Pergamon

0031-3203(94)00168-5

GENERALIZED CONVEXITY: CP_3 AND BOUNDARIES OF CONVEX SETS

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Abstract—A set S is *convex* if for every pair of points $P, Q \in S$, the line segment PQ is contained in S. This definition can be generalized in various ways. One class of generalizations makes use of k-tuples, rather than pairs, of points—for example, Valentine's property P_3: For every triple of points P, Q, R of S, at least one of the line segments PQ, QR, or RP is contained in S. It can be shown that if a set has property P_3, it is a union of at most three convex sets. In this paper we study a property closely related to, but weaker than, P_3. We say that S has *property* CP_3 ("collinear P_3") if P_3 holds for all *collinear* triples of points of S. We prove that a closed curve is the boundary of a convex set, and a simple arc is part of the boundary of a convex set, iff they have property CP_3. This result appears to be the first simple characterization of the boundaries of convex sets; it solves a problem studied over 30 years ago by Menger and Valentine.

Convexity　　Convex arcs　　Convex curves　　Generalized convexity　　Boundaries

1. INTRODUCTION

A set S is *convex* if for every pair of points $P, Q \in S$, the line segment PQ is contained in S. This definition can be generalized in various ways. For example,[1] S is called *starshaped from* $P_0 \in S$ if $P_0 Q$ is contained in S for all $Q \in S$; thus S is convex if it is starshaped from all of its points. As another example,[2] S is called *ortho-convex* if PQ is contained is S for all $P, Q \in S$ such that PQ is horizontal or vertical.

One class of generalizations of convexity, due to Valentine,[3] makes use of triples (or k-tuples), rather than pairs, of points. A set satisfies Valentine's property P_3 if for every triple of points P, Q, R of S, at least one of the line segments PQ, QR, or RP is contained in S. For example, a polygonal arc consisting of two non-collinear line segments (Fig. 1) is not convex, but is easily seen to have property P_3. (Note that the three-segment polygonal arc in Fig. 1b does not even have property P_3.) It can be shown that if a set has property P_3, it is a union of at most three convex sets.

In this paper we study a property closely related to, but weaker than, P_3. We say that S has *property* CP_3 ("collinear P_3") if P_3 holds for all *collinear* triples of points of S. For example, the three-segment arc in Fig. 1b has property CP_3. This property turns out to characterize (parts of) the boundaries of convex sets; in fact, we shall prove in this paper that a closed curve is the boundary of a convex set, and a simple arc is part of the boundary of a convex set (e.g. Fig. 1b), iff they have property CP_3. This result appears to be the first simple characterization of parts of the boundaries of convex sets; it solves a problem studied over 30 years ago by Menger[4] and Valentine[5] (For arcs and closed curves, convexity is a very strong property; in fact, a closed curve or a nonsimple arc cannot be convex, and a simple arc is convex iff it is a straight line segment. The weaker property CP_3, on the other hand, will be shown in this paper to define very useful classes of arcs and curves.)

In Section 2 we describe the partial characterizations of boundaries of convex sets given by Menger and by Valentine. In Section 3 we define property CP_3. In Section 4 we prove the main theorems of the paper: A simple closed curve has property CP_3 iff it is the boundary of a convex set, and an arc has property CP_3 iff it is a connected subset of such a boundary. Finally, in Section 5 we establish some additional results about property CP_3: an arc has property CP_3 iff there is at least one supporting line (= line such that the arc lies on one side of it) through each of its points; and a path having property CP_3 is a simple closed curve, provided it does not have infinitely many multiple points. Section 6 briefly discusses the possibility of establishing analogous results for digital objects, and it also poses the problem of extending our results to three dimensions (i.e. characterizing surface patches which are subsets of the surfaces of convex sets).

For completeness, in the Appendix we summarize basic definitions and propositions about arcs, curves, and convex sets that are used in this paper. Using

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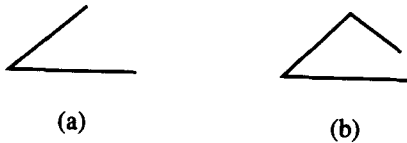
(a)   **(b)**

Fig. 1.

Fig. 3.

well-known characterizations of bounded convex sets,[6] we prove that the boundary of a bounded convex set (with nonempty interior) is a simple closed curve.

2. CHARACTERIZATIONS OF BOUNDARIES OF CONVEX SETS

Menger[4] gave a rather complicated characterization of the boundary of a convex set which was simplified by Valentine[5] (p. 106, T8.1) essentially as follows:

Let S be a compact set in the plane containing at least three points. Suppose that for each triple of non-collinear points x_i $(i = 1, 2, 3)$ of S we have

$$S \cap \text{int } \Delta = \varnothing$$
$$V_i \cap S = \varnothing \quad i = 1, 2, 3$$
$$W_{jk} \cap S \neq \varnothing \quad j, k = 1, 2, 3; \; j \neq k$$

where (see Fig. 2) Δ is the closed triangle determined by x_1, x_2, x_3; int Δ is the interior of Δ; V_i is the open V-shaped unbounded region abutting Δ at vertex x_i; and W_{ij} is an unbounded three-sided set abutting the edge $x_i x_j$. We define W_{ij} to contain the open line segment $x_i x_j$, and to be disjoint from the lines $x_i x_k$, $x_j x_k$ $(k \neq i, j)$, so that it is neither open nor closed. Also if x_1, x_2, x_3 are three distinct collinear points of S suppose that

$$S \cap \text{intv } x_i x_j \neq \varnothing \quad i, j = 1, 2, 3; \; i \neq j$$

where intv $x_i x_j$ is the interior of the interval $x_i x_j$. If all of these conditions are satisfied, S is the boundary of a convex set. The converse is also true for compact sets.

