [a,b]$ with $t_1 = a$ and $t_m = b$ such that $T_i = T([t_i, t_{i+1}])$ is supported for $i = 0, \ldots, m-1$. Let $\bigcirc$ denote concatenation of polygonal arcs. With every T_i we can associate a simple polygonal arc $P(T_i)$ such that $\bigcirc_{i=0}^{m-1} P(T_i)$ is also a polygonal arc (not necessarily simple). This means that all the $P(T_i)$ have a consistent order, i.e., we can traverse them from $T(a)$ to $T(b)$ following the directions of their vectors.This associates a polygonal arc

$$P(T) = \bigcirc_{i=0}^{m-1} P(T_i)$$

with T. An example is given in Fig. 13; here T is divided into two supported arcs $T = T_1 \cup T_2$ at the

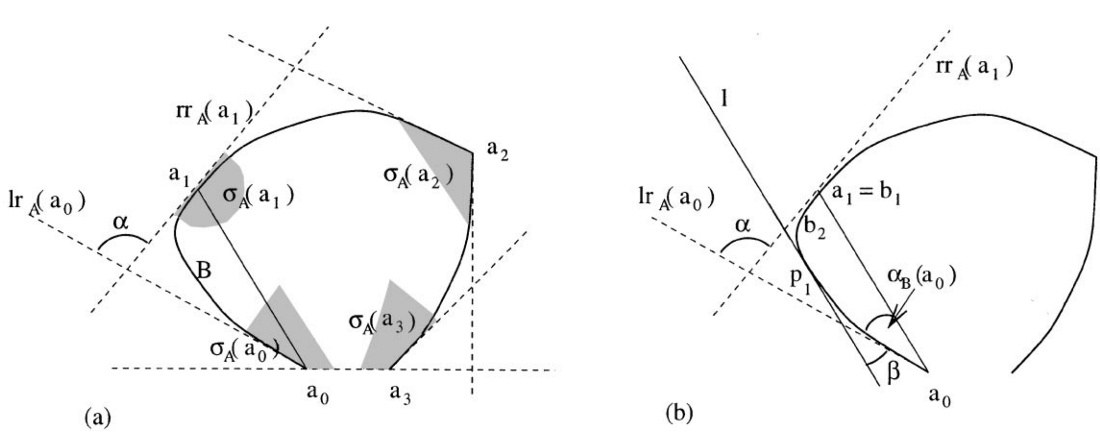

Fig. 12. Steps in the proof of Proposition 10.

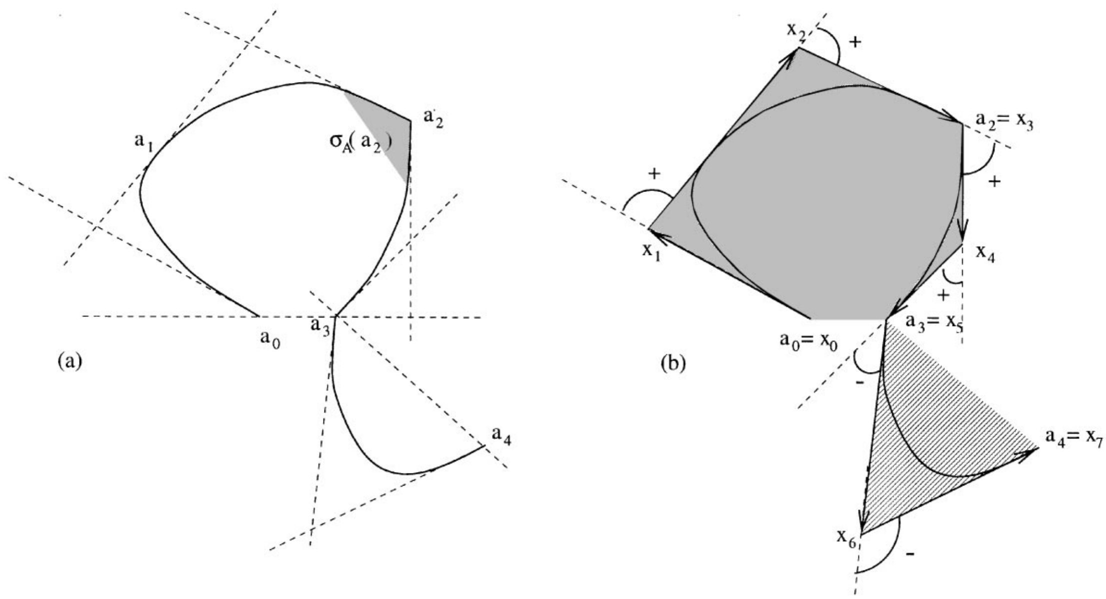

Fig. 13. A polygonal arc associated with a tame arc.

