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Topologies for the digital spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$

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Abstract

We show that there are only two topologies in $\mathbb{Z}^2$ and five topologies in $\mathbb{Z}^3$ whose connected sets are connected in the intuitive sense. Both topologies for $\mathbb{Z}^2$ are well known (e.g., one is presented in D. Marcus, F. Wyse et al., Amer. Math. Monthly 77 (1979) 1119, and the second in E. Khalimsky et al., Topology and its Applications 36 (1990) 1) and found applications in computer graphics and computer vision (e.g., A. Rosenfeld, Amer. Math. Monthly 77 (1979) pp. 621, and T.Y. Kong et al., Amer. Math. Monthly 98 (1991) 901). Two of the five topologies for $\mathbb{Z}^3$ are products of the topologies known from $\mathbb{Z}^1$ and $\mathbb{Z}^2$. The remaining three topologies for $\mathbb{Z}^3$ are also generated from the two topologies for $\mathbb{Z}^2$, however, they are not product topologies.

Keywords: Digital topology; Alexandroff topology; Digital plane; Digital space

1. Introduction

The *digital d-space* $\mathbb{Z}^d$ is the set of d-tuples $x = (x_1, x_2, \ldots, x_d)$ of the real Euclidean d-space having integer coordinates. A point with integer coordinates is called a *digital point*. The problem of finding a topology for the digital plane and the digital 3-space is of importance in image processing and more generally in all situations where spatial relations are modelled on a computer. In all these applications it is essential

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to have a data structure on the computer which shares as many as possible features with the real topological situation.

Usually the digital d-space is equipped with a graph structure based on the local adjacency relations of digital points. According to the number of neighbors that can be adjacent to a given point in the digital plane, one speaks of 8-, 4-, and 6-*neighborhoods* (see Fig. 1), 8-, 4-, and 6-*adjacency relations*, and 8-, 4-, and 6-*connectedness relations* (for definitions, see Section 2). Similarly one speaks about 6-,...,26-neighborhoods in the digital 3-space, generally $2d$-,...,(3^d-1)-neighborhoods in d-space.

The digital (2- or 3-) space is the most commonly used representation space in computer graphics and computer vision, where the points are assigned some gray-level or color values. Based on these values and the adjacency relations, the points in the digital space are grouped into connected components, which are used in computer graphics to display and in computer vision to recognize (parts of) real objects. Due to the applications, the adjacency relations, which play a primary role in the grouping process, must be consistent with human spatial intuition about nearness of points in the digital space. The following two conditions reflect this intuition with respect to nearness of two points in the digital plane having the same color:

1. If a set in $\mathbb{Z}^2$ is 4-connected, then it is (topologically) connected.
2. If a set in $\mathbb{Z}^2$ is not 8-connected, then it is not (topologically) connected.

These conditions are satisfied by all definitions of adjacency relations that are most commonly used in computer graphics and computer vision. If we interpret the concept of connectedness in the formal topological sense, then these conditions relate the intuitive concept of neighborhoods described in the graph theory to the topological concept of connectedness.

We show in this paper that there are only two topologies on $\mathbb{Z}^2$ which satisfy the conditions 1 and 2. One is homeomorphic to the cellular-complex topology for the digital plane in [2, Erster Teil, erstes Kapitel, §1.1, Beispiel 4°]. In the context of computer graphics and computer vision, the same topology for the digital plane is described in [5,6,8]. The second topology was introduced by Marcus and Wyse et al. [10] and applied in computer vision by Rosenfeld [13]. In this topology, every set is connected if and only if it is 4-connected.

From our result it also follows that there does not exist any topology for the digital plane with the property that the connected sets with respect to this topology are

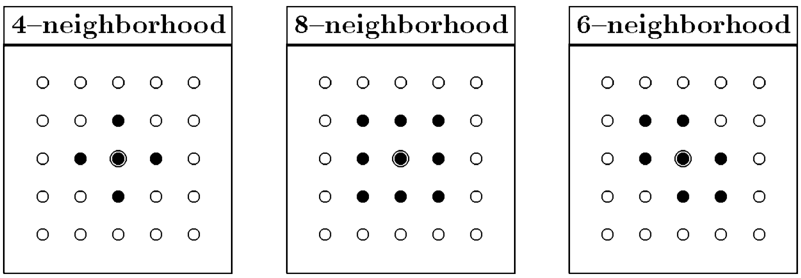


Fig. 1. 4-, 8-, and 6-neighborhoods of a given point.

exactly the 6-connected or exactly the 8-connected sets. This result was proved separately for 8-connected sets in [4]. Latecki [9] gave a much simpler proof of this fact. An even shorter proof was presented by Nogly and Schladt [11]. For 6-connected sets this result was proved by Ptak et al. [12]. Quite recently, a more general and even more simpler proof was published [3]. The 6-adjacency is not that commonly used in computer vision and computer graphics as 4- or 8-adjacency, but it also received a considerable attention (e.g. [8]). It is the main adjacency relation for the digital plane considered in mathematical morphology (see [14]).

In the last part of the paper we investigate the digital 3-space. The two conditions given above translate into

1^3. If a set in $\mathbb{Z}^3$ is 6-connected, then it is (topologically) connected.

2^3. If a set in $\mathbb{Z}^3$ is not 26-connected, then it is not (topologically) connected.

