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Digitization

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Synonyms

Relation between objects and their digital images

Definition

Digitization is a mathematical model of converting continuous subsets of the plane or space (representing real objects) to digital sets in $\mathbb{Z}^2$ or $\mathbb{Z}^3$ or similar grids (representing segmented images of these objects). This definition can be generalized to any dimension $n > 3$: Digitization converts (transforms) continuous subsets of $\mathbb{R}^n$ to digital sets in $\mathbb{Z}^n$ or, equivalently, to functions from $\mathbb{Z}^n$ to $\{0, 1\}$.

Background

A fundamental task of knowledge representation and processing is to infer properties of real objects or situations given their representations. In spatial knowledge representation and, in particular, in computer vision and medical imaging, real objects are represented in a pictorial way as finite and discrete sets of pixels or voxels. The discrete sets result in a quantization process, in which real objects are approximated by discrete sets. In computer vision, this process is called sampling or digitization and is naturally realized by technical devices like computer tomography scanners, CCD cameras, or document scanners. Digital images obtained by digitization are suitable to estimate the real object properties like volume and surface area. Therefore, a fundamental question addressed in spatial knowledge representation is: Which properties inferred from discrete representations of real objects correspond to properties of their originals, and under what conditions this is the case? While this problem is well-understood in the 2D case with respect to topology [1–5], it is not as simple in 3D, as shown in [6]. Only recently a first comprehensive answer to this question with respect to important topological and geometric properties of 3D objects has been presented in [7, 8].

Some recent works done for the general case are shown below. It is proven in [9] that although Gauss digitized boundaries of subsets of $\mathbb{R}^n$, for $n \geq 3$ may not be manifolds, non-manifoldness may only occur in places where the normal vector is almost aligned with some digitization axis, showing that although an object and its digitization are close in the Hausdorff sense through the projection map, they may not be homeomorphic. Nevertheless, in that entry, the authors prove the validity of the digital surface integral as a

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multigrid convergent integral estimator of subsets of $\mathbb{R}^n$, for $n \geq 3$, as long as the digital normal estimator is also multigrid convergent. In addition, [10] is a short survey on digital analytical geometry where the main idea is to analytically characterize digital sets to describe its continuous counterpart in $\mathbb{R}^n$, for $n \geq 3$ and related transform. This way, digital subsets of $\mathbb{Z}^n$ are defined by a list of inequalities and not by an enumeration of points in $\mathbb{Z}^n$. Finally, in [11], a modus operandi is proposed to model a digital subset of $\mathbb{Z}^n$ as a cubical complex proving that the digital fundamental group of a digital subset of $\mathbb{Z}^n$ is isomorphic to the fundamental group of its corresponding cubical complex, ensuring the topological correctness of the approach. Thus, properties of digital subsets of $\mathbb{Z}^n$ can be computed on their corresponding cubical complexes using powerful algebraic-topological tools. Observe that this last approach "closes" a loop: starting from a continuous subset of $\mathbb{R}^n$, a digital subset of $\mathbb{Z}^n$ is obtained and used to compute a cubical complex whose embedding in $\mathbb{R}^n$ is again a continuous subset of $\mathbb{R}^n$.

The description of geometric and, in particular, topological features in discrete structures is based on graph theory, which is widely accepted in the computer science community. A graph is obtained when a neighborhood relation is introduced into a discrete set, e.g., a finite subset of $\mathbb{Z}^2$ or $\mathbb{Z}^3$, where $\mathbb{Z}$ denotes the integers. On the one hand, graph theory allows investigation into connectivity and separability of discrete sets (e.g., for a simple and natural definition of connectivity, see [12, 13]). On the other hand, a finite graph is an elementary structure that can be easily implemented on computers. Discrete representations are analyzed by algorithms based on graph theory, and the properties extracted are assumed to represent properties of the original objects. Since practical applications, for example, in image analysis, show that this is not always the case, it is necessary to relate properties of discrete representations to the corresponding properties of the originals. Since such relations can describe and justify the algorithms on discrete graphs, their characterization contributes directly to the computational investigation of algorithms on discrete structures. This computational investigation is an important part of the research in computer science and, in particular, in computer vision [14], where it can contribute to the development of more suitable and reliable algorithms for extracting the required shape properties from discrete representations.

