# Linear Algebra Review 

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## Good Review Materials

http://www.imageprocessingbook.com/DIP2E/dip2e downloads/review material downloads.htm
(Gonzales \& Woods review materials)

Chapt. 1: Linear Algebra Review
Chapt. 2: Probability, Random Variables, Random Vectors


$$
u+v=\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]+\left[\begin{array}{l}
v_{1} \\
v_{2}
\end{array}\right]=\left[\begin{array}{l}
u_{1}+v_{1} \\
u_{2}+v_{2}
\end{array}\right]
$$

## Vector Subtraction

$u-v=\left[\begin{array}{l}u_{1} \\ u_{2}\end{array}\right]-\left[\begin{array}{l}v_{1} \\ v_{2}\end{array}\right]=\left[\begin{array}{l}u_{1}-v_{1} \\ u_{2}-v_{2}\end{array}\right]$


## Scaling a vector

$$
\boldsymbol{z}=\alpha \boldsymbol{x}
$$

for a scalar $\alpha$ then

$$
\boldsymbol{z}=\alpha\left(\begin{array}{l}
3 \\
2 \\
5
\end{array}\right)=\left(\begin{array}{c}
3 \alpha \\
5 \alpha \\
2 \alpha
\end{array}\right)
$$

(This is just like stretching/shrinking the vector by a factor $\alpha$

Example (on board)

## Inner product (dot product) of two vectors

$$
\begin{aligned}
a=\left[\begin{array}{c}
6 \\
2 \\
-3
\end{array}\right] \quad b & =\left[\begin{array}{l}
4 \\
1 \\
5
\end{array}\right] \\
a \cdot b & =a^{T} b \\
& =\left[\begin{array}{lll}
6 & 2 & -3
\end{array}\right]\left[\begin{array}{c}
4 \\
1 \\
5
\end{array}\right] \\
& =6 \cdot 4+2 \cdot 1+(-3) \cdot 5 \\
& =11
\end{aligned}
$$

## Inner (dot) Product



## Matrices

A matrix is an $n \times M$ element array

$$
A=\left[\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1 m} \\
a_{21} & & & \\
\cdot & & & \\
\cdot & & & \\
\cdot & & & \\
a_{N 1} & \ldots & \ldots & a_{N M}
\end{array}\right]
$$

transpose of matrix A (written $A^{T}$ ) is $a_{j i}$ ( A with rows and columns flipped)

## Transpose of a Matrix

Transpose:

$$
\begin{aligned}
C_{m \times n} & =A^{T}{ }_{n \times m} & (A+B)^{T} & =A^{T}+B^{T} \\
c_{i j} & =a_{j i} & (A B)^{T} & =B^{T} A^{T}
\end{aligned}
$$

Examples:

$$
\left[\begin{array}{ll}
6 & 2 \\
1 & 5
\end{array}\right]^{T}=\left[\begin{array}{ll}
6 & 1 \\
2 & 5
\end{array}\right] \quad\left[\begin{array}{ll}
6 & 2 \\
1 & 5 \\
3 & 8
\end{array}\right]^{T}=\left[\begin{array}{lll}
6 & 1 & 3 \\
2 & 5 & 8
\end{array}\right]
$$

If $A^{T}=A$, we say $\mathbf{A}$ is symmetric.
Example of symmetric matrix

## Transpose in Matlab

In Matlab use $A^{\prime}$
$\gg A=\left[\begin{array}{llll}1 & 2 & ; & 4\end{array}\right]$
$\mathrm{A}=$

| 1 | 2 |
| :--- | :--- |
| 3 | 4 |

> $A^{\prime}$
ans $=$
13
24

A square matrix is symmetric iff $A=A^{T}$

## Matrix Multiplication Example

$$
\left[\begin{array}{lll}
1 & 2 & 3 \\
3 & 2 & 1
\end{array}\right]\left[\begin{array}{cc}
4 & 5 \\
10 & 2 \\
2 & 10
\end{array}\right]=\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right]
$$

## Matrix Multiplication Example

$\left[\begin{array}{lll}1 & 2 & 3 \\ 3 & 2 & 1\end{array}\right]\left[\begin{array}{cc}4 & 5 \\ 10 & 2 \\ 2 & 10\end{array}\right]=\left[\begin{array}{ll}a_{11} & a_{12} \\ a_{21} & a_{22}\end{array}\right]$ $a_{11}=1 * 4+$