We need, however, an additional condition since there are infinitely many topologies in $\mathbb{Z}^3$ fulfilling these two conditions. Let S_0 be the subset of all points in $\mathbb{Z}^3$ such that the sum of its three coordinates is odd. Then a topology on $\mathbb{Z}^3$ is given by definition of smallest open neighborhoods in the following way: the singleton set $\{x\}$ is open for each $x \in S_0$ and for $x \notin S_0$ the smallest neighborhood is the 15-point set consisting of x itself and all 26-neighbors of x which are in S_0. It is easily seen that the set of all unions of such smallest open neighborhoods is a topology in $\mathbb{Z}^3$. The smallest neighborhoods of points not in S_0 are not 6-connected. We can generate in this way an infinite number of different topologies by requiring that the neighborhood of a point x not in S_0 contains x and all 6-neighbors of x and some of the other 26-neighbors of x which are in S_0.

We therefore introduce in $\mathbb{Z}^3$ a third condition:

3^3. For each $x \in \mathbb{Z}^3$ the smallest open set containing x is 6-connected.

Under these conditions we can show that there are five topologies for the digital 3-space. Two of them can be generated as topological products from the known topologies of $\mathbb{Z}^1$ and $\mathbb{Z}^2$. The other three topologies for $\mathbb{Z}^3$ are generated in a sandwich-like manner from the two topologies known for $\mathbb{Z}^2$.

2. Graph-theoretic and topological concepts

We begin with the definitions of some graph-theoretic concepts. Given a point $x = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$. The $(3^d - 1)$-*neighbors* of x are all points with integer coordinates $y = (m_1, m_2, \ldots, m_d)$ such that

$$\max_{i=1,\cdot,d} |n_i - m_i| = 1.$$

In the digital plane ($d = 2$) we get the 8-neighbors of a point (see Fig. 1).

The $2d$-*neighbors* (or *direct neighbors*) of a point $x \in \mathbb{Z}^d$ are all points $y \in \mathbb{Z}^d$ such that

$$\sum_{i=1}^{d} |n_i - m_i| = 1.$$

In the digital plane ($d = 2$) we get the 4-neighbors of a point (see Fig. 1).

The (3^d-1)-*neighborhood* $\mathcal{N}_{3^d-1}(x)$ of x is the set of all (3^d-1)-neighbors of x (excluding x), the $2d$-*neighborhood* $\mathcal{N}_{2d}(x)$ of x is the set of all $2d$-neighbors of x (excluding x). If two different points $x, y \in \mathbb{Z}^d$ are n-neighbors for $n = 2d$ or $n = 3^d - 1$, then x and y are also said to be n-*adjacent*. The $2d$- and (3^d-1)-connected sets are defined based on the graph structure induced by $2d$- or (3^d-1)-adjacency relations:

A digital set $X \subseteq \mathbb{Z}^d$ is n-*connected* for $n = 2d, 3^d - 1$ if for every pair of points $x, y \in X$, there is an n-path contained in X from x to y, where by an n-*path* from x to y we mean a sequence of points $x = x_1, x_2, \ldots, x_k = y$ such that x_i is n-adjacent to x_{i-1} for $1 < i \leqslant k$.

From topological concepts, we need only the definition of topology and topological connectedness.

A *topological space* is a pair $(X, \mathcal{T})$ that consists of a set X and a collection $\mathcal{T}$ of subsets of X which are called *open sets* and satisfy the following axioms:
(1) X and $\emptyset$ are open.
(2) The union of any collection of open sets is open.
(3) The intersection of any finite collection of open sets is open.
The collection of all open sets $\mathcal{T}$ is termed a *topology* on X.

Let S be a subset of X. In the *relative* (*or subset*) topology induced in S by the topology in X, the open sets are all sets $U \cap S$ with U open in X. One easily sees that this is indeed a topology in S. A set which is open in the relative topology of S is a *relatively open set*.

The set $S \subseteq X$ is called *connected* if there is no decomposition $S = S_1 \cup S_2$ such that $S_1 \cap S_2 = \emptyset$, both S_1, S_2 are nonempty and relatively open with respect to S.

Generally in this paper we do not distinguish between two topologies which differ from each other by a motion (i.e., a translation, rotation, or a reflection) of the digital space.

In this more general setting, conditions 1 and 2 above read:
1^d If a set in $\mathbb{Z}^d$ is $2d$-connected, then it is (topologically) connected.
2^d If a set in $\mathbb{Z}^d$ is not (3^d-1)-connected, then it is not (topologically) connected.

Lemma 2.1. *For any topology satisfying conditions* 1^d *and* 2^d *every point* x *in* $\mathbb{Z}^d$ *has a neighborhood* (*i.e., an open set containing* x) *that is contained in* $\mathcal{N}_{3^d-1}(x) \cup \{x\}$.

Proof. Assume that there exists a point x in $\mathbb{Z}^d$ none of whose neighborhoods is contained in $\mathcal{N}_{3^d-1}(x) \cup \{x\}$. Let $A = \mathbb{Z}^2 \setminus (\mathcal{N}_{3^d-1}(x) \cup \{x\})$. Yet, the set A is $2d$-connected and hence topologically connected by condition 1^d, the set $A \cup \{x\}$ is $3^d - 1$-disconnected, but $A \cup \{x\}$ is topologically connected, since every open set containing x intersects A. This contradicts condition 2^d. $\square$

To state it differently, condition 1^d implies that there always exists an open (topological) neighborhood of x which is contained in $\{x\} \cup \mathcal{N}_{3^d-1}(x)$. Since $\{x\} \cup \mathcal{N}_{3^d-1}(x)$ is a finite set, we conclude that there always exists a uniquely determined smallest neighborhood of x which is the intersection of all open sets containing x. As a consequence, if condition 1^d is true we get a topology with a modified third axiom

(3′) The intersection of any collection of open sets is open.
A topology having this property is called an *Alexandroff-topology* [1].