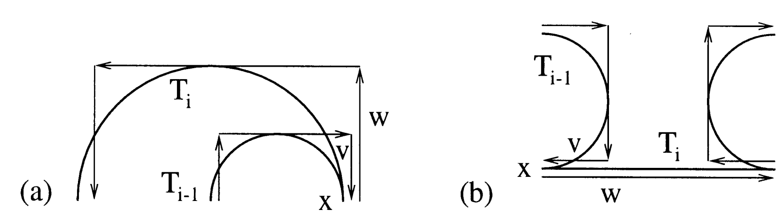

Fig. 14. For illustration purposes, the associated polygonal arcs are slightly translated.

joint a_3. $P(T_1)$ has vertices $x_0, \ldots, x_5$, and $P(T_2)$ has vertices x_5, x_6, x_7. The two polygonal arcs $P(T_1)$ and $P(T_2)$ have a consistent order in the sense that we can traverse them from x_0 to x_7 by following their vectors. The resulting polygonal arc associated with T is $P(T) = P(T_1) \circ P(T_2)$.

Definition. We define the absolute turn of the tame arc T as

$$\tau(T) = |\tau(P(T))|,$$

where $P(T)$ is a polygonal arc associated with T.

This definition is easy to apply if the right supporting ray of T_{i-1} and the left supporting ray of T_i do not coincide at their common endpoint x. When the rays are collinear, however, as shown in Figs 14(a) and (b), then vectors v and w of $P(T)$ that are contained in these rays and that have x as their common vertex are collinear and point in opposite directions. In this case, neither of the rules shown in Fig. 7 applies to x as a vertex of $P(T)$. It is clear that the absolute value of the turn $\tau(x)$ with respect to $P(T)$ is 180°, but it is not immediately clear how to determine the sign of $\tau(x)$.

We shall now define the sign of $\tau(x)$ with respect to $P(T)$. Let v be the last vector of $P(T_{i-1})$ and w the first vector of $P(T_i)$. Clearly, $x \in v \cap w$ and $\tau(x)$ [in $P(T)$] is the angle between v and w.

At least one of the arcs T_{i-1} and T_i is not a line segment, since otherwise T would not be a simple arc (we would have $v = T_{i-1}$, $w = T_i$, and v and w would coincide near x).

If one of T_{i-1} and T_i is a line segment, say T_{i-1}, then $\tau(x)$ is defined to have the opposite sign to the sign of $\tau(T_i)$ [see Fig. 15(a)].

Suppose now that both T_{i-1} and T_i are not line segments. Let l be the straight line containing v and w. Since l contains the right supporting ray of T_{i-1} and the left supporting ray of T_i, both T_{i-1} and T_i are contained in closed half-planes determined by l. If T_{i-1} and T_i are contained in two different closed half-planes of l, then T_{i-1} and T_i must have the same sign of turn [see Fig. 15(b)]. In this case, $\tau(x)$ is defined to have the opposite sign to the sign of $\tau(T_i)$.

It remains only to consider the case in which T_{i-1} and T_i are contained in the same closed half-plane of l. In this case $\tau(T_{i-1})$ and $\tau(T_i)$ have opposite signs. If the convex hull of T_{i-1} contains a subarc of T_i beginning at x, then $\tau(x)$ is defined to have the same

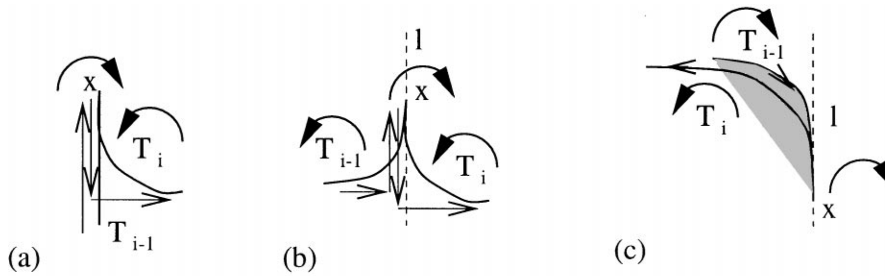

Fig. 15. Defining the sign of $\tau(x)$ in the collinear case.

Fig. 16. A regular point (b) and three inflection points (a), (c), and (d).

sign as the sign of $\tau(T_{i-1})$; otherwise, $\tau(x)$ is defined to have the same sign as $\tau(T_i)$ [see Fig. 15(c)].

