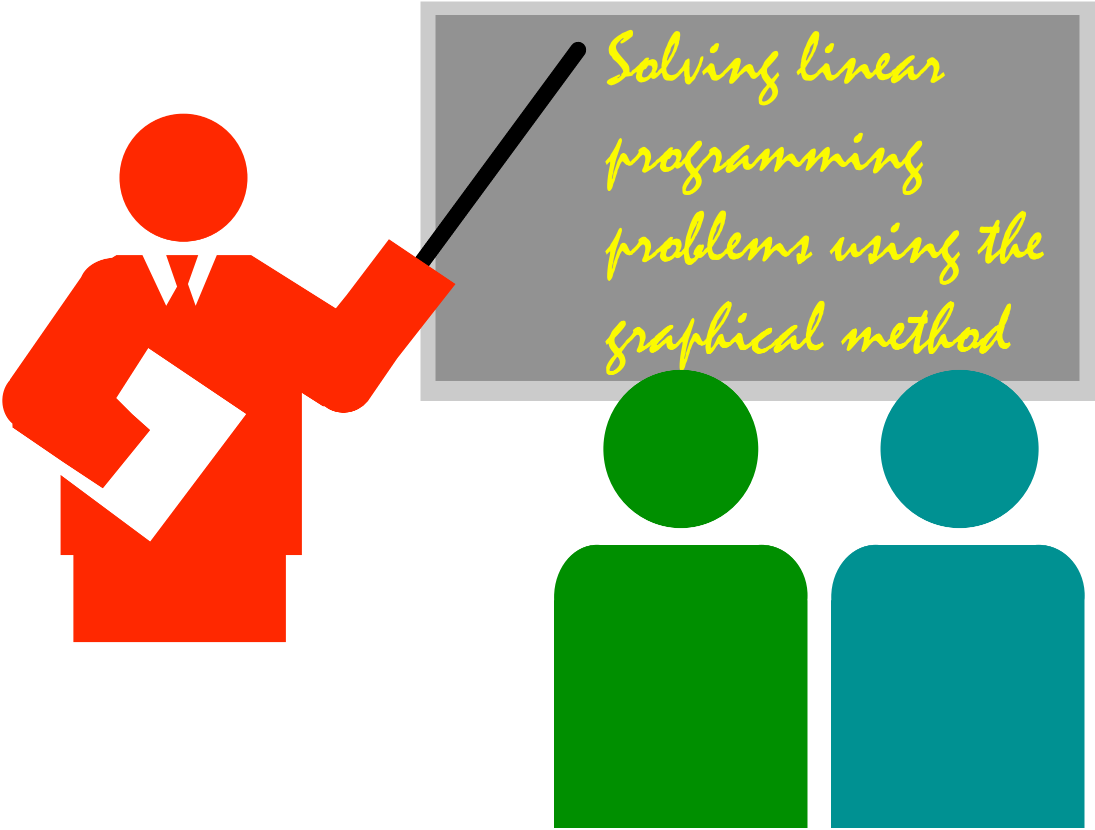

Solving linear programming problems using the graphical method

Example - designing a diet

A dietitian wants to design a breakfast menu for certain hospital patients. The menu is to include two items **A** and **B**. Suppose that each ounce of **A** provides 2 units of vitamin C and 2 units of iron and each ounce of **B** provides 1 unit of vitamin C and 2 units of iron. Suppose the cost of **A** is 4¢/ounce and the cost of **B** is 3¢/ounce. If the breakfast menu must provide at least 8 units of vitamin C and 10 units of iron, how many ounces of each item should be provided in order to meet the iron and vitamin C requirements for the least cost? What will this breakfast cost?