It is clear that no discrete representation can exhibit all features of the real original. Thus one has to accept compromises. The compromise chosen depends on the specific application and on the questions which are typical for that application. Real objects and their spatial relations can be characterized using geometric features. Therefore, any useful discrete representation should model the geometry faithfully in order to avoid false conclusions. Topology deals with the invariance of fundamental geometric features like connectivity and separability. Topological properties play an important role, since they are the most primitive object features and human visual system seems to be well-adapted to cope with topological properties.

However, one does not have any direct access to spatial properties of real objects. Therefore, real objects are represented as bounded subsets of the Euclidean space $\mathbb{R}^3$ and their 2D views (projections) as bounded continuous subsets of the plane $\mathbb{R}^2$. Hence, from the theoretical point of view of knowledge representation, the goal is to relate two different pictorial representations of objects in the real world: a discrete and a continuous representation.

Already two of the first entries in computer vision deal with the relation between the continuous object and its digital images obtained by modeling a digitization process. Pavlidis [1] and Serra [2] proved independently in 1982 that an r-regular continuous 2D set S (the definition follows below) and the continuous analog of the digital image of S have the same shape in a topological sense. Pavlidis used 2D square grids and Serra used 2D hexagonal sampling grids.

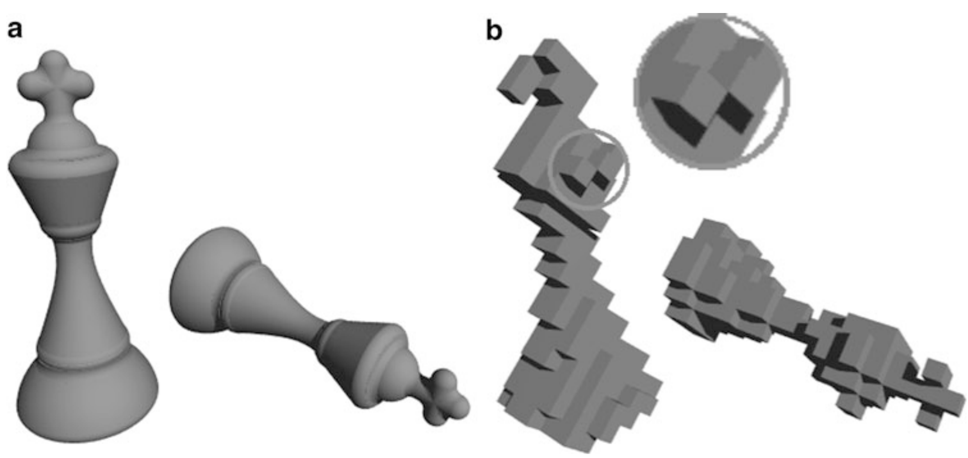
In 3D this problem is much more complicated. In 2005 it has been shown in [6] that the connectivity properties are preserved when digitizing a 3D r-regular object with a sufficiently dense sampling grid, but the preservation of connectivity is much weaker than topology. Stelldinger and Köthe [6] also found out that topology preservation can even not be guaranteed with sampling grids of arbitrary density if one uses the straightforward voxel reconstruction, since the surface of the continuous analog of the digital image may not be a 2D manifold. The question on how to guarantee topology preservation during digitization in 3D remained unsolved until 2007.

The solution was provided in [7], where the same digitization model as Pavlidis and Serra is used, and also r-regular sets (but in $\mathbb{R}^3$) are used to model the continuous objects. As already shown in [6], the generalization of Pavlidis' straightforward reconstruction method to 3D fails since the reconstructed surface may not be a 2D manifold. For example, Fig. 1a, b shows a continuous object and its digital reconstruction whose surface is not a 2D manifold. However, it is possible to use several other reconstruction methods that all result in a 3D object with a 2D manifold surface. Moreover it is also shown in [7] that these reconstructions and the original continuous object are homeomorphic and their surfaces are close to each other.