Proposition 11. The turn $\tau(T)$ of a tame arc T is uniquely defined, i.e. $\tau(T)$ does not depend on the subdivision of T into supported subarcs T_i.

Proof. Since each inflection point of a tame arc T is necessarily a joint of T, it is sufficient to show that the turn of a tame arc T that does not contain inflections is uniquely defined. We prove this by induction on the minimal number of optional joints of T. If the minimal number is zero, then T is supported, and the uniqueness of $\tau(T)$ follows from the fact that the turn of a supported arc is uniquely defined. We now assume that the turn is uniquely defined for any tame arc whose minimal number of optional joints is at most $n-1$.

Let $T:[a, b] \to \mathbb{R}^2$ be a tame arc with the minimal number of optional joints n. Let $t_1, \ldots, t_n$ be all the joints of T. Let $s_1, \ldots, s_m$ be any different set of joints of T. There exists an i such that the subarc $T_i = T[t_i, t_{i+1})$ contains some of the joints $s_1, \ldots, s_m$, say $s_k, \ldots, s_l$. Since T_i is supported, the turn of T_i determined with respect to joints $s_k, \ldots, s_l$ is equal to $\tau(T_i)$. By the induction assumption, the turn of $T[a, t_i)$ determined with respect to $s_1, \ldots, s_{k-1}$ is equal to the turn of $T[a, t_i)$ determined with respect to $t_1, \ldots, t_{i-1}$, and the same applies to the turn of $T[t_{i+1}, b)$. Consequently, the turn $\tau(T)$ is uniquely defined. □

7. CLASSIFICATION OF POINTS OF A TAME ARC

In this section we use total turn concepts to classify inflection points and cusps of a tame arc T. When we classify a given point $x \in T$, we can always assume that x is a joint of T and is a vertex of an associated polygonal arc $P(T)$. We will characterize x using the turn angle $\tau(x)$ in $P(T)$ and the signs of the turns of the supported subarcs T_{i-1} and T_i (determined with respect to $P(T)$) such that x is the endpoint of T_{i-1} and the beginning point of T_i.

We recall that x is a cusp if the union of the left supporting ray of T_{i-1} at x, $lr_{T_{i-1}}(x)$, and the right supporting ray of T_i at x, $rr_{T_i}(x)$, is not a straight line, i.e. either $lr_{T_{i-1}}(x)$ and $rr_{T_i}(x)$ are not collinear or $lr_{T_{i-1}}(x) = rr_{T_i}(x)$. We will show that $x \in T$ is a cusp of a tame arc T iff $\tau(x) \neq 0$. Then we will show that $x \in T$ is a regular point of a tame arc T iff $\tau(T_{i-1})$, $\tau(T_i)$, and $\tau(x)$ [in $P(T)$] have the same sign. For example, this is the case for x in Fig. 16(b), while x in Figs. 16(a), (c), or (d) is an inflection point.

Theorem 5. $x \in T$ is a cusp of a tame arc T iff $\tau(x) \neq 0$.

Proof. Let $P(T)$ be a polygonal arc associated with T. Let v be the vector of $P(T)$ whose endpoint is x, and let w be the vector of $P(T)$ whose beginning point is x. Then clearly $x \in v \cap w$ and $\tau(x)$ (in $P(T)$) is the angle between v and w. The theorem follows from the fact that v is contained in the left supporting ray at x and w is contained in the right supporting ray at x. □

Theorem 6. $x \in T$ is a regular point of a tame arc T iff there exist supported subarcs T_{i-1} and T_i of T such that x is the endpoint of T_{i-1} and the beginning point of T_i, and $\tau(T_{i-1})$, $\tau(T_i)$, and $\tau(x)$ [in $P(T)$] have the same sign* [see Fig. 16(b)].

* Here we consider the sign of a turn of 0° as both positive and negative.

Proof. $\Rightarrow$: There exists a supported subarc $A \subseteq T$ such that x is interior to A. Let T_{i-1} and T_i be two subarcs of A such that x is the endpoint of T_{i-1} and the beginning point of T_i and $A = T_{i-1} \cup T_i$. Since T_{i-1} and T_i are supported (non-degenerate) subarcs of A, $\tau(T_{i-1})$, $\tau(T_i)$, and $\tau(x)$ have the same sign as $\tau(A)$.

$\Leftarrow$: If $|\tau(x)| = 180°$, it follows from the definition of $\tau(x)$ in Section 6.3 that $\tau(T_{i-1})$, $\tau(T_i)$, and $\tau(x)$ cannot have the same sign. Therefore, we can assume that $|\tau(x)| < 180°$.