For any Alexandroff topology we can get a new topology simply by interchanging the meaning of the concepts 'open' and 'closed.'

A set $S \subseteq X$ is *closed* if its complement (with respect to X) is open. The *closure* of a set $S \subseteq X$ is the smallest closed set containing S.

Lemma 2.2. *For any topology satisfying conditions* 1^d *and* 2^d, *the closure of every point* x *in* $\mathbb{Z}^d$ *is contained in* $\mathcal{N}_{3^d-1}(x) \cup \{x\}$.

Proof. As in the proof of Lemma 2.1 we observe that the sets $\{x\}$ and $A = \mathbb{Z}^d \setminus (\mathcal{N}_{3^d-1}(x) \cup \{x\})$ are not connected while A is $2d$-connected. Hence there exist open sets S_1 and S_2 such that $x \in S_1$ and $A \subseteq S_2$. Moreover, $x \notin S_2$ and S_1 is contained in $\{x\} \cup \mathcal{N}_{3^d-1}(x)$. The complement of S_2, however, is a closed set which is contained in $\{x\} \cup \mathcal{N}_{3^d-1}(x)$ and which contains x. $\square$

3. Two topologies on $\mathbb{Z}^2$

As the first main theorem of this paper we prove in this section the following:

Theorem 3.1. *There are only two topologies on* $\mathbb{Z}^2$ *which satisfy the conditions* 1 *and* 2.

Let $\mathcal{T}$ be any topology on the digital plane $\mathbb{Z}^2$ that satisfies conditions 1 and 2 (we do not assume explicitly any separation axioms). For any pair of points $x, y \in \mathbb{Z}^2$, we define

$x \to y$ if and only if every open set containing x also contains y.

The relation "$\to$" will be our main tool in the proof of Theorem 3.1. It is easy to check that the relation "$\to$" is *reflexive* (i.e., $x \to x$ for every $x \in \mathbb{Z}^2$) and *transitive* (i.e., $x \to y$ and $y \to z$ implies $x \to z$ for every $x, y, z \in \mathbb{Z}^2$).

Let $x \nrightarrow y$ denote the negation of $x \to y$. Observe that $x \nrightarrow y$ holds if and only if there exists an open set containing x that does not contain y.

The following two conditions are simple consequences of the corresponding conditions 1 and 2:
1′ If two points $x, y \in \mathbb{Z}^2$ are 4-neighbors, then either $x \to y$ or $y \to x$.
2′ If two points $x, y \in \mathbb{Z}^2$ are not 8-neighbors, then $x \nrightarrow y$ and $y \nrightarrow x$.

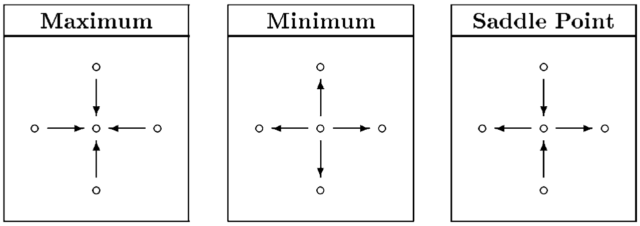
Lemma 3.1. *Given three points* x_0, x_1, *and* x_2 *which are on the same horizontal or vertical grid line of* $\mathbb{Z}^2$ *such that* $x_1 \in \mathcal{N}_4(x_0)$ *and* $x_2 \in \mathcal{N}_4(x_1)$.

If $x_0 \to x_1$, *then* $x_2 \to x_1$. *Analogously*, $x_1 \to x_0$ *implies* $x_1 \to x_2$.

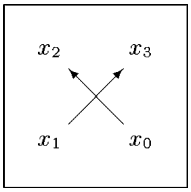
(*Thus, we have either* $x_0 \to x_1 \leftarrow x_2$ *or* $x_0 \leftarrow x_1 \to x_2$ *on a grid line.*)

Proof. By condition 1′, we have either $x_0 \to x_1$ or $x_1 \to x_0$. Let $x_0 \to x_1$. By condition 1', either $x_1 \to x_2$ or $x_2 \to x_1$. If $x_1 \to x_2$, then by transitivity of "$\to$," also $x_0 \to x_2$, which contradicts condition 2′. Hence $x_2 \to x_1$. $\square$

As a direct consequence of this lemma, we obtain that we distinguish only three types of neighborhoods $\mathcal{N}_4(x)$ of any point $x \in \mathbb{Z}^2$, which we will call neighborhoods of *maxima*, *minima*, and *saddle points* (the last one up to rotations by 90°):

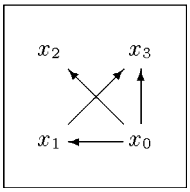


Lemma 3.2. *In the digital plane there is no cross of the form*

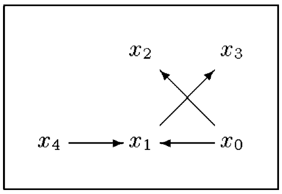


(*up to rotations by multiples of* 90°), *such that x_i is a direct neighbor of $x_{i+1\ (\mathrm{mod}\ 4)}$.*

Proof. By condition 1, either $x_0 \to x_1$ or $x_1 \to x_0$. Without loss of generality assume that the first relation holds. By transitivity of "$\to$," this implies $x_0 \to x_3$.