## Matrix Multiplication Example

$$
\begin{gathered}
{\left[\begin{array}{lll}
1 & 2 & 3 \\
3 & 2 & 1
\end{array}\right]\left[\begin{array}{cc}
4 & 5 \\
10 & 2 \\
2 & 10
\end{array}\right]=\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right]} \\
a_{11}=1 * 4+2 * 10+
\end{gathered}
$$

> Matrix Multiplication Example
> $\left[\begin{array}{lll}1 & 2 & 3 \\ 3 & 2 & 1\end{array}\right]\left[\begin{array}{cc}4 & 5 \\ 10 & 2 \\ 2 & 10\end{array}\right]=\left[\begin{array}{cc}30 & a_{12} \\ a_{21} & a_{22}\end{array}\right]$ $a_{11}=1 * 4+2 * 10+3 * 2=30$

## Matrix Multiplication Example

$$
\begin{gathered}
{\left[\begin{array}{lll}
1 & 2 & 3 \\
3 & 2 & 1
\end{array}\right]\left[\begin{array}{cc}
4 & 5 \\
10 & 2 \\
2 & 10
\end{array}\right]=\left[\begin{array}{cc}
30 & 39 \\
a_{21} & a_{22}
\end{array}\right]} \\
a_{12}=1 * 5+2 * 2+3 * 10=39
\end{gathered}
$$

## Matrix Multiplication Example

$\left[\begin{array}{lll}1 & 2 & 3 \\ 3 & 2 & 1\end{array}\right]\left[\begin{array}{cc}4 & 5 \\ 10 & 2 \\ 2 & 10\end{array}\right]=\left[\begin{array}{cc}30 & 39 \\ 34 & a_{22}\end{array}\right]$ $a_{12}=3 * 4+2 * 10+1 * 2=34$

$$
\begin{aligned}
& \text { Matrix Multiplication Example } \\
& {\left[\begin{array}{lll}
1 & 2 & 3 \\
3 & 2 & 1
\end{array}\right]\left[\begin{array}{cc}
4 & 5 \\
10 & 2 \\
2 & 10
\end{array}\right]=\left[\begin{array}{ll}
30 & 39 \\
34 & 29
\end{array}\right]} \\
& a_{12}=3 * 5+2 * 2+1 * 10=29
\end{aligned}
$$

## Matrix Product

Product:

$$
\begin{array}{cl}
C_{n \times p}=A_{n \times m} B_{m \times p} & \begin{array}{l}
\text { A and B must have } \\
\text { compatible dimensions }
\end{array} \\
c_{i j}=\sum_{k=1}^{m} a_{i k} b_{k j} & \text { In Matlab: >> A*B }
\end{array}
$$

Examples:
$\left[\begin{array}{ll}2 & 5 \\ 3 & 1\end{array}\right] \cdot\left[\begin{array}{ll}6 & 2 \\ 1 & 5\end{array}\right]=\left[\begin{array}{ll}17 & 29 \\ 19 & 11\end{array}\right] \quad\left[\begin{array}{ll}6 & 2 \\ 1 & 5\end{array}\right] \cdot\left[\begin{array}{ll}2 & 5 \\ 3 & 1\end{array}\right]=\left[\begin{array}{ll}18 & 32 \\ 17 & 10\end{array}\right]$
Matrix Multiplication is not commutative:

$$
A_{n \times n} B_{n \times n} \neq B_{n \times n} A_{n \times n}
$$

## Matrix Sum

Sum:

$$
A_{n \times m}=\left[\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1 m} \\
a_{21} & a_{22} & \ldots & a_{2 m} \\
& \ldots & & \\
a_{n 1} & a_{n 2} & \ldots & a_{n m}
\end{array}\right]^{\text {Sum: }} \begin{gathered}
C_{n \times m}=A_{n \times m}+B_{n \times m} \\
c_{i j}=a_{i j}+b_{i j}
\end{gathered}
$$