By Proposition 10, there exists a subarc T'_{i-1} of T_{i-1} containing x whose absolute turn is arbitrarily small, and the same holds for T_i. Therefore, we can assume that $|\tau(T_{i-1}) + \tau(T_i) + \tau(x)| < 180°$.

We show that the arc $R = T_{i-1} \circ T_i$ is supported, i.e. that there is a supporting line of R at any point y of R. Without loss of generality, we can assume that $y \in T_{i-1}$.

Let $P(T_{i-1})$ and $P(T_i)$ be polygonal arcs associated with T_{i-1} and T_i such that $y \in P(T_{i-1})$. Then $P(R) = P(T_{i-1}) \circ P(T_i)$ is associated with R. We know that $|\tau(P(R))| < 180°$ and that $P(R)$ turns in one direction.Therefore, $P(R)$ is a convex (i.e. supported) polygonal arc.

Let *conv* denote the convex hull operator. From Proposition 9 it follows that $T_{i-1} \subseteq conv(P(T_{i-1}))$ and $T_i \subseteq conv(P(T_i))$. Since $conv(P(T_{i-1})) \subseteq conv(P(R))$ and $conv(P(T_i)) \subseteq conv(P(R))$, we obtain $R = (T_{i-1} \circ T_i) \subseteq conv(P(R))$. Since $y \in P(R)$, there is a supporting line of $conv(P(R))$ at y, and therefore there is a supporting line of R at y. □

8. DIGITAL SUPPORTEDNESS

In this section we consider supported digital arcs. Every digital arc is tame, since it contains a finite number of points, and therefore can be decomposed into a finite number of supported digital arcs (e.g. straight line segments). The total turn of a digital arc can therefore be defined in the analogous way to our continuous definition for tame arcs. This definition differs from the standard methods of estimating the curvature of a digital arc based on local neighborhoods. The classification of points of an arc in Theorems 5 and 6 can also be applied to shape analysis of digital arcs.

Let $\mathbb{Z}^2$ be the set of points with integer coordinates in the plane $\mathbb{R}^2$. Any finite subset $S \subseteq \mathbb{Z}^2$ will be called a *digital set*. The 4-*boundary* $bd_4 A$ of a digital set A is the set of points of A which have at least one 4-neighbor not in A, i.e. $bd_4 A = \{a \in A: \mathcal{N}_4(a) \cap A^c = \emptyset\}$, where A^c denotes the complement of A in $\mathbb{Z}^2$, and where, for any point $(x, y) \in \mathbb{Z}^2$, $\mathcal{N}_4((x, y)) = \{(x, y), (x+1, y), (x-1, y), (x, y+1), (x, y-1)\}$.

The *subset digitization* of a planar set X is defined as the set of points with integer coordinates that are contained in X:

$$SD(X) = \{s \in \mathbb{Z}^2 : s \in X\}.$$

The *object boundary quantization* of the boundary of a planar set is defined by

$$D_{OBQ}(bdX) = bd_4\, SD(X),$$

where bdX is the standard topological boundary of X.

Definition. A set $P \subseteq \mathbb{Z}^2$ is called a *digital half-plane* if there exists a real closed half-plane $HP \subseteq \mathbb{R}^2$ such that $SD(HP) = P$. A set $L \subseteq Z^2$ is called a *digital straight line* if $L = bd_4 P$ for some digital half-plane P.

$L \subseteq \mathbb{Z}^2$ is a digital straight line if there exists a real straight line M such that $L = D_{OBQ}(M)$. The black points in Fig. 17(a) represent a digital half-plane, since they are obtained by the subset digitization of the gray half-plane. The black points in Fig. 17(b) represent a digital line, since they are obtained by the digitization D_{OBQ} of the straight line which is the boundary of the gray half-plane.

We can restate the definition of a (continuous) supporting half-plane given in Section 2 in the following way:

Let S be a subset of the plane, and p a point of S. A closed half-plane P is a *supporting half plane* of S at p if $S \subseteq P$ and $p \in bdP$, where bdP denotes the boundary of P (i.e. its boundary line).

We use exactly this definition to define a supporting half plane in the digital case:

Definition. Let S be a subset of $\mathbb{Z}^2$ and p a point of S. A digital half-plane P is called a (*digital*) *supporting half-plane* of S at p if $S \subseteq P$ and $p \in bd_4 P$. In this case, point p belongs to the digital line $L = bd_4 P$. The line L is called a (*digital*) *supporting line* of S at p.