Valentine[5] (p. 108, T8.3) stated a condition slightly stronger than our property CP_3, and tried to relate it to the property of being a "convex curve", i.e. a (proper or improper) subset of the boundary of a convex set. Let S be a closed connected set in the plane. Suppose that for each triple of distinct collinear points in S, the
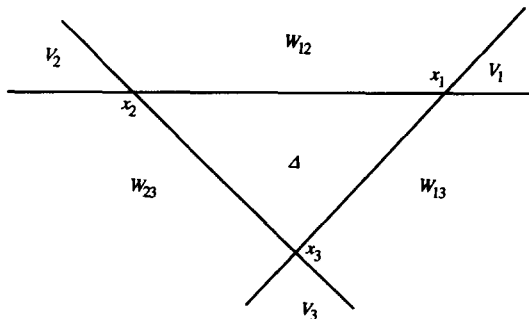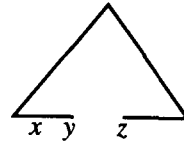
minimal line segment containing them belongs to S. Then the set S satisfies at least one of the following four statements: S is closed convex set; S is a convex curve; S is the union of two linear elements R_i $(i = 1, 2)$ with $R_1 \cap R_2 \neq \varnothing$, where a linear element is either a closed line segment, a closed half line (ray), or a line; S is the union of three linear elements R_1, R_2, R_3 having a common end point x such that $x \in \text{int conv } (R_1 \cup R_2 \cup R_3)$ [where int conv(A) means the interior of the convex hull of the set A]. (Hence, S is a kind of three-legged star.)

Note that Valentine's condition does not imply that S is a convex curve. Conversely, Fig. 3 (suggested by David Mount) shows a convex curve which does not fulfill Valentine's condition (consider the triple of collinear points x, y and z). In this paper, we will show that a slightly weaker property, which we call CP_3, does completely characterize convex curves.

3. CP_3-CONVEXITY

Definition

A set $S \subseteq R^2$ will be said to have property CP_3 if for every three collinear points in S, at least two of them are joined by a line segment contained in S.

The main result of this paper is that property CP_3 characterizes convex curves. We first need to establish some properties of CP_3-convex sets.

Definition

A set S will be said to have property C_3 if for each triple of collinear points in S, the minimal line segment containing them belongs to S.

Note that this is the condition given by Valentine (see Section 2). It is clear that property C_3 implies property CP_3.

Proposition 1

The boundary of a bounded convex set has property C_3.

Proof. Let S be the boundary of a bounded convex set C, and let L be any straight line. If L contains an interior point of C, then L intersects S in exactly two points (Theorem 30), so it cannot contain three collinear points of S. If L does not contain any interior point of C and $L \cap (C \cup S) \neq \varnothing$, then $L \cap (C \cup S) = L \cap S$. But $L \cap (C \cup S)$ is a convex subset of L, since $C \cup S$ is convex (Proposition 29). Hence $L \cap (C \cup S)$ is a line segment, so that if L contains three collinear points of S, the

Fig. 2.

minimal line segment containing them belongs to $L \cap S \subseteq S$, which proves that S has property C_3.

Corollary 2

The boundary of a bounded convex set has property CP_3

Lemma 3

Let S be an arc with endpoints a and b such that $S \neq ab$. Let $L(a, b)$ be the straight line passing through points a and b. If S has property CP_3, then $S \cap L(a, b)$ has exactly two connected components, one containing a and the other containing b, and when these components are deleted, S lies in one of the open half planes into which $L(a, b)$ divides R^2.

Proof. The assumption that $S \neq ab$ implies that ab cannot be contained in S; otherwise ab would be a proper subarc of S with the same endpoints, which is impossible (Proposition 25). Hence $S \cap L(a, b)$ has at least two connected components, since the connected components $C(a)$ and $C(b)$ containing a and b cannot be the same. On the other hand if $S \cap L(a, b)$ had a third component, S could not have property CP_3.

$C(a)$ and $C(b)$ are subarcs of S (Proposition 17), and so must be the images of initial and final subintervals $[f^{-1}(a), u]$ and $[v, f^{-1}(b)]$ of I, respectively, where $u < v$. Let x, y be distinct points of S that do not lie on $L(a, b)$, where (say) $f^{-1}(x) < f^{-1}(y)$; then we must have $u < f^{-1}(x) < f^{-1}(y) < v$. Let A be the subarc of S joining x and y; then $A = f([f^{-1}(x), f^{-1}(y)])$, i.e. A is the image of a subinterval of I that is disjoint from $[f^{-1}(a), u]$ and $[v, f^{-1}(b)]$. Thus A cannot intersect $L(a, b)$; but this means that x and y must lie in the same open half plane defined by $L(a, b)$.