$A$ and $B$ must have the same dimensions
Example:

$$
\left[\begin{array}{ll}
2 & 5 \\
3 & 1
\end{array}\right]+\left[\begin{array}{ll}
6 & 2 \\
1 & 5
\end{array}\right]=\left[\begin{array}{ll}
8 & 7 \\
4 & 6
\end{array}\right]
$$

## Determinant of a Matrix

Determinant: A must be square
$\operatorname{det}\left[\begin{array}{ll}a_{11} & a_{12} \\ a_{21} & a_{22}\end{array}\right]=\left|\begin{array}{ll}a_{11} & a_{12} \\ a_{21} & a_{22}\end{array}\right|=a_{11} a_{22}-a_{21} a_{12}$
$\operatorname{det}\left[\begin{array}{lll}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{array}\right]=a_{11}\left|\begin{array}{ll}a_{22} & a_{23} \\ a_{32} & a_{33}\end{array}\right|-a_{12}\left|\begin{array}{ll}a_{21} & a_{23} \\ a_{31} & a_{33}\end{array}\right|+a_{13}\left|\begin{array}{ll}a_{21} & a_{22} \\ a_{31} & a_{32}\end{array}\right|$
Example: $\quad \operatorname{det}\left[\begin{array}{ll}2 & 5 \\ 3 & 1\end{array}\right]=2-15=-13$

## Determinant in Matlab

```
In matlab use det(A)
>>A=[[llll}12; 3 4] 
A =
    1 2
>> det(A)
ans =
-2
```


## Inverse of a Matrix

If A is a square matrix, the inverse of $A$, called $A^{-1}$, satisfies

$$
A A^{-1}=\mathrm{I} \quad \text { and } \quad A^{-1} A=\mathrm{I},
$$

Where $I$, the identity matrix, is a diagonal matrix with all 1's on the diagonal.

$$
I_{2}=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] \quad I_{3}=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

## Inverse of a 2D Matrix

For a 2-D matrix

$$
\text { if } \begin{aligned}
A & =\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \\
A^{-1} & =\frac{\left[\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right]}{|A|}
\end{aligned}
$$

Example: $\quad\left[\begin{array}{ll}6 & 2 \\ 1 & 5\end{array}\right]^{-1}=\frac{1}{28}\left[\begin{array}{cc}5 & -2 \\ -1 & 6\end{array}\right]$

$$
\left[\begin{array}{ll}
6 & 2 \\
1 & 5
\end{array}\right]^{-1} \cdot\left[\begin{array}{ll}
6 & 2 \\
1 & 5
\end{array}\right]=\frac{1}{28}\left[\begin{array}{cc}
5 & -2 \\
-1 & 6
\end{array}\right] \cdot\left[\begin{array}{cc}
6 & 2 \\
1 & 5
\end{array}\right]=\frac{1}{28}\left[\begin{array}{cc}
28 & 0 \\
0 & 28
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]
$$

## Inverses in Matlab

```
in Matlab use inv(A)
> A = [1 2 ; 3 4]
A =
    1 2
    3 4
>> inv(A)
ans =
\begin{tabular}{rr}
-2.0000 & 1.0000 \\
1.5000 & -0.5000
\end{tabular}
```


## Other Terms

trace of a Matrix is $\operatorname{Tr}(A)=$

$$
\sum_{i=1}^{N} a_{i i}
$$

(the sum of the diagonal entries)
In matlab use trace $(A)$
a matrix $A$ is orthonormal if

$$
A^{T}=A^{-1}
$$

and in this case

$$
A A^{T}=I
$$

## Matrix Transformation

Multiplying a vector $x$ by a matrix $y=A x$ transforms $x$ to a new vector $y$ (which may have a different number of dimensions if $A$ is not square). We can talk about different properties that the transformation given by A represents.
rotation matrix is given by

$$
A=\left[\begin{array}{cc}
\cos (\theta) & -\sin (\theta) \\
\sin (\theta) & \cos (\theta)
\end{array}\right]
$$

So to rotate vector

$$
\binom{1}{0}
$$

by 30 deg we multiply

$$
\left[\begin{array}{cc}
.8660 & -.5 \\
.5 & .8660
\end{array}\right]\binom{1}{0}=\binom{.8660}{.5}
$$



## Matrix Transformation: Scale

A square diagonal matrix scales each dimension by the corresponding diagonal element.