Thus if P is a (digital) supporting half plane, there exists a real half-plane $H(p)$ such that $S \subseteq H(p)$ and

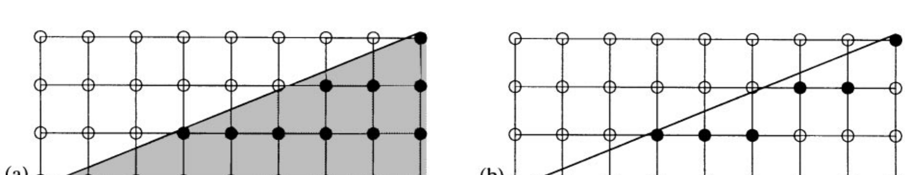

Fig. 17. The black points represent a *digital half-plane* in (a) and a *digital line* in (b).

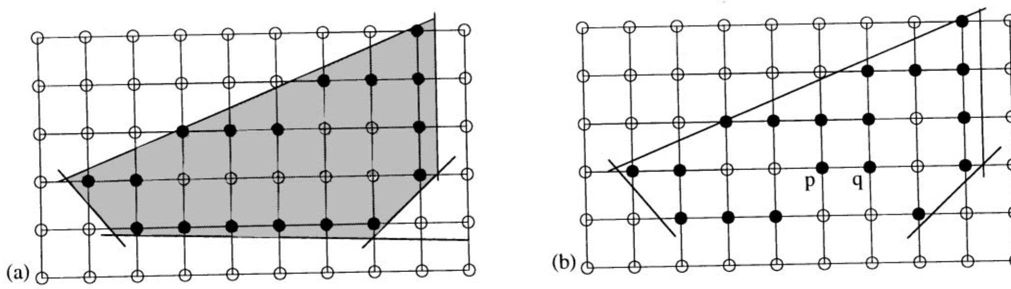

Fig. 18. The set of black points is *digitally supported* in (a) but not in (b).

$p \in bd_4 P$, where $P = SD(H(p))$. Similarly, we can directly transform the definition of a supported set to the digital domain:

Definition: A subset S of $\mathbb{Z}^2$ is (*digitally*) *supported* if, for every $p \in S$, there exists at least one digital supporting half-plane of S at p.

For example, the set of black points in Fig. 18(a) is digitally supported, but the set of black points in Fig. 18(b) is not, since there do not exist supporting half-planes at points p and q.

The following theorem allows us to give a new definition of a *convex digital set* as a digital set whose 4-boundary is digitally supported [e.g. see Fig. 18(a)].

Theorem 7. A finite set $S \subseteq \mathbb{Z}^2$ is digitally supported iff there exists a compact and convex set $B \subseteq \mathbb{R}^2$ with nonempty interior such that $S \subseteq bd_4 SD(B)$.

Proof. $\Rightarrow$: Let $S \subseteq \mathbb{Z}^2$ be digitally supported. For every $p \in S$, there exists a real half-plane $H(p)$ such that $S \subseteq SD(H(p))$ and $p \in bd_4 SD(H(p))$. Note that $S \subseteq SD(H(p))$ iff $S \subseteq H(p)$.

Let $B = \bigcap \{H(p) : p \in S\}$. B is a closed and convex set as a finite intersection of real half-planes, and $S \subseteq B$. Since S is finite, we can assume that B is bounded. (If this is not the case, we can always find a convex, closed, bounded subset of B that contains S.)

If the interior of B is empty, then B is a line segment, since a bounded convex set with empty interior is a line segment. In this case, we can replace B by a compact and convex set with nonempty interior (e.g. a rectangle) that contains exactly the same points of $\mathbb{Z}^2$ as B. Therefore, we can assume that the interior of B is nonempty.

Thus B is a compact and convex set with nonempty interior such that $S \subseteq B$. Consequently, $S \subseteq SD(B)$. Since for every $p \in S$, $p \in bd_4 SD(H(p))$, there exists $q \in \mathbb{Z}^2$, a 4-neighbor of p, such that $q \notin SD(H(p))$, and therefore $q \notin SD(B)$. We obtain $p \in bd_4 SD(B)$ for every $p \in S$. Hence $S \subseteq bd_4 SD(B)$.

$\Leftarrow$: Let $S \subseteq bd_4 SD(B)$, where B is a compact and convex set with nonempty interior. It is sufficient to show that $D_{OBQ}(bdB) = bd_4 SD(B)$ is digitally supported. If $SD(B)$ is the empty set, then $bd_4 SD(B)$ is trivially digitally supported. Therefore, we assume that $SD(B)$ is non-empty.

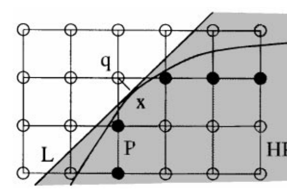

Fig. 19. *HP* is a (continuous) supporting half-plane at *x*.