Lemma 4

Let S be an arc or a simple closed curve. Let $x, z, y \in S$ be three different points, and let $L(x, z) = L$ be the straight line containing x and z [Fig. 4(a)]. Let the subarcs $\text{arc}(x, z)$ and $\text{arc}(z, y)$ of S be such that $\text{arc}(x, z) \cap \text{arc}(z, y) = \{z\}$ and $\text{arc}(x, z) \neq xz$. If there exists a point

$p \in \text{arc}(x, z)$ with nonzero distance to L such that p and y lie in one of the closed half planes into which L divides R^2, then S does not have property CP_3.

Proof. Let M be any straight line intersecting line segments xp, pz and zy but not passing through points x, z, p or y. Such a line exists, since x and z are different points of line L, p and y are two different points in one of the closed half planes into which L divides R^2, and p is at nonzero distance from L. (The cases in which y is also at a positive distance from L, and y lies on L, are illustrated in Fig. 4(b) and (c)). By Proposition 34, M intersects S in at least three points lying on the following subarcs of S: $\text{arc}(x, p)$, $\text{arc}(p, z)$, and $\text{arc}(z, y)$. One of these points, say q, lies between the other two on M. Let J and K be two straight lines different from M and from each other which pass through q and satisfy the same conditions as M [see Fig. 4(d)]; the lines J and K can evidently be obtained by slightly rotating line M around point q so that the rotated lines still intersect line segments xp and zy. Each of the lines M, J and K intersects S in at least two points different from q in such a way that q lies between these two points (on M, J and K, respectively).

We now have six rays emanating from q and intersecting S. By Proposition 21, initial segments of at most two of these rays can be contained in S. Therefore, for at least one of the three lines, neither of its two intersection points with S different from q can be joined with q by a line segment contained in S. This implies that S does not have property CP_3.

Proposition 5

Let S be an arc with endpoints x and y. If S has property CP_3 and $S \neq xy$, then $S \cap xy = \{x, y\}$.

Proof. Let $z \in S \cap xy$ be different from x and y. Since S has property CP_3, at least one of xz and zy, say zy, is contained in S. Then xz cannot be contained in S. Indeed if xz were contained in S, then xy would be contained in S, so that xy would be a proper subarc of S with the same endpoints as S, which is impossible (Proposition 25). Since S is arc-connected, there exists
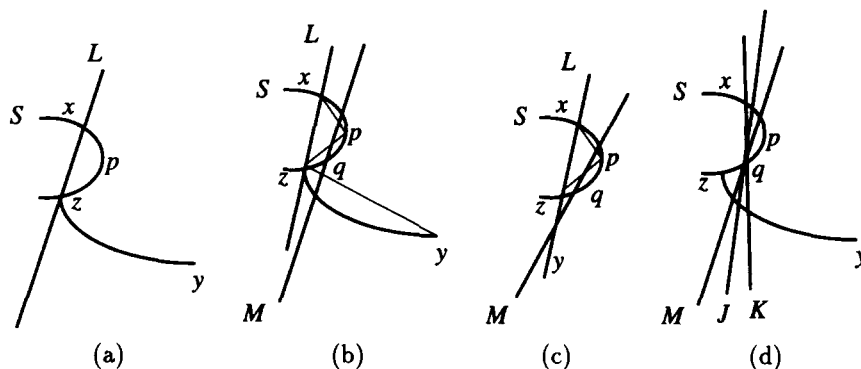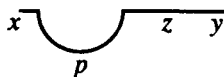

Fig. 4.

Fig. 5.

an arc $\text{arc}(x, z) \subseteq S$ joining x and z. Let L be the straight line containing x and z and let $p \in \text{arc}(x, z)$ be any point whose distance to L is greater than 0. Such a point exists, since $\text{arc}(x, z) \neq xz$ (see Fig. 5.).

Then the assumptions of Lemma 4 are fulfilled for x, z, y, and p: x and z lie on L; $\text{arc}(x, z)$ and $\text{arc}(z, y) = zy$ are subarcs of S such that $\text{arc}(x, z) \cap \text{arc}(z, y) = \{z\}$ (Proposition 23); $\text{arc}(x, z) \neq xz$; and points p and y lie in one of the closed half planes into which L divides R^2, since y lies on L. Hence by Lemma 4, S cannot have property CP_3, contradiction; it follows that $S \cap xy = \{x, y\}$.

The following corollary makes use of Proposition 26.

Corollary 6

Let S be an arc with endpoints x and y. If S has property CP_3 and $S \neq xy$, then $S \cup xy$ is a simple closed curve.

4. THE MAIN THEOREMS

In this section we prove the main theorems of this paper.

Theorem 7

A simple closed curve has property CP_3 iff it is the boundary of a convex set.

Theorem 8

An arc has property CP_3 iff it is a connected subset of the boundary of a convex set.