The first reconstruction method, majority interpolation, is a voxel-based representation on a grid with doubled resolution. It always leads to a well-composed set in the sense of [15], which implies that a lot of problems in 3D digital geometry become relatively simple.

The second method is the most simple one. It just uses balls with a certain radius instead of cubical voxels. When choosing an appropriate radius, the topology of an r-regular object will not be destroyed during digitization.

The third method is a modification of the well-known marching cubes algorithm [16]. The original marching cubes algorithm does not always construct a topologically sound surface due to several ambiguous cases [17, 18]. As shown in [7] and [8], most of the ambiguous cases can not occur in the digitization of an r-regular object and that the only remaining ambiguous case always occurs in an unambiguous way, which can be dealt with by a slight modification of the original marching cubes algorithm. Thus the generated surface is not only topologically sound, but it also has exactly the same topology as the original object before digitization. Moreover it is shown that one can use trilinear interpolation and that one can even blend the trilinear patches smoothly into each other such that one gets smooth object surfaces with the correct topology. Each of these methods has its own advantages making the presented results applicable to many different image analysis algorithms.



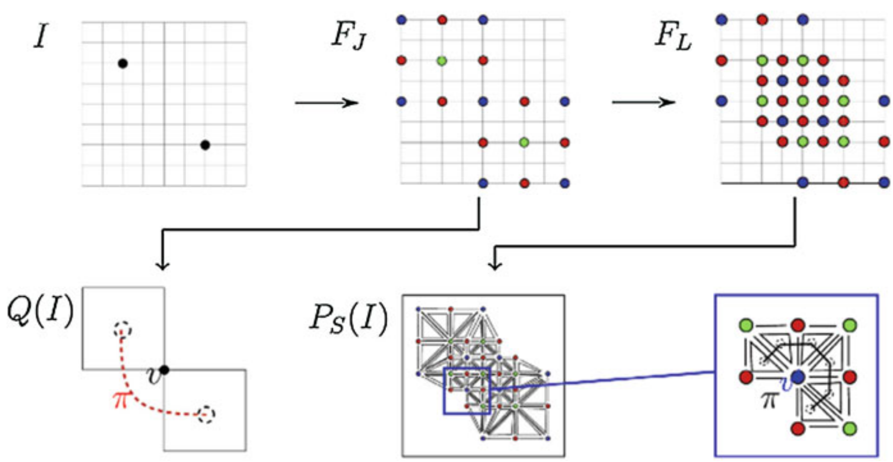
Digitization, Fig. 1 The digital reconstruction (**b**) of an r-regular object (**a**) may not be well-composed, i.e., its surface may not be a 2D manifold as can be seen in the magnification

In the general case, well-composed digital subsets of $\mathbb{Z}^n$ do not present topological paradoxes. They also have very interesting properties and practical applications. Different "flavors" of well-composedness (WC) are present in the literature: WC based on equivalence of connectivities (EWC), digital WC (DWC), WC in the Alexandrov sense (AWC), and WC in the continuous sense (CWC). All these definitions are equivalent in 2D. For the 3D case, we have DWC $\Leftrightarrow$ AWC $\Leftrightarrow$ CWC. For the nD case, $n \geq 3$, we have EWC $\Leftarrow$ DWC. The rest of equivalences for the case $n > 3$ are open problems nowadays. The definition of well-composedness has also been extended to arbitrary grids and multivalued images (AGWC) (see [19]).

Methods for repairing digital subsets of $\mathbb{Z}^n$, for $n > 3$, to convert them in well-composed ones are a complicated open problem. A first step in this direction is done in [20] in which a combinatorial method is given for computing a simplicial complex homotopy equivalent to the cubical complex associated to a given digital subset of $\mathbb{Z}^n$. This simplicial complex is continuously well-composed for $n \leq 3$ and weakly well-composed for $n > 3$ in the sense that for any two n-simplices incident to a common vertex v, there always exists a face-connected path of n-simplices incident to v. A graphical diagram of the method is given in Fig. 2. Observe that cubical and simplicial complexes derived from that method are also stored as digital subsets of $\mathbb{Z}^n$, so that later calculations on the elements of the complex can be done efficiently.