Let $p \in bd_4 SD(B)$ be any point. Then there exists a 4-neighbor $q \in \mathbb{Z}^2$ of p such that $q \notin SD(B)$, and therefore $q \notin B$.

Let x be a closest point to q in B. Let L be the straight line through x perpendicular to line segment xq, and let HP be the real closed half-plane of L that does not contain q (see Fig. 19).

Since no point of B is closer to q than x and B is convex, B is contained in HP. Thus, HP is a supporting half-plane at x of B.

Since $p \in HP$ and $q \notin HP$, we have $p \in bd_4 SD(HP)$. We also have $SD(B) \subseteq SD(HP)$, and consequently $bd_4 SD(B) \subseteq SD(HP)$. Thus, $SD(HP)$ is a digital supporting half-plane of $bd_4 SD(B)$ at p. □

We recall that C^* denotes the closed bounded set surrounded by a simple closed curve C in the plane. In particular, we have $C = bdC^*$. We define $D_{OBQ}(C) = bd_4 SD(C^*)$.

Corollary 1. A finite set $S \subseteq \mathbb{Z}^2$ is digitally supported iff there exists a supported simple closed curve $C \subseteq \mathbb{R}^2$ such that $S \subseteq D_{OBQ}(C)$.

Proof. $\Rightarrow$: By Theorem 7, there exists a compact and convex set $B \subseteq \mathbb{R}^2$ with nonempty interior such that $S \subseteq bd_4 SD(B)$. Since the boundary of a bounded convex set with non-empty interior is a simple closed curve [e.g., Theorem 32 of reference (8)], we see that $C = bdB$ is a supported simple closed curve. Since

$C^* = B$, and consequently $D_{OBQ}(C) = bd_4 SD(B)$, we obtain $S \subseteq D_{OBQ}(C)$.

$\Leftarrow$: This is a special case of Theorem 7. □

By Corollary 1, supported digital sets correspond to supported continuous sets. Thus, we can extend our theory of tame arcs to digital arcs.

9. CONCLUSIONS AND EXTENSIONS

The results of this paper can be summarized as follows:

An arc (not necessarily simple) is called *tame* if it is the concatenation of a finite set of supported (simple) arcs; for example, a polygonal arc is tame. For such arcs we have given simple definitions of significant points that extend the classical definitions in differential geometry. For example, a non-endpoint of a tame arc which is not interior to any supported subarc is called an *inflection.* If a tame arc is differentiable, its curvature must change sign at an infection. Since a tame arc need not be differentiable at an inflection, our definition extends the corresponding definition in differential geometry. We have also shown that a tame arc can have only finitely many inflections.

Although we do not use the standard tools of differential geometry that are based on limits, we are able to compute the total curvature [which we call the (total) turn] of a tame arc, and we are able to show that the total absolute turn of a tame arc must be finite.

We have also extended our theory of tame arcs to digital arcs. Since every digital arc contains a finite number of points, it is tame. We can therefore use our definition of total turn to calculate the total curvature of a digital arc, by decomposing the arc into convex subarcs and calculating the total curvature of each subarc. The total curvature calculated in this way may be more reliable than that calculated using local curvature operators in digital image analysis, since we use a global definition of total curvature which is not based on local curvature operators. Our classification of significant points of a tame arc can also be applied to shape analysis of digital arcs.

The ideas in this paper can be further extended in several ways; we plan to pursue these extensions in future papers. In particular, we plan to extend our results to three-dimensional (either Euclidean or digital) space. We can define a *supporting plane* to a set S at a point p as a plane P through p such that S lies in one of the closed half-spaces bounded by P. It is not hard to see that a set which has a supporting half-plane at every point must be contained in the boundary of its convex hull. There are still many possibilities for such a set; even if we require it to be closed, bounded, and connected, it can be arc-like, surface-patch-like, or a combination. It would be of considerable interest to develop a theory of supported sets and digital supported sets in three dimensions, analogous to the planar theory developed in this paper.

APPENDIX

Here we give precise definitions of the left and right supporting rays (see Section 3).