Let x_4 be the 4-neighbor of x_1 which is on the line through x_1 and x_0 and which is different from x_0. By Lemma 3.1, we have $x_4 \to x_1$:



Transitivity implies $x_4 \to x_3$, which contradicts condition 2. □

Theorem 3.2. *By one single saddle point configuration all "$\to$"-relations in the digital plane are determined (see Fig.* 2*).*

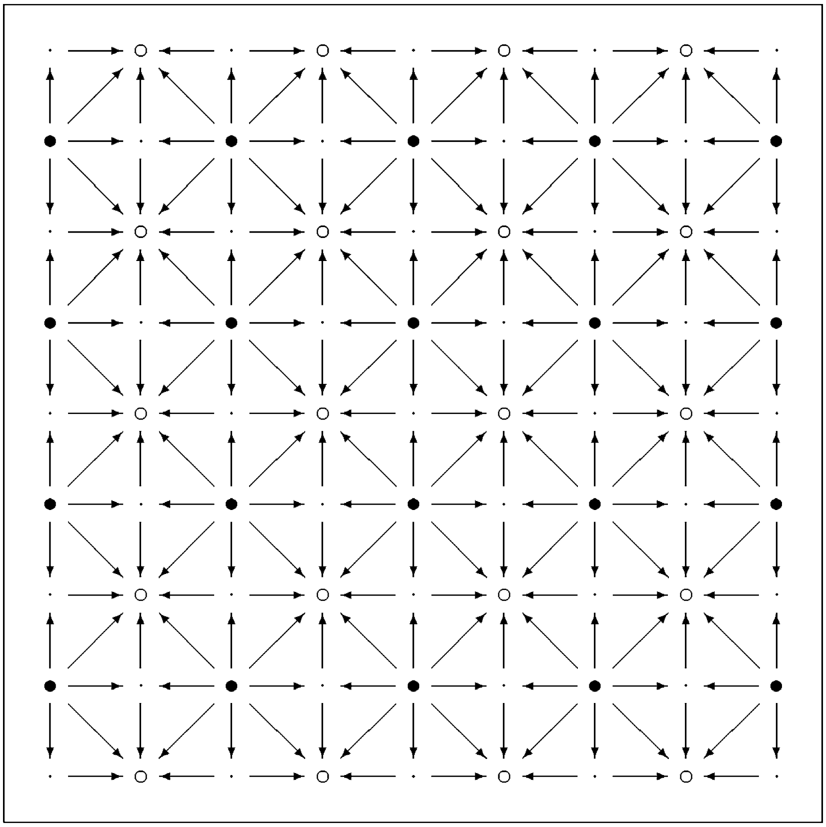
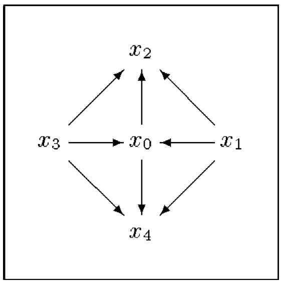
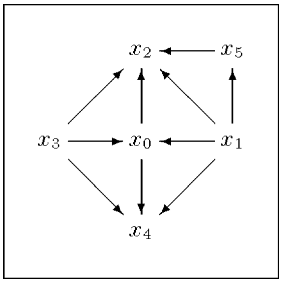


Fig. 2. Maxima are marked by ○, minima by ●, and saddle points by ·.

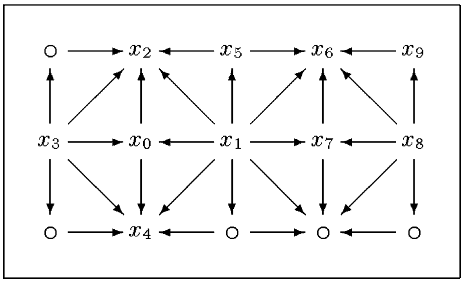
Proof. By transitivity, the given saddle point configuration looks as follows:



Let x_5 be the common direct neighbor of x_1 and x_2 which is different from x_0. $x_5 \to x_1$ would imply $x_5 \to x_0$, which is not possible by Lemma 3.2. Hence, $x_1 \to x_5$. Analogously, $x_2 \to x_5$ is not possible. Thus, we obtain:



Consider now points x_6, x_7, x_8, x_9 located as shown below:

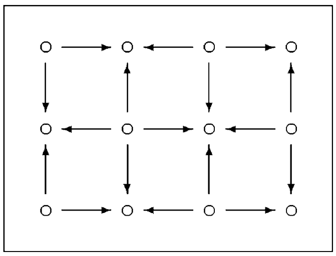


$x_1 \to x_5 \to x_6$ implies $x_1 \to x_6$. If $x_6 \to x_7$, then by $x_5 \to x_6$ also $x_5 \to x_7$, which is not possible by Lemma 3.2. $x_8 \to x_7 \to x_6$ implies $x_8 \to x_6$. Finally, $x_9 \to x_8$ would imply that $x_9 \to x_7$, hence, by Lemma 3.2 $x_8 \to x_9$.