By Theorems 7 and 32, a set is a simple closed curve and has property CP_3 iff it is the boundary of a bounded convex set with nonempty interior. Similarly, a set is an arc and has property CP_3 iff it is a closed, connected subset of the boundary of a convex set.

Proof of Theorem 7. "$\Leftarrow$": This follows from the Corollary to Proposition 1. "$\Rightarrow$": Let S be a simple closed curve. By the Jordan Curve Theorem, S separates R^2 into exactly two components, one bounded and the other unbounded, and S is the boundary of each of these two components. Let C be the bounded components together with S. Then S is the boundary of C, and C is closed.

We will show that if C is not convex, then S does not have property CP_3. Let L be a straight line passing through an interior point of C and intersecting S (the boundary of C) in at least three distinct points, say x, z and y (Theorem 30). We can assume that z is between x and y on L, and that the interior point is between x and z. Therefore, xz cannot be contained in S. If zy is

also not contained in S, then S does not have property CP_3; so, it remains only to consider the case where zy is contained in S. Since $(S \backslash zy) \cup \{z, y\}$ is an arc containing z, y and x, there exists an arc joining x and z: $\text{arc}(x, z) \subseteq (S \backslash zy) \cup \{z, y\} \subseteq S$. Therefore, $\text{arc}(x, z) \cap zy = \{z\}$. Let $p \in \text{arc}(x, z)$ be any point with nonzero distance to L; such a point exists, since $\text{arc}(x, z) \neq xz$, because xz is not contained in S.

Hence the assumptions of Lemma 4 are satisfied for x, z, y, and p: x and z lie on L; $\text{arc}(x, z)$ and $\text{arc}(z, y) = zy$ are subarcs of S such that $\text{arc}(x, z) \cap \text{arc}(z, y) = \{z\}$; $\text{arc}(x, z) \neq xz$; and points p and y lie in one of the closed half planes into which L divides R^2, since y lies on L. Thus by Lemma 4, S does not have property CP_3.

Remark. Theorem 7 can also be proved along the same lines as the proof of Theorem 30 given in (Ref. 6 pp. 114–116, solutions 1–4 and 1–5).

From Theorem 30 we also have the following.

Corollary 9

A simple closed curve S has property CP_3 iff every straight line passing through an arbitrary interior point C (the set bounded by S) intersects S in exactly two points.

Corollary 10

A simple closed curve has property CP_3 iff it is C_3-convex.

Proof. This follows from Theorem 7 and from the Corollary to Proposition 1.

In order to prove Theorem 8, we first prove

Theorem 11

Let S be an arc with a and y as endpoints. If S has property CP_3, so has $S \cup ay$.

Proof. If $S = ay$, the theorem is trivially true; therefore we assume that $S \neq ay$. Note that in this case ay cannot be completely contained in S (Proposition 25). Let L be the line containing ay. By Lemma 3, S lies in one of the closed half planes into which L divides R^2.

Suppose $S \cup ay$ did not have property CP_3, and let M be a straight line intersecting $S \cup ay$ in three different points x, z and d in such a way that no line segment joining two of them is contained in $S \cup ay$. It is easy to see that two of these points must belong to $S \backslash ay$ and the third one to ay, say $x, z \in S \backslash ay$ and $d \in ay$. Furthermore, d cannot be between x and z on M, since then x and z could not lie in the same closed half plane defined by L [see Fig. 6(a)]. We will now show that this situation contradicts the assumption that S has property CP_3.

Since x and z lie on the same side of d on line M, the distances from x and z to d cannot be equal. Let x be farther from d than z. Since xz is not contained in S, the subarc $\text{arc}(x, z) \subseteq S$ joining x and z is not contained in M [see Fig. 6(a)]. Therefore, there exists a point $p \in$
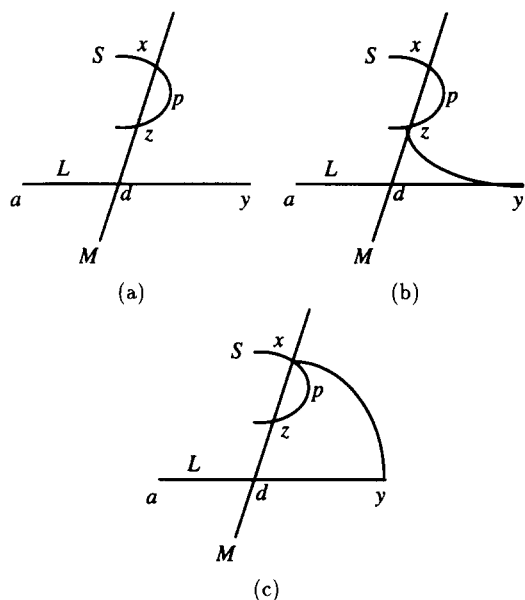
(a)   (b)

(c)

Fig. 6.