Theory

The (Euclidean) *distance* between two points x and y in $\mathbb{R}^n$ is denoted by $d(x, y)$, and the (Hausdorff) *distance between two subsets of* $\mathbb{R}^n$ is the maximal distance between each point of one set and the nearest point of the other. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets. A function $f : A \to B$ is called *homeomorphism* if it is bijective, and both it and its inverse are continuous. If f is a homeomorphism, then A and B are *homeomorphic*. Let A and B be the two subsets of $\mathbb{R}^n$ (particularly, $n = 2$ or 3). Then a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(A) = B$ and $d(x, f(x)) \leq r$, for all $x \in \mathbb{R}^n$, is called an r-*homeomorphism* of A to B, and A and B are r-*homeomorphic*. A *Jordan curve* is a set $J \subset \mathbb{R}^n$ which is homeomorphic to a circle.



Digitization, Fig. 2 We start from $I = (\mathbb{Z}^n, F_I)$ being F_I a digital subset of $\mathbb{Z}^n$ (in fact, $F_I \subset 4\mathbb{Z}^n$). The digital subset F_J of $\mathbb{Z}^n$ encodes the cells of the associated cubical complex $Q(I)$ (blue is used for 0-cells, red for 1-cells, and green for 2-cells). Now, we "repair" F_J to obtain the digital subset F_L of $\mathbb{Z}^n$ by "thickening" the critical points of F_J. Then, we compute the simplicial complex $P_S(I)$ whose set of vertices is F_L, satisfying that there exists a face-connected path of n-simplices in $P_S(I)$ joining any two n-simplices incident to a common vertex in $P_S(I)$, that is, $P_S(I)$ is weakly well-composed (see [20])

Let A be any subset of $\mathbb{R}^n$. The *complement* of A is denoted by A^c. All points in A are *foreground*, while the points in A^c are called *background*. The *open ball* in $\mathbb{R}^n$ of radius r and center c is the set $\mathcal{B}_r^0(c) = \{x \in \mathbb{R}^n \mid d(x,c) < r\}$, and the *closed ball* in $\mathbb{R}^n$ of radius r and center c is the set $\overline{\mathcal{B}}_r(c) = \{x \in \mathbb{R}^n \mid d(x,c) \leq r\}$. The *boundary* of A, denoted ∂A, consists of all points $x \in \mathbb{R}^n$ with the property that if B is any open set of $\mathbb{R}^n$ such that $x \in B$, then $B \cap A \neq \emptyset$ and $B \cap A^c \neq \emptyset$.

An open ball $\mathcal{B}_r^0(c)$ is *tangent* to ∂A at a point $x \in \partial A$ if $\partial A \cap \partial B_r^0(c) = \{x\}$. An open ball $\mathcal{B}_r^0(c)$ is an *osculating open ball of radius r to ∂A at point $x \in \partial A$* if $\mathcal{B}_r^0(c)$ is tangent to ∂A at x and either $\mathcal{B}_r^0(c) \subseteq A^0$ or $\mathcal{B}_r^0(c) \subseteq (A^c)^0$, where A^0 is a maximal open subset of A, i.e., A without its boundary.

Definition 1 A set $A \subset \mathbb{R}^n$ is called *r-regular* if, for each point $x \in \partial A$, there exist two osculating open balls of radius r to ∂A at x such that one lies entirely in A and the other lies entirely in A^c. Examples illustrating 2D and 3D cases are shown in Fig. 3.