x = #oz. of **A**
y = #oz. of **B**

vit. C: $2x + y \geq 8$
iron: $2x + 2y \geq 10$
$x \geq 0,\ y \geq 0$

Cost = C = $4x+3y$

x = #oz. of **A**

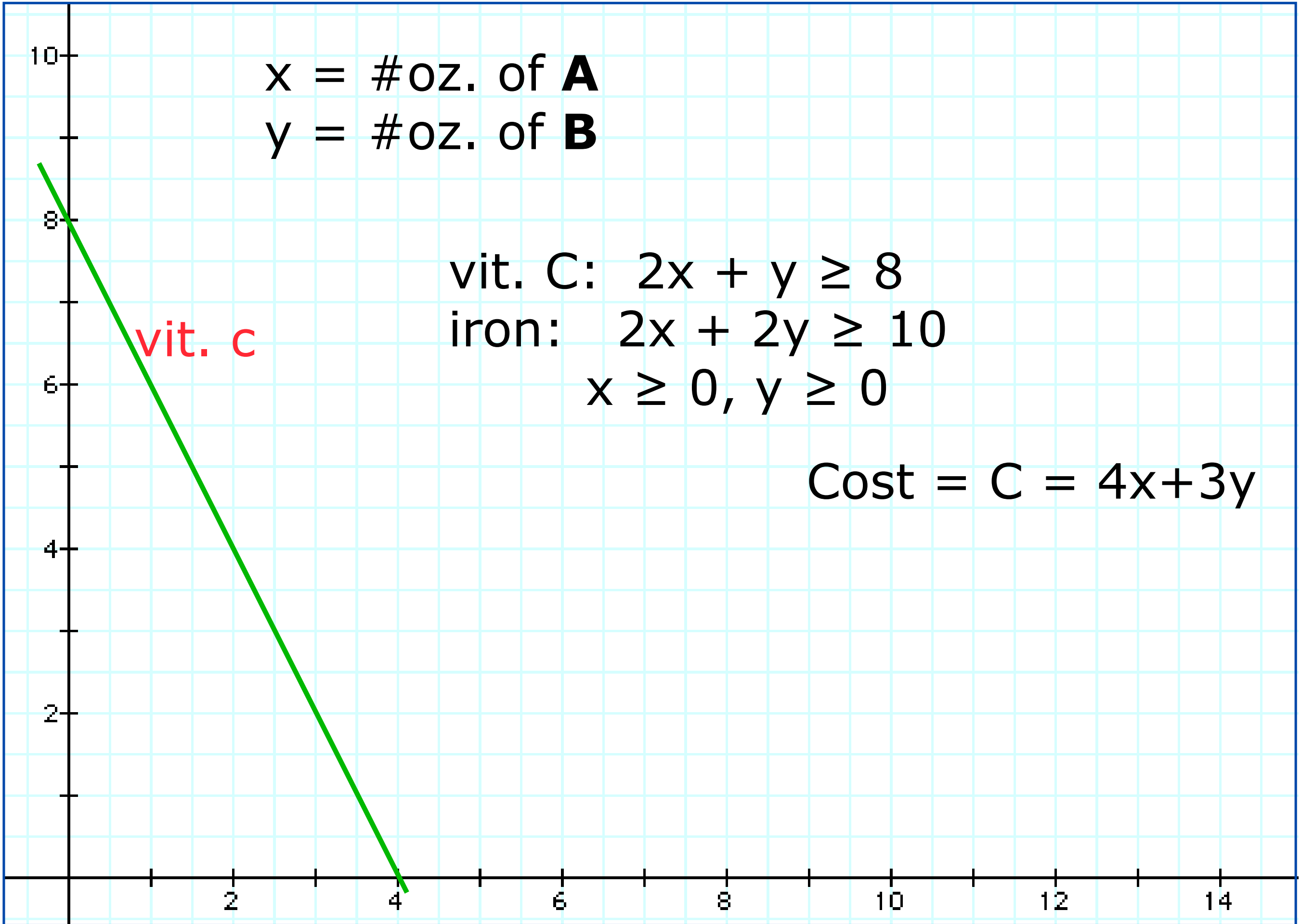
y = #oz. of **B**

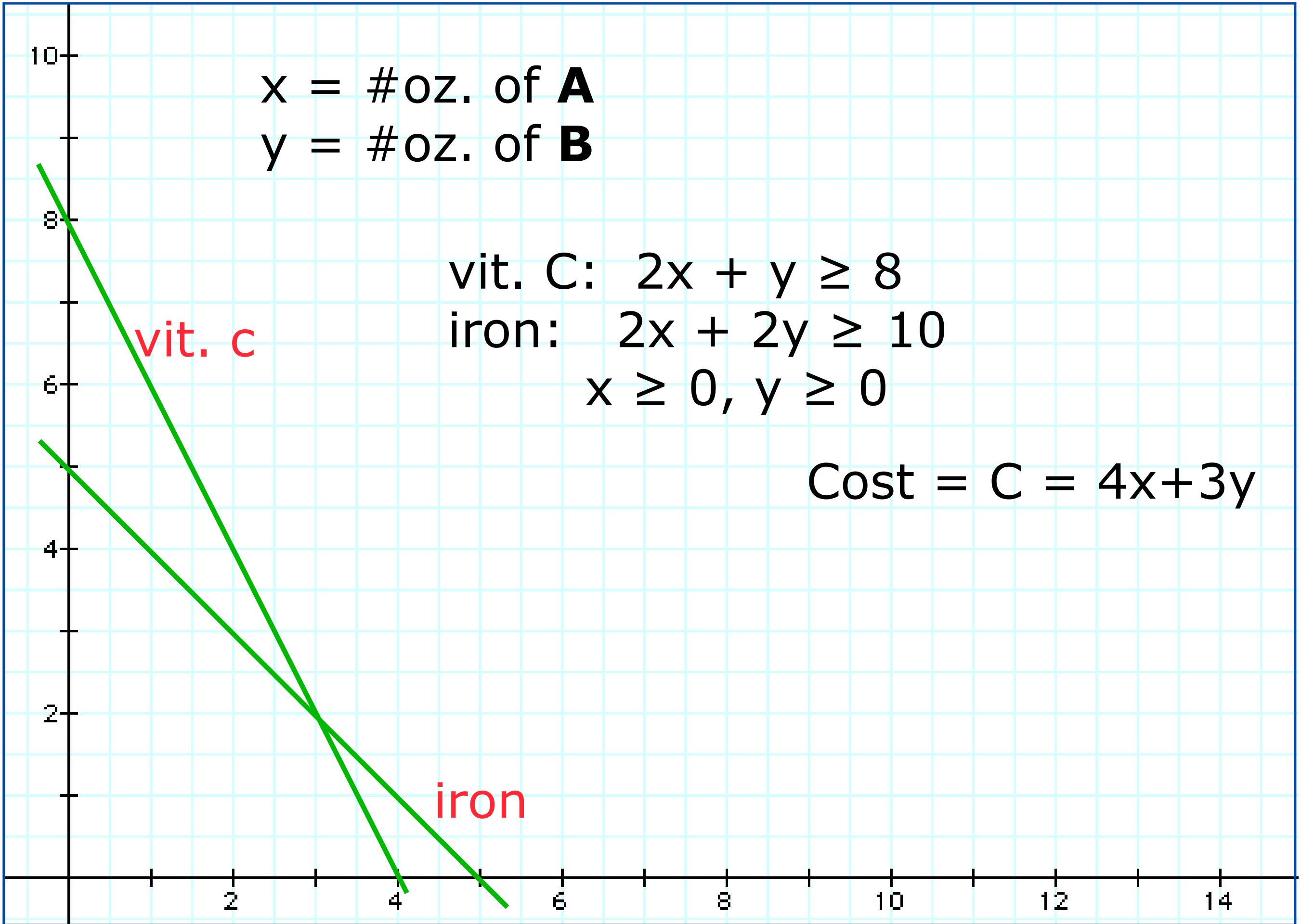
vit. C: $2x + y \geq 8$