arc(x, z) with nonzero distance to M. Since d is on ay, either a or y must lie in the same closed half plane defined by M as point p does; suppose, as shown in Fig. 6(a), y and p lie in the same closed half plane. There exists a subarc arc$(z, y) \subseteq S$ joining z and y [see Fig. 6(b) and (c)]. By Proposition 24, either arc$(x, z) \subseteq$ arc(z, y) or arc$(x, z) \cap$ arc$(z, y) = \{z\}$. If arc$(x, z) \subseteq$ arc(z, y) [Fig. 6(c)], then the set arc$(x, y) = ($arc$(z, y)\backslash$arc$(z, x)) \cup \{x\}$ is evidently an arc joining x and y with the property arc$(x, y) \cap$ arc$(z, x) = \{x\}$. Therefore we know that there is an arc joining either x or z to y [Fig. 6(c) and (b), respectively] such that arc$(x, y) \cap$ arc$(z, x) = \{x\}$ or arc$(x, y) \cap$ arc$(z, x) = \{z\}$, respectively. In either case, the assumptions of Lemma 2 are fulfilled: Points x and z lie on a straight line M; arc(z, x) and arc(z, y) (or arc(x, y)) are subarcs of S such that arc$(z, y) \cap$ arc$(z, x) = \{z\}$ (or arc$(x, y) \cap$ arc$(z, x) = \{x\}$) and arc$(z, x) \neq zx$; $p \in$ arc(z, x) has nonzero distance to M; and p and y lie in one of the closed half planes defined by M. Hence by Lemma 4, S cannot have property CP_3, contradiction.

Proof of Theorem 8. "⇒": Let S be an arc with a and b as endpoints. If $S = ab$, the theorem is trivially true. If $S \neq ab$, then by Proposition 5, $S \cap ab = \{a, b\}$; hence $S \cup ab$ is a simple closed curve (Proposition 26). Since S has property CP_3, Theorem 11 implies that $S \cup ab$ also has property CP_3. Thus Theorem 7 implies that $S \cup ab$ is a boundary of a convex set. Therefore, S is a connected subset of the boundary of a convex set.

"⇐": Let S be the boundary of a convex set C. We prove this part of the theorem for every connected bounded proper subset of S, and therefore for every arc. Let T be a connected bounded proper subset of S. If the interior of C is empty, then T is a line segment, and the theorem is trivially true. If the interior of C is nonempty, then S is a simple closed curve (Theorem

32). Let a, b, $c \in T$ be three collinear points with b between a and c. By Proposition 1, S has property C_3; therefore the minimal line segment ac containing a, b, c belongs to S. Since T is a connected subset of a simple closed curve containing a, b, c, at least one of line segments ab and bc must be contained in T; indeed, if neither of them were contained in T, then T would not be connected (Proposition 27).

While proving Theorem 8, we also have proved.

Theorem 12

Let S be an arc with endpoints a and b. S has property CP_3 iff $S \cup ab$ is the boundary of a convex set.

Corollary 13

If a simple arc or curve S has property CP_3, so has any arc-connected subset of S.

Proof. This follows from Theorems 7 and 8 and Propositions 17 and 18.

5. SOME OTHER RESULTS ABOUT PROPERTY CP_3

5.1. Supporting lines

Theorem 14. An arc S has property CP_3 iff through each of its points there passes at least one supporting line.

Proof. "⇒": If S has property CP_3, by Theorem 8 it is part of the boundary of a convex set. Hence Theorem 31 implies that through each point of S there passes at least one supporting line.

"⇐": Let S be an arc such that through each of its points there passes at least one supporting line. Let S' be the intersection of all closed half planes containing S. Then S' is convex. Since through each point of S there passes at least one supporting line, every point of S is a boundary point of S'. Since S is an arc that is contained in the boundary of a convex set S', it follows from Theorem 8 that S has property CP_3.

5.2. Simplicity

Theorem 15. Let $f(I)$ be a path defined by $f : I \rightarrow R^2$ such that the preimage of the set of multiple points of f is a finite nonempty subset of I. If $f(I)$ has property CP_3, then $f(I)$ is a simple closed curve.

Note that when $f(I)$ is a simple closed curve, only the endpoints of I are mapped into a multiple point, which is the only such point; and that $f(I)$ is an arc if f has no multiple points.

Lemma 16. Let $f(I)$ be a path defined by $f : I \rightarrow R^2$, and suppose $f(x) = f(y)$ for some $x, y \in I$, where $x < y$. Let $f([x, y]) = S$, and let K be a line segment in R^2 which does not contain any multiple point of f. If K intersects S and is not contained in S, then K is not contained in $f(I)$.

Proof. Since K does not contain any multiple point of f and $f(x) = f(y)$ is a multiple point of f, we have $K \cap S \subseteq f((x, y))$, where (x, y) is an open interval. We show that the assumption $K \subseteq f(I)$ leads to inconsistency.

Let I be the unit interval $[0, 1]$. If $K \subseteq f[0, 1]$), then $K \backslash S \neq \varnothing$ and $K \backslash S \subseteq f([0, x] \cup [y, 1])$. Since $[0, x] \cup [y, 1]$ is a compact set, $f[0, x] \cup [y, 1]$) is also compact, and therefore closed. Hence $\mathrm{cl}(K \backslash S) \subseteq f([0, x] \cup [y, 1])$, where cl is the usual closure operator in R^2. Since S is closed (as an image of a compact set) and K is a line segment (and therefore closed), $K \backslash S$ is not closed. Therefore, there exists $p \in \mathrm{cl}(K \backslash S)$ such that $p \notin (K \backslash S)$.