Example:

$$
\left[\begin{array}{lll}
2 & 0 & 0 \\
0 & .5 & 0 \\
0 & 0 & 3
\end{array}\right]\left[\begin{array}{c}
6 \\
8 \\
10
\end{array}\right]=\left[\begin{array}{c}
12 \\
4 \\
30
\end{array}\right]
$$

## Eigenvalues and Eigenvectors

When a Matrix multiplies a vector in general the direction and magnitude of the vector will change.

BUT there are special vectors where only the magnitude changes (on multiplication by the Matrix). These are called eigenvectors The value by which the length changes is the associated eigenvalue

We say that $x$ is an eigenvector of A iff

$$
A x=\lambda x
$$

In other words, $x$ is an eigenvector if when you multiply it by $A$ it returns a multiple of itself. $\lambda$ is called the associated eigenvalue.
on-line demo
http://www.math.ubc.ca/~cass/courses/m309-8a/java/m309gfx/eigen.html

## Eigenvalues and Eigenvectors in Matlab

In Matlab use $[\mathrm{V}, \mathrm{D}]=\operatorname{eig}(\mathrm{A})$ to get a matrix V whose columns are the eigenvectors of A and a diagonal matrix D whose entries on the diagonal are the corresponding eigenvalues.
>> A
$A=$

12
34
>> [V,D] = eig(A)
$\mathrm{V}=$
$\begin{array}{rr}-0.8246 & -0.4160 \\ 0.5658 & -0.9094\end{array}$

D $=$
$-0.3723 \quad 0$

## Some Properties of Eigenvalues and Eigenvectors

- If $\lambda_{1}, \ldots, \lambda_{n}$ are distinct eigenvalues of a matrix, then the corresponding eigenvectors $e_{1}, \ldots, e_{n}$ are linearly independent.
- A real, symmetric square matrix has real eigenvalues, with eigenvectors that can be chosen to be orthonormal.


## Linear Independence

- A set of vectors is linearly dependent if one of the vectors can be expressed as a linear combination of the other vectors.
Example:

$$
\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right],\left[\begin{array}{l}
2 \\
1 \\
0
\end{array}\right]
$$

- A set of vectors is linearly independent if none of the vectors can be expressed as a linear combination of the other vectors.
Example:

$$
\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right],\left[\begin{array}{l}
2 \\
1 \\
3
\end{array}\right]
$$

## Rank of a matrix

- The rank of a matrix is the number of linearly independent columns of the matrix.
Examples:

$$
\left[\begin{array}{lll}
1 & 0 & 2 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{array}\right] \quad \text { has rank } 2
$$

$$
\left[\begin{array}{ccc}
1 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right] \quad \text { has rank } 3
$$

- Note: the rank of a matrix is also the number of linearly independent rows of the matrix.


## Singular Matrix

All of the following conditions are equivalent. We say a square $(n \times n)$ matrix is singular if any one of these conditions (and hence all of them) is satisfied.

- The columns are linearly dependent
- The rows are linearly dependent
- The determinant $=0$
- The matrix is not invertible
- The matrix is not full rank (i.e., rank $<n$ )


## Linear Spaces

A linear space is the set of all vectors that can be expressed as a linear combination of a set of basis vectors. We say this space is the span of the basis vectors.

- Example: $\mathbf{R}^{3}$, 3-dimensional Euclidean space, is spanned by each of the following two bases:

$$
\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right] \quad\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
1 \\
2
\end{array}\right],\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]
$$

## Linear Subspaces

A linear subspace is the space spanned by a subset of the vectors in a linear space.

- The space spanned by the following vectors is a two-dimensional subspace of $\mathbf{R}^{3}$.

$$
\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right] \quad \text { What does it look like? }
$$

- The space spanned by the following vectors is a two-dimensional subspace of $\mathbf{R}^{3}$.

$$
\left[\begin{array}{l}
1 \\
1 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right] \quad \text { What does it look like? }
$$

## Orthogonal and Orthonormal Bases

$n$ linearly independent real vectors
span $\mathbf{R}^{n}$, $n$-dimensional Euclidean space

- They form a basis for the space.
- An orthogonal basis, $a_{1}, \ldots, a_{n}$ satisfies

$$
a_{i} \cdot a_{j}=0 \quad \text { if } i \neq j
$$

- An orthonormal basis, $a_{1}, \ldots, a_{n}$ satisfies

$$
\begin{array}{ll}
a_{i} \cdot a_{j}=0 & \text { if } i \neq j \\
a_{i} \cdot a_{j}=1 & \text { if } i=j
\end{array}
$$

- Examples.