Note that the boundary of a 3D r-regular set is a 2D manifold surface. Any set S which is a translated and rotated version of the set $\frac{2 \cdot r'}{\sqrt{3}}\mathbb{Z}^3$ is called a *cubic r'-grid* and its elements are called *sampling points*. Note that the distance $d(x,p)$ from each point $x \in \mathbb{R}^3$ to the nearest sampling point $s \in S$ is at most r'. The *voxel* $\mathcal{V}_S(s)$ of a sampling point $s \in S$ is its *Voronoi region* $\mathbb{R}^3$: $\mathcal{V}_S(s) = \{x \in \mathbb{R}^3 \mid d(x,s) \leq d(x,q),\ \forall q \in S\}$, i.e., $\mathcal{V}_S(s)$ is the set of all points of $\mathbb{R}^3$ which are at least as close to s as to any other point in S. In particular, note that $\mathcal{V}_S(s)$ is a cube whose vertices lie on a sphere of radius r' and center s.

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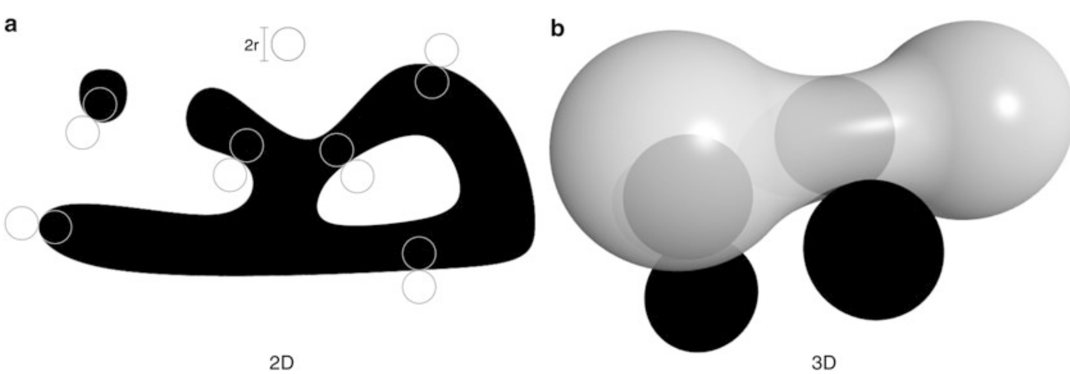
Definition 2 Let S be a cubic r'-grid, and let A be any subset of $\mathbb{R}^3$. The union of all voxels with sampling points lying in A is the *digital reconstruction of A with respect to S*, $\hat{A} = \bigcup_{s \in (S \cap A)} \mathcal{V}_S(s)$.

This method for reconstructing the object from the set of included sampling points is the 3D generalization of the 2D *Gauss digitization* (see [13]) which has been used by Gauss to compute the area of discs and which has also been used by [1] in his sampling theorem.

For any two points p and q of S, $\mathcal{V}_S(p) \cap \mathcal{V}_S(q)$ is either empty or a common vertex, edge, or face of both. If $\mathcal{V}_S(p) \cap \mathcal{V}_S(q)$ is a common face, edge, or vertex, then $\mathcal{V}_S(p)$ and $\mathcal{V}_S(q)$ are *face-adjacent*, *edge-adjacent*, or *vertex-adjacent*, respectively. Two voxels $\mathcal{V}_S(p)$ and $\mathcal{V}_S(q)$ of $\hat{A}$ are *connected in $\hat{A}$* if there exists a sequence $\mathcal{V}_S(s_1), \ldots, \mathcal{V}_S(s_k)$, with $k \in \mathbb{Z}$ and $k > 1$, such that $s_1 = p$, $s_k = q$, and $s_i \in A$ (or equivalently, $\mathcal{V}_S(s_i) \subset \hat{A}$), for each $i \in \{1, \ldots, k\}$, and $\mathcal{V}_S(s_j)$ and $\mathcal{V}_S(s_{j+1})$ are face-adjacent, for each $j \in \{1, \ldots, k-1\}$. A (connected) *component* of $\hat{A}$ is a maximal set of connected voxels in $\hat{A}$.