Now $p \in \mathrm{cl}(K \backslash S)$ implies that $p \in f([0, x] \cup [y, 1])$. On the other hand $p \in K \cap S$, since $p \in \mathrm{cl}(K \backslash S) \subseteq K$ and $p \notin K \backslash S$. Hence $p \in f((x, y))$, because $K \cap S \subseteq f((x, y))$. Thus p is in the image (under f) of both (x, y) and its complement, and so is a multiple point; but $p \in K$, contradiction.

Proof of Theorem 15. Let $J \subseteq I$ be the preimage of the set of multiple points of f. Let $x, y \in J$, where $x < y$, be points of I such that $f(x) = f(y)$ and such that there exists no pair of points of J strictly between x and y with the same property, i.e. there do not exist $a, b \in J$, where $x < a < b < y$, such that $f(a) = f(b)$. Such points x, y must exist, because otherwise J would be infinite. The restriction $f|_{(x,y)}$ of f to the open interval (x, y) is an injection and f is continuous. Therefore, $S = f([x, y])$ is a simple closed curve. If x and y are the endpoints of I, we are done; hence we can assume that at least one of them is not an endpoint. We show that this assump-tion leads to inconsistency with property CP_3 of $f(I)$.

By the Jordan Curve Theorem, S separates R^2 into exactly two components, one bounded and the other unbounded, and S is the boundary of each of these components. Let C be the bounded component together with S; then S is the boundary of C. Since at least one of x and y is not an endpoint, there exists a point $z \in I \backslash [x, y]$ such that $z \notin J$.

Let L be a straight line passing through $f(z)$ and through an interior point of C, but not intersecting $f(J)$, i.e. L does not contain any multiple point of f [see Fig. 7(a), (b)]; such a line exists since J, hence $f(J)$, is finite. Then L intersects S in at least two distinct

points $f(u), f(v)$ such that the line segment $f(u)f(v)$ contains an interior point c of C. Therefore, $f(u)f(v)$ is not contained in S. By Lemma 16, $f(u)f(v)$ cannot be contained in $f(I)$.

Evidently $f(u)f(z)$ is not contained in S, since $f(z) \notin S$. Hence by Lemma 16, $f(u)f(z)$ cannot be contained in $f(I)$. In exactly the same way, we can show that $f(v)f(z)$ cannot be contained in $f(I)$. Since the line segment joining any two of the three collinear points $f(z), f(u)$ and $f(v)$ in $f(I)$ cannot be contained in $f(I)$, $f(I)$ does not have property CP_3. This contradiction proves the theorem.

6. CONCLUDING REMARKS

In this paper we have introduced a new generalization of convexity, and have shown that it characterizes arcs which are subsets of boundaries of convex sets in the plane. This is the first simple characterization of such arcs; it solves a problem first stated over 30 years ago. Incidentally, we could give somewhat shorter proofs of Theorems 7 and 8 by first showing (using the proof of Lemma 4) that if an arc or curve is CP_3-convex, it is contained in the boundary of its convex hull. However, if we took this approach, it would be harder to prove Theorem 12.

Our characterization of parts of the boundaries of convex sets could be used in (digital) image analysis to determine whether a region could be convex, given only an image of part of its boundary. Note, however, that we have not yet established analogs of our results for digital images; we plan to do so in a forthcoming paper. It should be pointed out that there exist classical characterizations of convex sets that do not hold in the digital case. For example, it is well known[7] that a set S is convex iff it is locally convex, i.e. every point of S has a neighborhood N such that $S \cap N$ is convex. This is not true for digital sets, as illustrated in Fig. 8; here every point of S intersects its 3×3 neighborhood in a digitally convex set, but evidently S is not digitally convex.

It would be of interest to find a similar characterization of surface patches which are subsets of the surfaces of convex sets in three dimensions; we plan to investigate this in a subsequent paper.
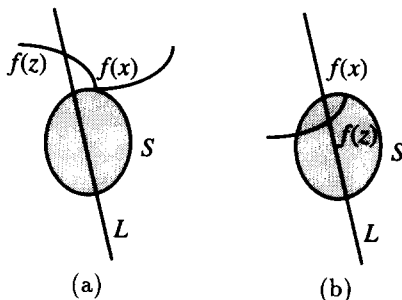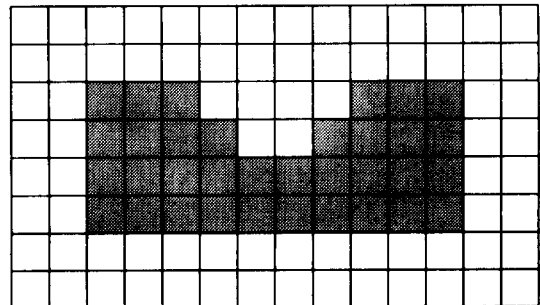

Fig. 7.

Fig. 8.