Consequently, we always have points of type x_2 and x_1 (maxima and minima, respectively) and of type x_7 (saddle points) in a checkerboard-like manner. In Fig. 2, this topology is illustrated. $\square$

By similar but simpler arguments as in the proof of Theorem 3.2, the following Theorem can be proved.

Theorem 3.3. *If there is no saddle point, then the "$\to$"-relation in the whole digital plane is determined (up to a translation) and is given by*



The proof of Theorem 3.1 follows directly from Theorems 3.3 and 3.2.

Lemma 2.1 guarantees for every point $x \in \mathbb{Z}^2$ the existence of a (topological) neighborhood which is contained in $\mathcal{N}_8(x) \cup \{x\}$. In Theorems 3.2 and 3.3 such a neighborhood is explicitly constructed for each point by means of the "$\rightarrow$"-relation. Moreover, it is shown that under the conditions of the theorems by this construction a topology for the whole digital plane is uniquely determined.

In the topology determined in Theorem 3.3, the smallest open set containing a minimum point x coincides with $\mathcal{N}_4(x) \cup \{x\}$ and the smallest open set containing a maximum point x is equal to $\{x\}$. Thus, the topology determined in Theorem 3.3 is the topology introduced by Marcus and Wyse et al. [10], in which every set is connected if and only if it is 4-connected.

Similarly, in the topology determined in Theorem 3.2, shown in Fig. 2, the smallest open set containing a minimum point x is uniquely determined and is equal to $\mathcal{N}_8(x) \cup \{x\}$. The smallest open set of a maximum point x is equal to $\{x\}$. The smallest open set of a saddle point x contains x and either its two horizontal direct neighbors or its two vertical direct neighbors.

Finally we show that the topology determined in Theorem 3.2 is homeomorphic to the cellular-complex topology for the digital plane introduced by Alexandroff and Hopf [2, Erster Teil, erstes Kapitel, §1.1, Beispiel 4°], which is homeomorphic to the topology in [5] as well as to the topology in [8].

The topology of Alexandroff and Hopf is based on the subdivision of the plane into cells: open squares, line segments without their endpoints (which are sides of the squares), and points (which are corners of the squares), as illustrated in Fig. 3. The smallest open neighborhood of each cell in this topology is the smallest open set in the standard planar topology that contains this cell. For example, open squares are (the only) one-point open sets in this topology.

The homeomorphism is determined by the following correspondence: a minimum point corresponds to an open square by Alexandroff and Hopf (and to an 'open–open point' in [5]), a maximum point corresponds to a vertex point by Alexandroff and Hopf (and to a 'closed–closed point' in [5]), and a saddle point corresponds to a side of a square (and to a 'mixed point' in [5]).

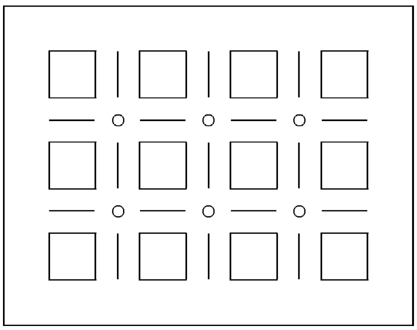


Fig. 3. The topological space of Alexandroff and Hopf.

4. Five topologies on $\mathbb{Z}^3$

Now we are going to investigate the three-dimensional digital space.

Conditions 1 and 2 and the third condition now read:

1^3 If a set in $\mathbb{Z}^3$ is 6-connected, then it is (topologically) connected.

2^3 If a set in $\mathbb{Z}^3$ is not 26-connected, then it is not (topologically) connected.

3^3 For each $x \in \mathbb{Z}^3$ the smallest open set containing x is 6-connected.

First we state some simple definitions and observations:

The *principal planes* in $\mathbb{Z}^3$ are the sets of all triples $(i, j, k) \in \mathbb{Z}^3$ with the property that one of the three coordinates is kept fixed whereas the others vary in $\mathbb{Z}$.

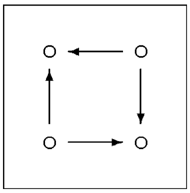
Observation 4.1. *Given a topology in* $\mathbb{Z}^3$ *fulfilling conditions* 1^3 *and* 2^3. *Then the relative topology induced in each principal plane is a topology of* $\mathbb{Z}^2$ *fulfilling conditions* 1 *and* 2.

We define: An *elementary square* of $\mathbb{Z}^2$ is the square whose vertices are points (i, j), $(i+1, j)$, $(i+1, j+1)$, and $(i, j+1)$, $i, j \in \mathbb{Z}$. An elementary cube in $\mathbb{Z}^3$ is the set whose vertices consist of a point $(i, j, k) \in \mathbb{Z}^3$ and all its neighbors of the form $(i', j', k') \in \mathbb{Z}^3$ such that $i' \in \{i, i+1\}$, $j' \in \{j, j+1\}$, and $k' \in \{k, k+1\}$.

Observe that elementary squares in $\mathbb{Z}^2$ (cubes in $\mathbb{Z}^3$) are the largest subsets with the property that each two points in them are 8-adjacent (26-adjacent). By this condition, it is possible to define n-dimensional cells for $n = 2, 3, \ldots$

Observation 4.2. *A topology in* $\mathbb{Z}^2$ *is completely determined by the relation* $\rightarrow$ *on the sides of a single elementary square. There are exactly two possibilities* (*up to rotations and translations*):

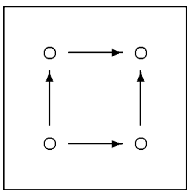
Antiparallel case



In this situation the topology is the Marcus–Wyse topology which is characterized by the fact that there is no saddle point (*Theorem* 3.3).