Let $A : [a, b] \to \mathbb{R}^2$ be an arc. Let $p = A(x)$ for some point $x \in (a, b)$. Let $r < \min\{d(A(x), A(a)), d(A(x), A(b))\}$, where d is Euclidean distance in the plane. Then the circle $C(p, r)$ with center p and radius r intersects arc A in at least two points [see Fig. 20(a)]. This follows from the fact that a circle is a Jordan curve and p is inside the bounded region enclosed by $C(p, r)$ while the arc endpoints $A(a)$ and $A(b)$ are outside of this region. Let $x_- \in (a, x)$ be a point such that $A(x_-) \in C(p, r) \cap A$ and $A((x_-, x)) \cap C(p, r) = \emptyset$ (i.e. $x_- = \sup\{y \in (a, x)\colon A(y) \in C(p, r) \cap A\}$; since the set $C(p, r) \cap A$ is compact, we have $A(x_-) \in C(p, r) \cap A\}$). Similarly, let $x_+ \in (x, b)$ be a point such that $A(x_+) \in C(p, r) \cap A$ and $A(x, x_+) \cap C(p, r) = \emptyset$.

Since $A([x_-, x_+])$ is a subarc of A, it is contained in the sector $\sigma_A(p)$. In particular, the points $A(x_-)$ and $A(x_+)$ lie on $C(p, r) \cap \sigma_A(p)$.

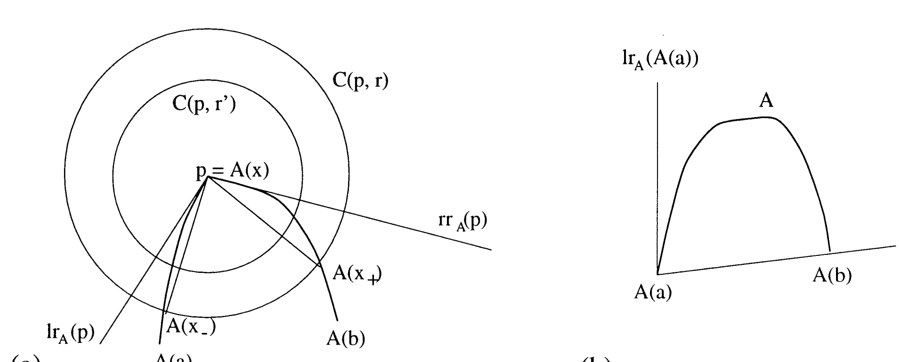

Fig. 20. Defining the left and right supporting rays.

The ray bounding $\sigma_A(p)$ that can be reached from point $A(x_-)$ while traversing $C(p,r) \cap \sigma_A(p)$ without going through $A(x_+)$ will be denoted by $lr_A(p)$ and called the *left supporting ray* of A at p [see Fig. 20 (a)]. Similarly, the ray bounding $\sigma_A(p)$ that can be reached from point $A(x_+)$ while traversing $C(p,r) \cap \sigma_A(p)$ without going through $A(x_-)$ will be denoted by $rr_A(p)$ and called the *right supporting ray* of A at p.

We next show that if A is supported, $lr_A(p)$ and $rr_A(p)$ do not depend on the radius of the circle $C(p,r)$. To see this, note that the subarc $A([x_-, x])$ is contained in the sector determined by the ray $lr_A(p)$ and the line segment $A(x)\,A(x_-)$ [see Fig. 20(a)], since the interior of triangle $A(x_-)\,A(x)\,A(x_+)$ cannot contain any points of arc A (Proposition 15, below). If $r' < r$, the point $A(x'_-)$ determined with respect to circle $C(p,r')$ must thus be contained in this sector; therefore, the ray $lr_A(p)$ can be reached from $A(x'_-)$ along $C(p,r')$ without going through point $A(x'_+)$. A similar argument applies for $rr_A(p)$.

It remains to define the left and right supporting rays at the endpoints of arc A. This can be done even if A is equal to the line segment $A(a)\,A(b)$ (where $a \neq b$); the angle $\alpha_A(p)$ is 0° if $p = A(a)$ or $A(b)$, and 180° otherwise. In the latter case, $\sigma_A(p)$ cuts off a semicircle on $C(p,r)$, and the subarcs $A(x)\,A(x_-)$, $A(x)\,A(x+)$ coincide respectively with rays $lr_A(p)$, $rr_A(p)$. In the former cases, $\sigma_A(p)$ is a ray, and it coincides with $lr_A(p) = rr_A(p)$ if $p = A(a)$ or $p = A(b)$.

Now suppose that A is different from line segment $A(a)\,A(b)$. As we see in Fig. 20(b), at $A(a)$ one of the bounding rays of $\sigma_A(S(a))$ is just the line segment $A(a)$ $A(b)$ by Proposition 4. We define this ray to be $lr_A(A(a))$, and the other bounding ray of the sector $\sigma_A(A(a))$ to be $rr_A(A(a))$; and vice versa at $A(b)$. The foregoing discussion gives us

Proposition 12. Let $A: [a,b] \rightarrow \mathbb{R}^2$ be a supported arc. Let A' be the subarc $A([p,c])$, where $a \leq p < c \leq b$. The subarc A' is contained in the sector defined by the right supporting ray $rr_A(A(p))$ and the line segment $A(p)\,A(c)$ (see Fig. 21). The analogous statement holds for the left supporting ray.