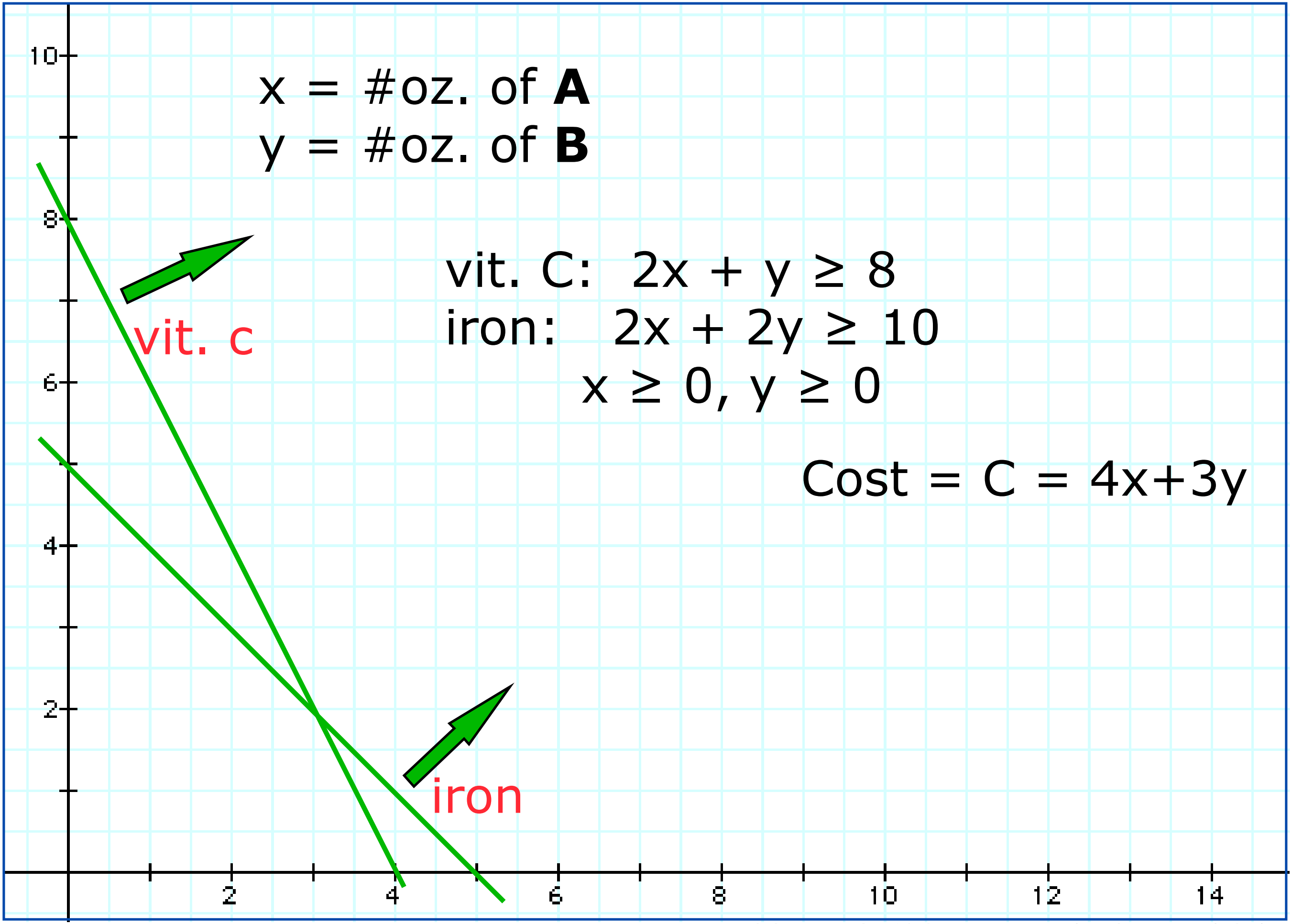
iron: $2x + 2y \geq 10$

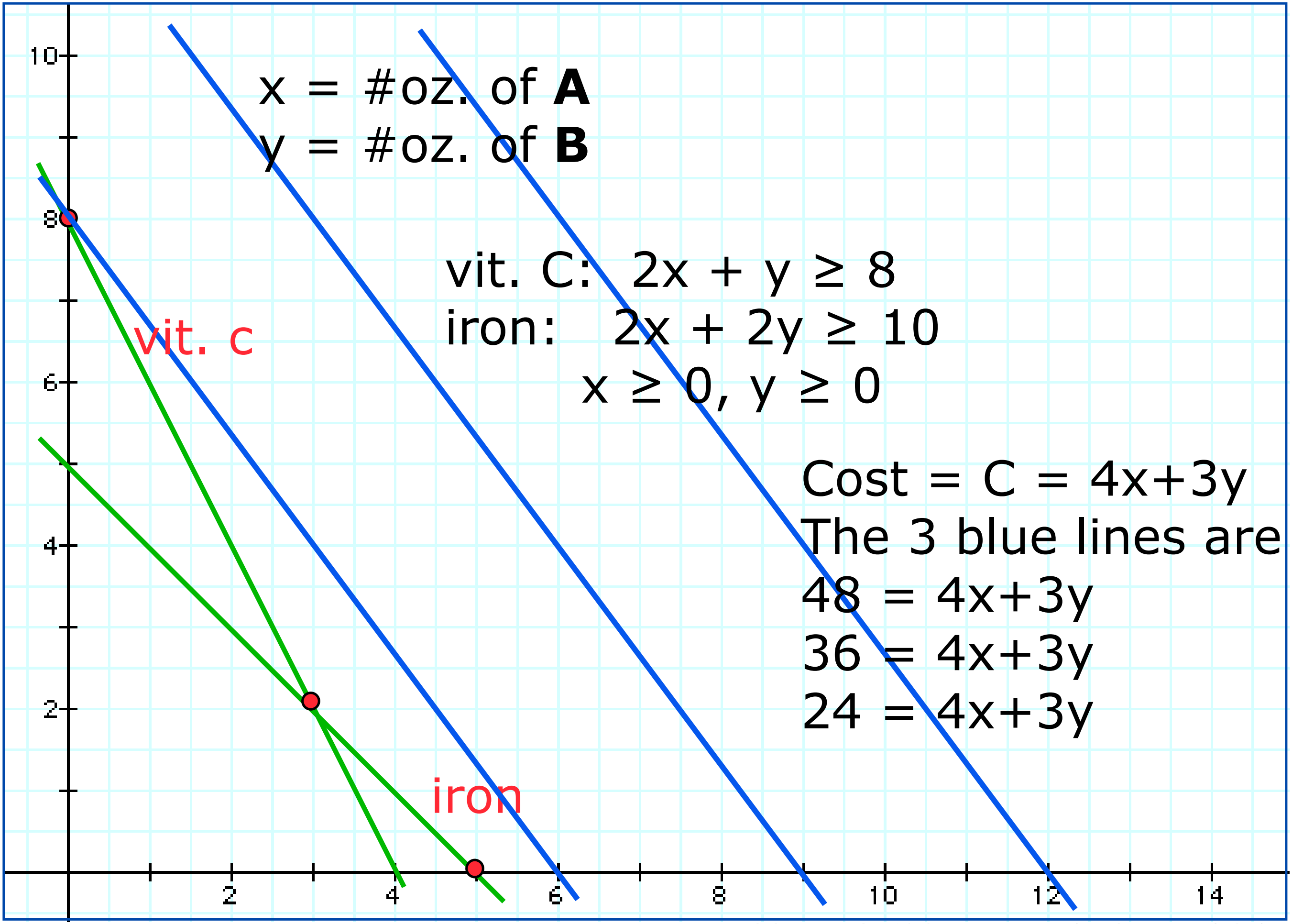
$x \geq 0,\ y \geq 0$