In this case the arrows of parallel sides point into opposite directions.

Parallel case



This is the Alexandroff–Hopf topology of Theorem 3.2 *which is characterized by the existence of saddle points.*

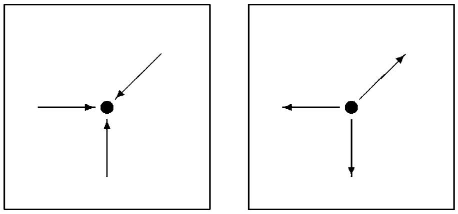
In this case the arrows of parallel sides of the elementary square have the same direction.

We note that in the parallel case a diagonal arrow is implied by transitivity. This arrow was not shown in the picture.

For convenience we represent the vertices, sides, and faces of an elementary cube in $\mathbb{Z}^3$ by its Schlegel diagram (see Fig. 4).

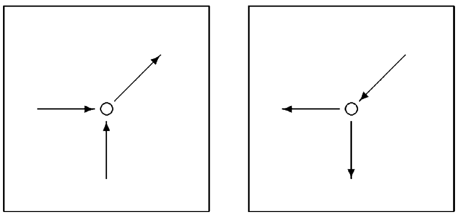
We now look at a single elementary cube of $\mathbb{Z}^3$. Each vertex of such a cube belongs to exactly one of the following two classes (modulo rotations):

Pure vertices. The arrows which are incident to the vertex and belong to the elementary cube either point all towards the vertex or point all away from the vertex:



In the two-dimensional case pure vertices correspond to minima and maxima.

Mixed vertices. In this case not all arrows incident to the vertex point in the same direction.



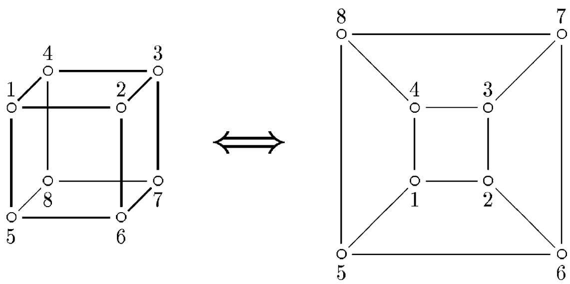


Fig. 4. Representation of a 3-cube by its Schlegel diagram.

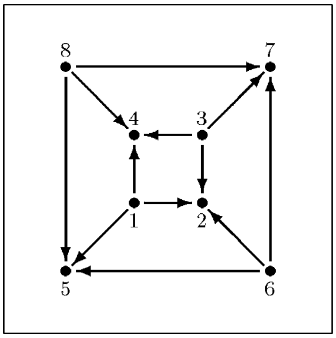
In the two-dimensional case mixed vertices are saddle points.

In the sequel we denote pure vertices by $\bullet$ and mixed vertices by $\circ$.

Lemma 4.1. *There are only two possibilities for an elementary cube to have only pure vertices. These two differ from each other only in the directions of the arrows. Both configurations define the same topology on the elementary cube.*

Proof. Assume that all vertices of an elementary cube are pure vertices and that vertex number 1 in Fig. 4 has only outgoing arrows. This means that these arrows point exactly to the direct neighbors in the cube of vertex 1. Then, since vertices 2 and 4 are pure, they have only incoming arrows coming from the direct neighbors of these points and so on. Obviously, by the requirement that all vertices are pure and that vertex 1 has outgoing arrows, the situation is completely determined as far as arrows joining direct neighbors are concerned.

In order to prove that this configuration of arrows indeed defines a topology on the elementary cube, we give together with its Schlegel diagram a table showing for each vertex its smallest neighborhood which is the intersection of all open sets containing the vertex.



Node	Neighborhood
1	1, 2, 4, 5
2	2
3	2, 3, 4, 7
4	4
5	5
6	2, 5, 6, 7
7	7
8	4, 5, 7, 8

It can be easily checked that all intersections of open sets on the elementary cube are open sets, hence the arrangement of arrows in this case is indeed a topology on the elementary cube. Moreover, all these open sets are 6-connected so that on the elementary cube condition 3^3 is fulfilled.

Each vertex of the elementary cube has exactly three direct neighbors on the cube. By condition 1^3 the smallest open set containing a vertex contains among the vertices of the cube at most these direct neighbors. Since no arrow joins any two points on the cube which are not direct neighbors, and since all arrows meeting a vertex are either pointing to or else coming from a direct neighbor, no other topology fulfilling condition 1^3 is possible.

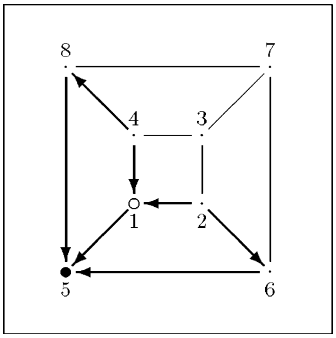
The other possibility for arranging pure vertices on the elementary cube is obtained by assuming that vertex 1 has only incoming arrows. This corresponds to the reversion of the arrows and obviously yields no new topology since this transformation corresponds to a 90°-rotation of the cube. $\square$

We now investigate the situation that one vertex, say vertex 1 in Fig. 4, is a mixed vertex. We assume further that the neighboring vertices are related to vertex 1 as follows: $1 \to 5$, $2 \to 1$, and $4 \to 1$. This is no loss of generality since we may rotate the elementary square suitably and also revert the directions of all arrows. The latter operation corresponds to the open–closed transformation of Alexandroff topologies. We have to prove in each case whether the topology is changed by it.