As a consequence of Proposition 12 we have:

Proposition 13. Let A and A' be as in Proposition 12. Then $rr_A(A(p)) \neq rr_{A'}(A(p))$. The analogous statement holds for the left supporting ray.

Proof. Clearly, $A(p) = A'(p)$. Since A' is a subarc of A, supporting sector $\sigma_{A'}(A(p))$ of A' is contained in $\sigma_A(A(p))$. Suppose $rr_A(A(p)) \neq rr_{A'}(A(p))$; then there would be a point $A(t)$ of A in $\sigma_A(A(p)) \backslash \sigma_{A'}(A(p))$ which lies between $rr_A(A(p))$ and $rr_{A'}(A(p))$. Consider the triangle $A(p)\,A(c)\,A(t)$ (see Fig. 22). Since A is supported, no point of A can lie in the interior of this triangle (Proposition 15, below). But by Proposition 12, $A([p,c])$ is contained in the angular sector spanned by line segment $A(p)\,A(c)$ and $rr_A(A(p))$; hence there exist parts of A in the interior of the triangle, contradiction. □

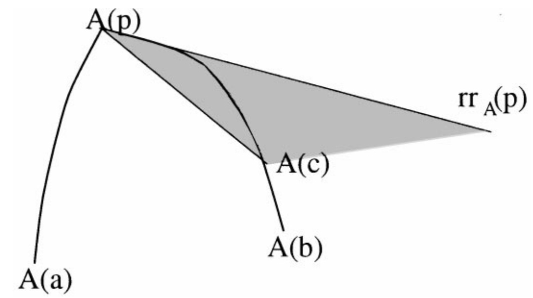

Fig. 21. The subarc is contained in the sector.

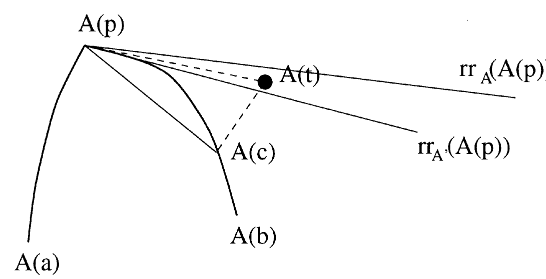

Fig. 22. The subarc has the same supporting ray as the arc.

As a simple consequence of Proposition 13, we obtain:

Proposition 14. Let A be a supported arc and A' a proper subarc of A. If p is not an endpoint of A', then $\sigma_{A'}(p) = \sigma_A(p)$. If p is an endpoint of A, then $\sigma_{A'}(p)$ is a proper subset of $\sigma_A(p)$.

The following simple but general characterization of supporting lines was used in proving the above results.

Proposition 15. A planar set S has a supporting line at $p \in S$ iff there do not exist three points $q, r, s \in S$ such that p lies in the interior of the triangle spanned by q, r, and s.

Proof. $\Rightarrow$: If p is in the interior of such a triangle, any line l through p must intersect an interior point of at least one side if the triangle, so that the endpoints of that side cannot lie in the same closed half-plane defined by l [see Fig. 23(a)].

$\Leftarrow$: Conversely, the set of rays joining p to all the other points of S is contained in some closed angular sector with vertex p, possibly with vertex angle 360°. The intersection of all such sectors is also a closed angular sector with vertex p, say with vertex angle α.

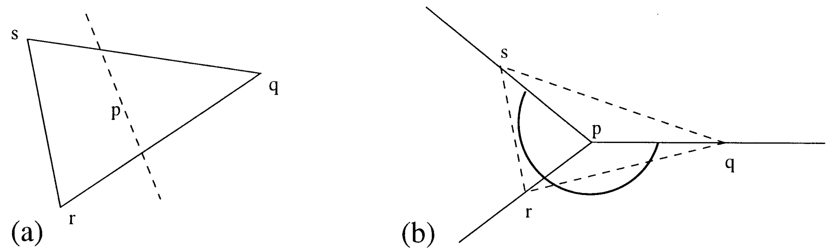

Fig. 23. A criterion for the existence of a supporting line at P.

If $\alpha \leq 180°$, there is a supporting line of S at p. If $\alpha > 180°$ [see Fig. 23(b)], then there exist three points $q, r, s \in S$ such that p is in the interior of the triangle spanned by q, r, and s. □

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