## Orthonormal Matrices

A square matrix is orthonormal (also called unitary) if its columns are orthonormal vectors.

- A matrix $A$ is orthonormal iff $A A^{\mathrm{T}}=I$.
- If $A$ is orthonormal, $A^{-1}=A^{\mathrm{T}}$

$$
A A^{\mathrm{T}}=A^{\mathrm{T}} A=I .
$$

- A rotation matrix is an orthonormal matrix with determinant $=1$.
- It is also possible for an orthonormal matrix to have determinant $=-1$. This is a rotation plus a flip (reflection).


## SVD: Singular Value Decomposition

Any matrix $A(m \times n)$ can be written as the product of three matrices:

$$
A=U D V^{\mathrm{T}}
$$

where

- $U$ is an $m \times m$ orthonormal matrix
- $D$ is an $m \times n$ diagonal matrix. Its diagonal elements, $\sigma_{1}, \sigma_{2}, \ldots$, are called the singular values of A, and satisfy $\sigma_{1} \geq \sigma_{2} \geq \ldots \geq 0$.
- $V$ is an $n \times n$ orthonormal matrix

Example: if $m>n$

$$
\left[\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array}\right]=\left[\begin{array}{cccc}
\uparrow & \uparrow & \uparrow & \\
\mid & \mathrm{I} & \mathrm{I} & \\
u_{1} & u_{2} & u_{3} & \cdots \\
\mathrm{I} & \mathrm{I} & \mathrm{I} & \\
\downarrow & \downarrow & \downarrow & \\
\downarrow & \downarrow
\end{array}\right]\left[\begin{array}{ccc}
\sigma_{1} & 0 & 0 \\
0 & \sigma_{2} & 0 \\
0 & 0 & \sigma_{n} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]\left[\begin{array}{ccc}
\leftarrow & v_{1}^{T} & \rightarrow \\
\vdots & \vdots & \vdots \\
\leftarrow & v_{n}^{T} & \rightarrow
\end{array}\right]
$$

## SVD in Matlab



## Some Properties of SVD

- The rank of matrix $A$ is equal to the number of nonzero singular values $\sigma_{i}$
- A square $(n \times n)$ matrix $A$ is singular iff at least one of its singular values $\sigma_{1}, \ldots, \sigma_{n}$ is zero.


## Geometric Interpretation of SVD

If $A$ is a square $(n \times n)$ matrix,

$$
\left.\begin{array}{cc}
A \\
{\left[\begin{array}{cc}
\bullet & \cdot \\
\bullet & \cdot \\
\bullet & \cdot \\
\cdot & \cdot
\end{array}\right]}
\end{array}\right]=\left[\begin{array}{ccc}
\uparrow & \cdots & \uparrow \\
u_{1} & \cdots & u_{n} \\
\downarrow & \cdots & \downarrow
\end{array}\right]\left[\begin{array}{ccc}
\sigma_{1} & 0 & 0 \\
0 & \sigma_{2} & 0 \\
0 & 0 & \sigma_{n}
\end{array}\right]\left[\begin{array}{ccc}
\leftarrow & v_{1}^{T} & \rightarrow \\
\vdots & \vdots & \vdots \\
\leftarrow & v_{n}^{T} & \rightarrow
\end{array}\right]
$$

- $U$ is a unitary matrix: rotation (possibly plus flip)
- $D$ is a scale matrix
- $V$ (and thus $V^{\mathrm{T}}$ ) is a unitary matrix

Punchline: An arbitrary $n$-D linear transformation is equivalent to a rotation (plus perhaps a flip), followed by a scale transformation, followed by a rotation
Advanced: $y=A x=U D V^{\mathrm{T}} x$

- $V^{\mathrm{T}}$ expresses $x$ in terms of the basis $V$.
- $D$ rescales each coordinate (each dimension)
- The new coordinates are the coordinates of $y$ in terms of the basis $U$