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APPENDIX A: BASIC DEFINITIONS AND PROPOSITIONS

Definition

A set $S \subseteq R^2$ is called a **(simple) arc** if it is a homeomorphic image of a closed interval. A set $S \subseteq R^2$ is called a **simple closed curve** (Jordan curve) if it is a homeomorphic image of a circle of nonzero radius. An arc is called *degenerate* if it consists of a single point.

Definition

A set $P \subseteq R^2$ is called a **path** if it is a continuous image of a closed interval I. A point p of a path P is a **multiple point** if p is the image of two distinct points of I. A point which is not a multiple point will be called a **simple point**.

For example, all points of an arc are simple points. Note also that a path P that contains only simple points is an arc, since then the function from I to P is also one-to-one, and a continuous one-to-one function on a closed interval is a homeomorphism onto its image.

Definition

Let P be an arc which is the image of the closed interval $[a, b] \subseteq R$ by the homeomorphism $f : [a, b] \to R^2$. The points $p = f(a)$ and $q = f(b)$ will be called the **endpoints** of P; P will be said to "join" p and q, and will sometimes be denoted by **arc(p, q)**.

Definition

A set S is **arc-connected** if for every pair of points $p, q \in S$, there is an arc joining p and q contained in S. A closed arc-connected subset of R^2 will be called a **figure**.

Definition

Let P be an arc which is the image of the closed interval $I \subseteq R$ by the homeomorphism $f : I \to R^2$, and let J be a closed subinterval of I. The image $f(J)$ will be called a **subarc** of P. Note that the restriction of f to J is a homeomorphism: $J \to R^2$; hence a subarc of an arc is an arc.

We now state some propositions which are useful in the proofs of the theorems in this paper. Most of them are stated without proof, because they are well-known, basic facts about arcs and closed curves.

Proposition 17

Every closed arc-connected subset of an arc is a subarc.

Proposition 18

Every closed arc-connected proper subset of a simple closed curve is an arc.

Proposition 19

A subset of an arc is connected iff it is arc-connected

Proposition 20

Three nondegenerate closed subarcs of an arc cannot pairwise intersect in a single point.

Proof. Let $S = f(I)$ be an arc, and let A, B, $C \subseteq S$ be nondegenerate closed subarcs which pairwise intersect in a single point. Then $f^{-1}(A), f^{-1}(B), f^{-1}(C)$ would be three nondegenerate closed subintervals of I which pairwise intersect in a single point, which is impossible.

Proposition 21

Let S be an arc, and let A, B, $C \subseteq S$ be three nondegenerate closed line segments. Then A, B, C cannot pairwise intersect in a single point.

Proof. This follows from Propositions 17 and 20.

Proposition 22

Let S be an arc which is the image of the closed interval $I \subseteq R$ by the homeomorphism $f : I \to R^2$. Then for every three points $a, b, c \in I$ we have $b \in [a, c]$ iff $f(b) \in f([a, c]) = \operatorname{arc}(f(a), f(c))$.

Note that by this proposition, the order of the points on S is the same as or the reverse of the order of the points on I.

Proposition 23

Let S be an arc which is the image of the closed interval $I \subseteq R$ by the homeomorphism $f : I \to R^2$. If $b \in \operatorname{arc}(a, c) \subseteq S$, then $\operatorname{arc}(a, b) \cap \operatorname{arc}(b, c) = \{b\}$.

Proof. By Proposition 22, $f^{-1}(b) \in [f^{-1}(a), f^{-1}(c)]$ (or its reversal). Hence the intervals $[f^{-1}(a), f^{-1}(b)]$ and $[f^{-1}(b), f^{-1}(c)]$ can have only $f^{-1}(b)$ in common. Applying f to both sides proves the proposition.

Proposition 24

Let S be an arc and let $\operatorname{arc}(a, b)$ and $\operatorname{arc}(a, c)$ be subarcs of S such that $b \notin \operatorname{arc}(a, c)$; then either $\operatorname{arc}(a, c) \subseteq \operatorname{arc}(a, b)$ or $\operatorname{arc}(a, c) \cap \operatorname{arc}(a, b) = \{a\}$ (see Fig. A1).

Proof. Let $S = f(I)$, where f is a homeomorphism. By Proposition 22, $f^{-1}(b)$ cannot lie between $f^{-1}(a)$ and $f^{-1}(c)$. If $f^{-1}(c)$ is between $f^{-1}(a)$ and $f^{-1}(b)$, then $\operatorname{arc}(a, c) \subseteq \operatorname{arc}(a, b)$. If $f^{-1}(a)$ is between $f^{-1}(b)$ and $f^{-1}(c)$, we have $\operatorname{arc}(a, c) \cap \operatorname{arc}(a, b) = \{a\}$ by Proposition 23.

Proposition 25

A proper subarc of an arc S cannot have the same endpoints as S.

Proposition 26

The union of two arcs which are disjoint except for one endpoint is an arc. If they are disjoint except for both endpoints, their union is a simple closed curve.
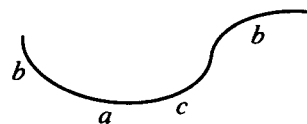

Fig. A1.

Proposition 27

Deleting one point (other than an endpoint) from an arc, or two points from a simple closed curve, disconnects it.