Yet an arrow $6 \to 2$ is not possible by Observation 4.2, hence we have $2 \to 6$ and, by a similar argument, $4 \to 8$. As a consequence, we get $6 \to 5$ and $8 \to 5$. We see that in this case necessarily vertex 5 becomes a pure vertex. We may state as a result:

Each mixed vertex has at least one neighbor vertex which is a pure vertex.

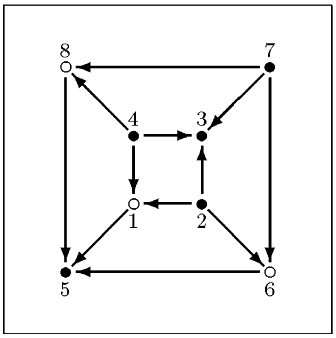
As a consequence, it is not possible to get an elementary cube having exclusively mixed vertices. We therefore have the following configuration:



Now we distinguish two main cases, depending on the direction of the arrow joining vertices 6 and 7.

Case a: $7 \to 6$. There are two subcases, depending on the direction of the arrow joining vertices 7 and 3.

Subcase a_1: $7 \to 3$. In this case, squares 7623 and 7348 must have antiparallel topologies:

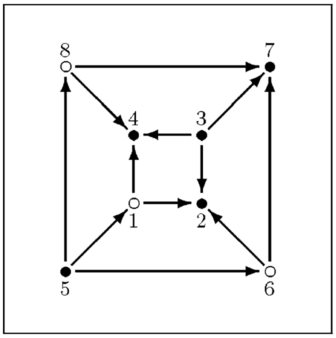


Node	Neighborhood
1	1, 5
2	1, 2, 3, 5, 6
3	3
4	1, 3, 4, 5, 8
5	5
6	5, 6
7	3, 5, 6, 7, 8
8	5, 8

Note that vertex 5 belongs to the neighborhood of vertices 2, 4, and 7 by transitivity (in our representation diagonal arrows are omitted). In a similar way as above we show that by this arrangement of arrows indeed a topology is given.

We have here five pure vertices in the elementary cube.

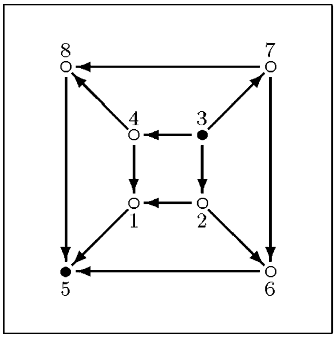
Subcase a_1'. When we revert the directions of all arrows of configuration a_1 we get a different topology on the unit cube.



Node	Neighborhood
1	1, 2, 4
2	2
3	2, 3, 4, 7
4	4
5	1, 2, 4, 5, 6, 7, 8
6	2, 6, 7
7	7
8	4, 7, 8

Whereas in configuration a_1 there are two open sets consisting of one point, in configuration a_1' there are three such points, hence topologies a_1 and a_1' are not homeomorphic.

Subcase a_2: $3 \rightarrow 7$. In this case, squares 7623 and 7348 must have parallel topologies:



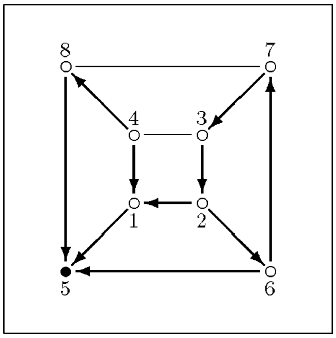
Node	Neighborhood
1	1, 5
2	1, 2, 5, 6
3	1, 2, 3, 4, 5, 6, 7, 8
4	1, 4, 5, 8
5	5
6	5, 6
7	5, 6, 7, 8
8	5, 8

This configuration is symmetric. Each face of the elementary cube contains exactly one pure point. Also in this case we have a topology on the elementary cube. When the directions of all arrows are reverted, we get a topology which is homeomorphic to the topology of subcase a_2.

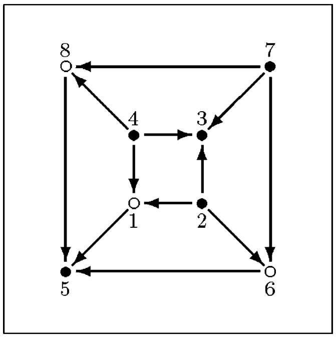
We have here two pure vertices in the elementary cube.

Case b: $6 \rightarrow 7$. Again, there are two subcases, depending on the direction of the arrow joining vertices 7 and 3.

Subcase b_1: $7 \rightarrow 3$. In this case, there exists no topology consistent with the arrows: by transitivity of arrows, we obtain that $3 \rightarrow 2$. This configuration of arrows is not consistent with any topology, since square 6732 has neither parallel nor antiparallel topology (see Observation 4.2):



Subcase b_2: $3 \rightarrow 7$.