Digitization, Fig. 3 For each boundary point of a **2D/3D**, r-regular set exists an outside and an inside osculating open r-disc/ball

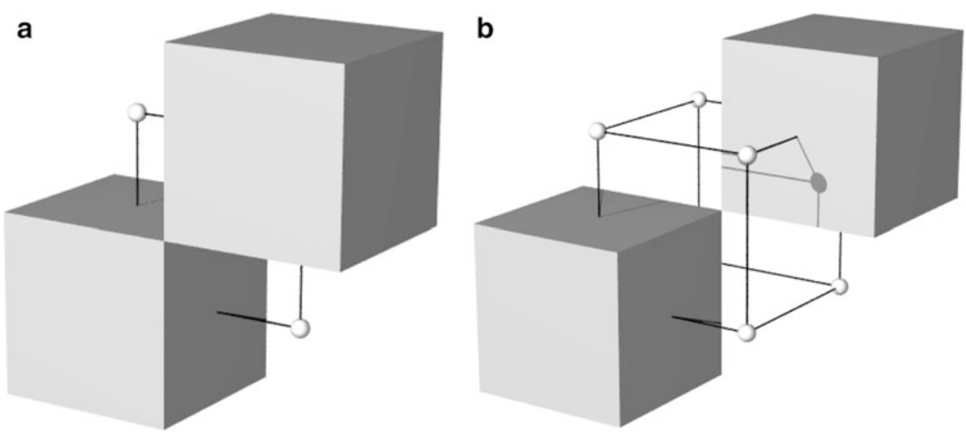
Definition 3 Let S be a cubic r'-grid, and let T be any subset of S. Then $\bigcup_{t \in T} \mathcal{V}_S(t)$ is *well-composed* if $\partial(\bigcup_{t \in T} \mathcal{V}_S(t))$ is a surface in $\mathbb{R}^3$ or, equivalently, if for every point $x \in \partial(\bigcup_{t \in T} \mathcal{V}_S(t))$, there exists a positive number r such that the intersection of $\partial(\bigcup_{t \in T} \mathcal{V}_S(t))$ and $\mathcal{B}_r^0(x)$ is homeomorphic to the open unit disk in $\mathbb{R}^2$, $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$.

Well-composed digital reconstructions can be characterized by two local conditions depending only on voxels of points of S. Let $s_1, \ldots, s_4$ be any four points of S such that $\bigcap_{i=1}^4 \mathcal{V}_S(s_i)$ is a common edge of $\mathcal{V}_S(s_1), \ldots, \mathcal{V}_S(s_4)$. The set $\{\mathcal{V}_S(s_1), \ldots, \mathcal{V}_S(s_4)\}$ is an instance of the *critical configuration (C1) with respect to* $\bigcup_{t \in T} \mathcal{V}_S(t)$ if two of these voxels are in $\bigcup_{t \in T} \mathcal{V}_S(t)$ and the other two are in $(\bigcup_{t \in T} \mathcal{V}_S(t))^c$ and the two voxels in $\bigcup_{t \in T} \mathcal{V}_S(t)$ (resp. $(\bigcup_{t \in T} \mathcal{V}_S(t))^c$) are edge-adjacent, as shown in Fig. 4a. Now, let $s_1, \ldots, s_8$ be any eight points of S such that $\bigcap_{i=1}^8 \mathcal{V}_S(s_i)$ is a common vertex of $\mathcal{V}_S(s_1), \ldots, \mathcal{V}_S(s_8)$. The set $\{\mathcal{V}_S(s_1), \ldots, \mathcal{V}_S(s_4)\}$ is an instance of the *critical configuration (C2) with respect to* $\bigcup_{t \in T} \mathcal{V}_S(t)$ if two of these voxels are in $\bigcup_{t \in T} \mathcal{V}_S(t)$ (resp. $(\bigcup_{t \in T} \mathcal{V}_S(t))^c$) and the other six are in $(\bigcup_{t \in T} \mathcal{V}_S(t))^c$ (resp. $\bigcup_{t \in T} \mathcal{V}_S(t)$) and the two voxels in $\bigcup_{t \in T} \mathcal{V}_S(t)$ (resp. $(\bigcup_{t \in T} \mathcal{V}_S(t))^c$) are vertex-adjacent, as shown in Fig. 4b. The following theorem from [15] establishes an important equivalence between well-composedness and the (non)existence of critical configurations (C1) and (C2).