Definition

The line segment joining two points x and y will be denoted by **xy**.

Definition

A set $S \subseteq R^2$ is called **convex** if for every two points in S the line segment joining them is contained in S.

Evidently, a convex set is arc-connected. Note also that if S is convex, then for any three points in S the triangle determined by them (with its interior, if any) is contained in S. Thus if S contains three noncollinear points, it has a nonempty interior. This proves

Proposition 28

A convex set with empty interior must be a line, a ray, or a line segment.

Proposition 29

The closure of a convex set is convex.

Proof. Let A be a convex set whose closure cl$A$ is not convex. Then there exist two points $a, b \in$ cl$A$ such that the line segment ab is not contained in cl$A$. Therefore, there exists a point $c \in ab$ and $\varepsilon > 0$ such that $B(c, \varepsilon) \cap A = \varnothing$ (see Fig. A2), where $B(c, \varepsilon)$ denotes the ball (i.e. a disk) having center c and radius ε.

If we take $0 < \delta < \varepsilon$, then two points x and y exist such that $x \in B(a, \delta) \cap A$ and $y \in B(b, \delta) \cap A$, because $a, b \in$ cl$A$. The line segment xy cannot be contained in A, since $xy \cap B(c, \varepsilon) \neq \varnothing$ and $B(c, \varepsilon) \cap A = \varnothing$. Hence A cannot be convex.

In the book of Yaglom and Boltyanskii,[6] the following two characterizations of convex sets are given.

Theorem 30

A bounded figure in R^2 is convex iff every straight line passing through an arbitrary interior point of the figure intersects the boundary of the figure in exactly two points (see Ref. 6, p. 7).

Definition

A straight line L passing through a boundary point p of a set $S \subseteq R^2$ is called a **supporting line** of S at p if S is contained in one of the closed half-planes into which L divides R^2.

Theorem 31

A bounded figure in R^2 is convex iff through each of its boundary points there passes at least one supporting line (see Ref. 6, p. 12).

In order to characterize connected subsets of the boundaries of convex sets, it is enough to give such a characterization for arcs and simple closed curves, since (Proposition 28) a bounded convex set with empty interior is a line segment, and the boundary of a bounded convex set with nonempty interior is a simple closed curve, as we shall now prove.
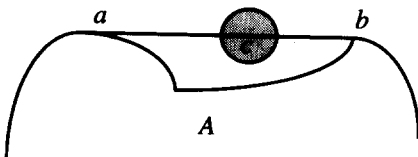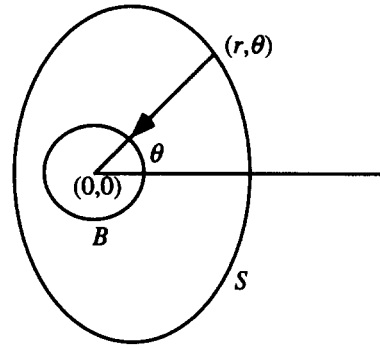

Fig. A3.

Theorem 32

The boundary of a bounded convex set with nonempty interior is a simple closed curve.

Proof. Every point in R^2 can be described as a pair (r, θ), where r is the distance to the origin and θ is an angle, $0 \leqslant \theta < 2\pi$. The function $f: R^2 \setminus \{(0, 0)\} \to B$, $f((r, \theta)) = (1, \theta)$, where B is the unit circle, is continuous and onto B.

Now let S be the boundary of a bounded convex set C with nonempty interior. Translate C so that the origin is in its interior. Function f restricted to S, $f: S \to B$, is "1–1" and "onto", by Theorem 30 (see Fig. A3). Since S is a boundary of a bounded set, it is bounded and closed, hence compact. Thus f is a homeomorphism between S and the unit circle, since it is a continuous bijection on a compact set.

By Propositions 18 and 28, we thus have

Corollary 33

A closed arc-connected proper subset of the boundary of a bounded convex set is an arc.

Proposition 34

For any pair of distinct points $x, y \in R^2$, every straight line that intersects line segment xy also intersects every arc(x, y) (see Fig. A4).

Proof. Let $f: I \to$ arc(x, y) be a continuous function from the closed interval I onto arc(x, y) mapping the endpoints of I onto x and y. Obviously, the straight line M that contains x and y intersects every arc(x, y) at least in x and y. Let L be any straight line that intersects xy in a single point p. Let π_L be the projection of R^2 along L onto M. The composition $\pi_L \circ f: I \to M$ is a continuous function mapping the endpoints of I onto x and y. Therefore, $\pi_L \circ f$ takes on every intermediate value between x and y on line M, i.e. every value on xy, and in particular value p. This implies that L intersects arc(x, y) in at least one point.
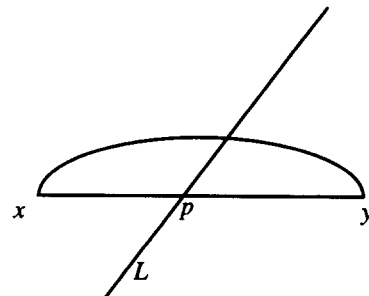

Fig. A2.

Fig. A4.

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