Node	Neighborhood
1	1, 5
2	1, 2, 3, 5, 6
3	3
4	1, 3, 4, 5, 8
5	5
6	5, 6
7	3, 5, 6, 7, 8
8	5, 8

Again we have a topology on the elementary cube. Here each face of the elementary cube contains exactly two pure points. We have here four pure vertices in the elementary cube. Also here reversion of the arrows yields not a new topology.

As a result we state that

Lemma 4.2. *There are exactly five possibilities to introduce an order on the vertices of the elementary cube in $\mathbb{Z}^3$ that is consistent with a topology on the cube such that each face either exhibits the Marcus–Wyse configuration or the Alexandroff–Hopf configuration.*

These are topologies given by Lemma 4.1 and Cases a_1, a_2, and b_2. The five topologies on the elementary cube are different, since they contain either 8, 5, 2, or 4 pure points.

We now investigate the task of joining elementary cubes. This is done by putting them together face by face, i.e., they have exactly one common face F and all arrows which are within face F necessarily coincide.

Let two edges E_1 and E_2 be perpendicular to face F and such that $E_1 \cap E_2 \cap F$ is a single vertex, say vertex v (see Fig. 5 for illustration). By the transitivity and condition 2^3, we obtain that E_1 and E_2 have mirror directions of arrows, e.g., if the arrow of side E_1 points towards vertex v, then the arrow of side E_2 also points towards vertex v.

Since the arrows on three sides of a square uniquely determine the direction of the arrow on the fourth side of the square, we obtain that

Lemma 4.3. *Two cubes that share a face have mirror directions of arrows.*

We obtain the smallest open sets in the configuration of two elementary cubes as follows: if a vertex is not common to both cubes, then its smallest neighborhood is that of the cube, the smallest neighborhood of a point common to both cubes is the union of the smallest neighborhoods of either cube. In this way we obviously get a topology on the union of two cubes.

It is easily seen that necessarily in the common face of the two elementary cubes common points have the same type (simple or mixed) with respect to either cube. This immediately implies that only cubes of the same type (only pure points or types a_1, a_2, or b_2) can be joined together. Although cubes of type a_1 have faces containing two pure points, the configuration of the arrows perpendicular to these faces are not mirror images of faces of type b_2.

Lemma 4.3 implies that a single cube determines the topology on a line of cubes in which each cube shares a side with exactly two others. Further, putting lines next to each other, we obtain a plane of cubes, and putting planes of cubes on top of each other, we fill the whole space in $\mathbb{Z}^3$ with cubes. We obtain that a single cube determines the topology on $\mathbb{Z}^3$. Hence, the topology of $\mathbb{Z}^3$ is uniquely determined by the topology of an elementary cube and each topologized elementary cube leads to a topology of $\mathbb{Z}^3$. Since we only have five topologically different cubes, we obtain the following theorem.

Theorem 4.1. *In $\mathbb{Z}^3$ there exist five topologies* (*up to motions of $\mathbb{Z}^3$*) *satisfying conditions 1^3 and 2^3. These are*:

1. *The Marcus–Wyse topology for $\mathbb{Z}^3$ which is characterized by the fact that all points in $\mathbb{Z}^3$ are pure vertices.*

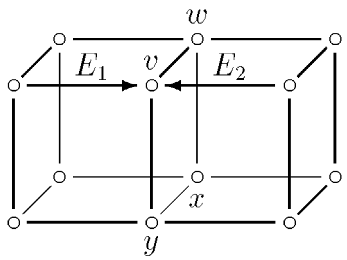


Fig. 5. Two elementary cubes with common face. The common face of the two cubes is determined by four vertices v, w, x, and y. Edges E_1 and E_2 meet in v and are perpendicular to F.

2. *A topology which induces in each principal plane the Alexandroff–Hopf topology as relative topology* (*Case* a_2).
3. *A topology which induces in two prinicipal planes the Alexandroff–Hopf topology and the Marcus–Wyse topology* (*Case* b_2) *in the third prinicipal plane perpendicular to the two others.*
4. *Two topologies which are characterized by the fact that parallel grid planes carry alternatively the Marcus–Wyse and the Alexandroff–Hopf topology* (*Cases* a_1 *and* a_2).

5. Conclusions

- The only topology for the digital plane which induces the 4-connectedness relation is the topology introduced in [10], by Theorem 3.3.
- The only topology for $\mathbb{Z}^2$ satisfying conditions 1 and 2 which is different from the Marcus–Wyse topology is the topology in Theorem 3.2.
- The topology of Theorem 3.2 is homeomorphic to the cellular-complex topology in [2, Erster Teil, erstes Kapitel, §1.1, Beispiel 4°] as well as to the topology proposed in [5,8].
- Neither 8- nor 6-connectedness in the digital plane can be induced by a topology. This extends the results in [4,9,11].
- In $\mathbb{Z}^3$ there exists two topologies which are obtained as a triple product of the Marcus–Wyse-topology for $\mathbb{Z}^1$ as the product of one Alexandroff–Hopf-topology for $\mathbb{Z}^2$ and one $\mathbb{Z}^1$-Marcus–Wyse-topology.
- Three of the topologies for $\mathbb{Z}^3$ are obtained by stacking $\mathbb{Z}^2$-planes but which are not topological products of lower-dimensional spaces.

Kong [7] recently published a paper treating the cases $d = 2, 3, 4$. He found 24 topologies in $\mathbb{Z}^4$. His results, obtained in a different way, also include the results obtained here.

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