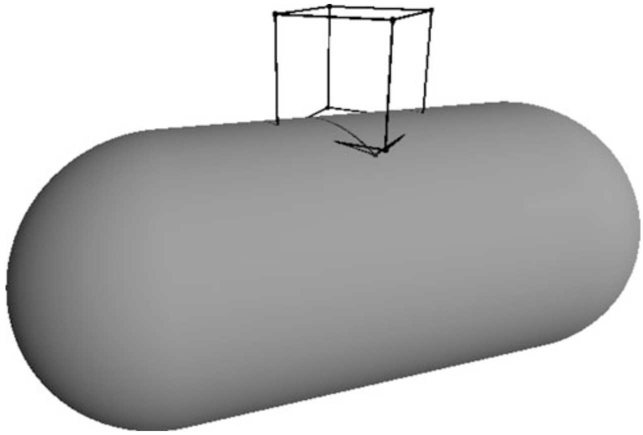
Theorem 1 ([15]) *Let S be a cubic r0-grid and let T be any subset of S. Then,* $\bigcup_{t \in T} \mathcal{V}_S(t)$ *is well-composed if the set of voxels* $\{\mathcal{V}(s) | s \in S\}$ *does not contain any instance of the critical configuration (C1) nor any instance of the critical configuration (C2) with respect to* $\bigcup_{t \in T} \mathcal{V}_S(t)$.

A simple consequence of the 2D digitization theorem by [1] is that the reconstruction of an r'-regular set is well-composed. The main difficulty of 3D digitization as compared to 2D lies in the fact that digital reconstruction $\hat{A}$ of A with respect to S is not guaranteed to be well-composed. An example is provided in Fig. 5. Therefore, it is necessary to repair $\hat{A}$ in order to ensure the topological equivalence between A and repaired $\hat{A}$. The first topology-preserving repairing method has been proposed in [7], where also the following theorem is proven. It is an interesting observation that it took 25 years to obtain this 3D theorem.

Theorem 2 ([7]) *If A is an r-regular object and S is a cubic* r'*-grid with* $2r' < r$*, then the result of the topology-preserving repairing method of the reconstruction* $\hat{A}$ *is r-homeomorphic to A.*



Digitization, Fig. 4 (**a**) Critical configuration (C1). (**b**) Critical configuration (C2). For the sake of clarity, only the voxels of foreground or background points are shown



Digitization, Fig. 5 The surface of an object only needs to have an arbitrarily small but nonzero curvature in order to make occurrences of the critical configuration (C1) possible in the digital reconstruction

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Application

A complete understanding of the loss of information due to the digitization process is fundamental for the justification of any computer vision application. If the relevant information is not contained in the digital image, there is no way to reconstruct it without using context knowledge. Thus, whenever one needs to have guarantees for the correct behavior of some computer vision algorithm, one has to be aware of what happens during digitization. This entry gives an exemplary insight to the topic, the related problems, and the way to solve them.

Open Problems

The analysis of the effect of digitization to the information being extractable from an image is a challenging research area. The newest results approximate real acquisition processes and thus give direct implications for many computer vision algorithms which rely on precise information of the structures being approximated by the digital image. However, in reality the digitization process is still more complicated than the models which are used for topological or geometric sampling theorems. The goal is to derive guarantees for digitization models approximating real digitization processes.

For the case $n \geq 3$, the equivalences between the different definitions of well-composedness (EWC, DWC, AWC, EWC, AGWC) is an open problem together with a general method for repairing non-well-composed digital sets in $\mathbb{Z}^n$. Besides, the study of which properties well-composed images own in $\mathbb{Z}^n$ that reflect the continuous world is a promising line of research, such as the link between critical points and Morse theory [21] or topological persistence [22] and tree of shapes. A more exhaustive list can be consulted in [19, Section 10].

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