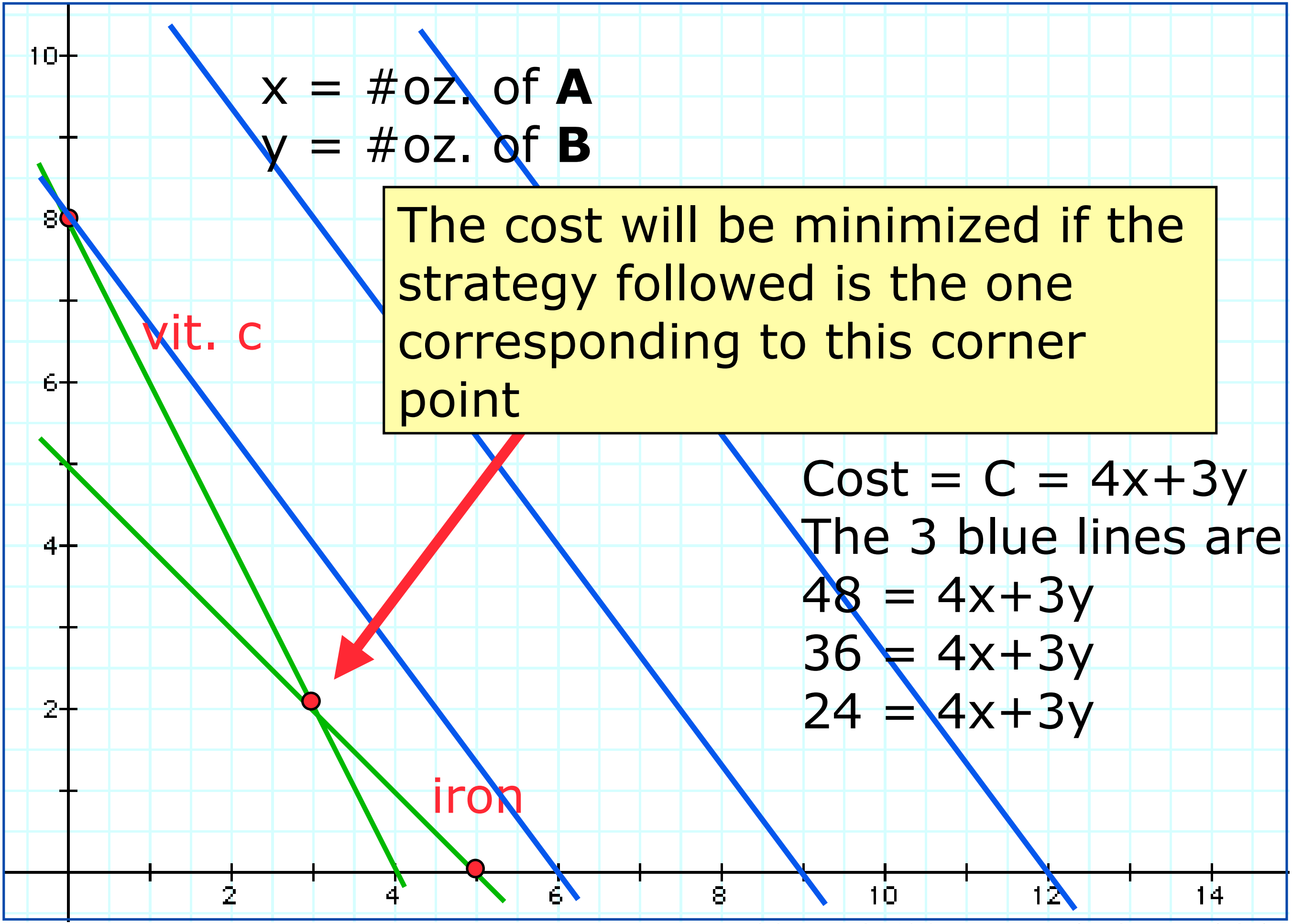
Cost = C = $4x+3y$

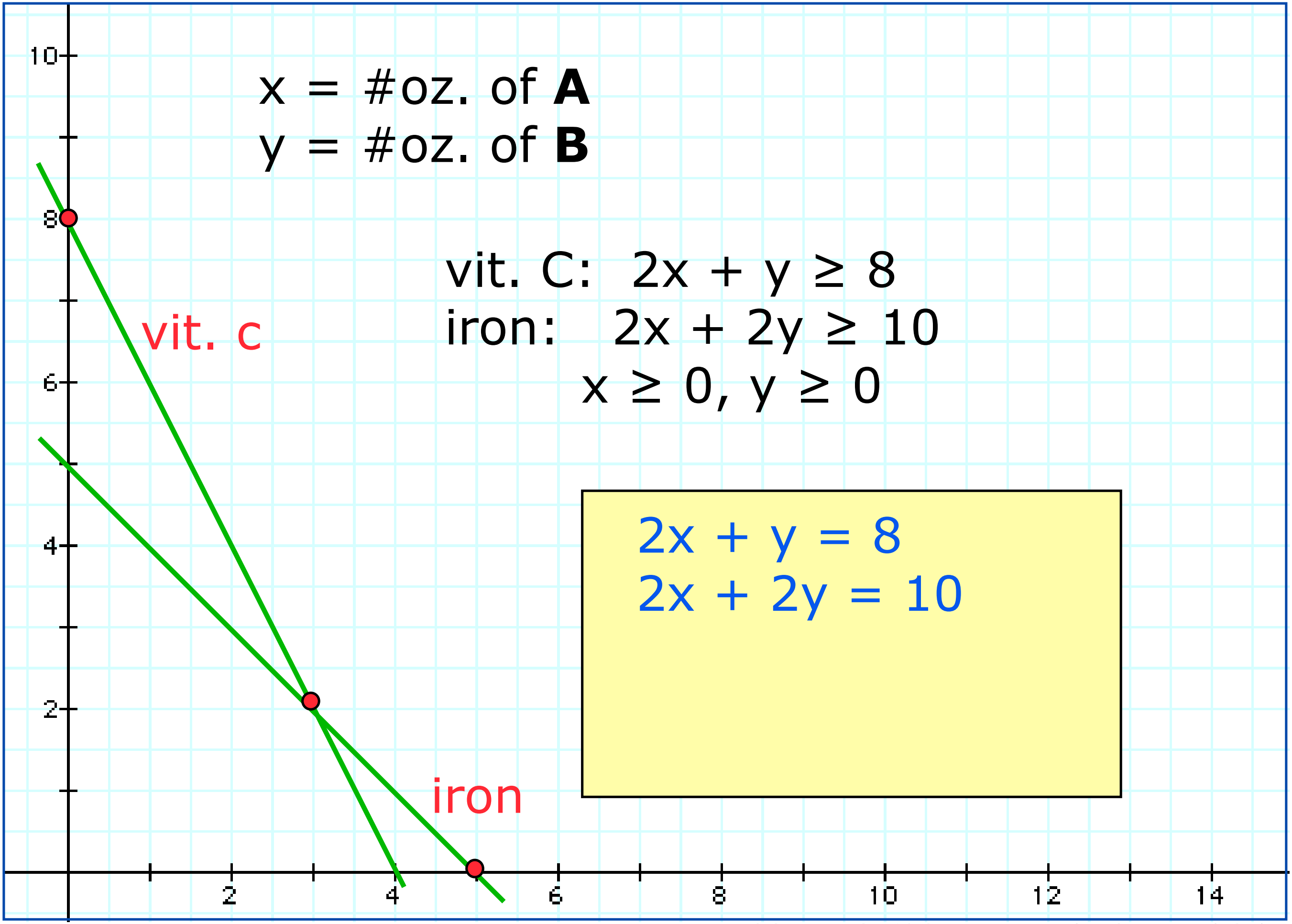



x = #oz. of **A**
y = #oz. of **B**
vit. C: 2x + y ≥ 8
iron: 2x + 2y ≥ 10
x ≥ 0, y ≥ 0
Cost = C = 4x+3y
vit. c
iron
10
8
6
4
2
2
4
6
8
10
12
14


x = #oz. of **A**
y = #oz. of **B**
vit. C: 2x + y ≥ 8
iron: 2x + 2y ≥ 10
x ≥ 0, y ≥ 0
Cost = C = 4x+3y
vit. c
iron
10
8
6
4
2
2
4
6
8
10
12
14

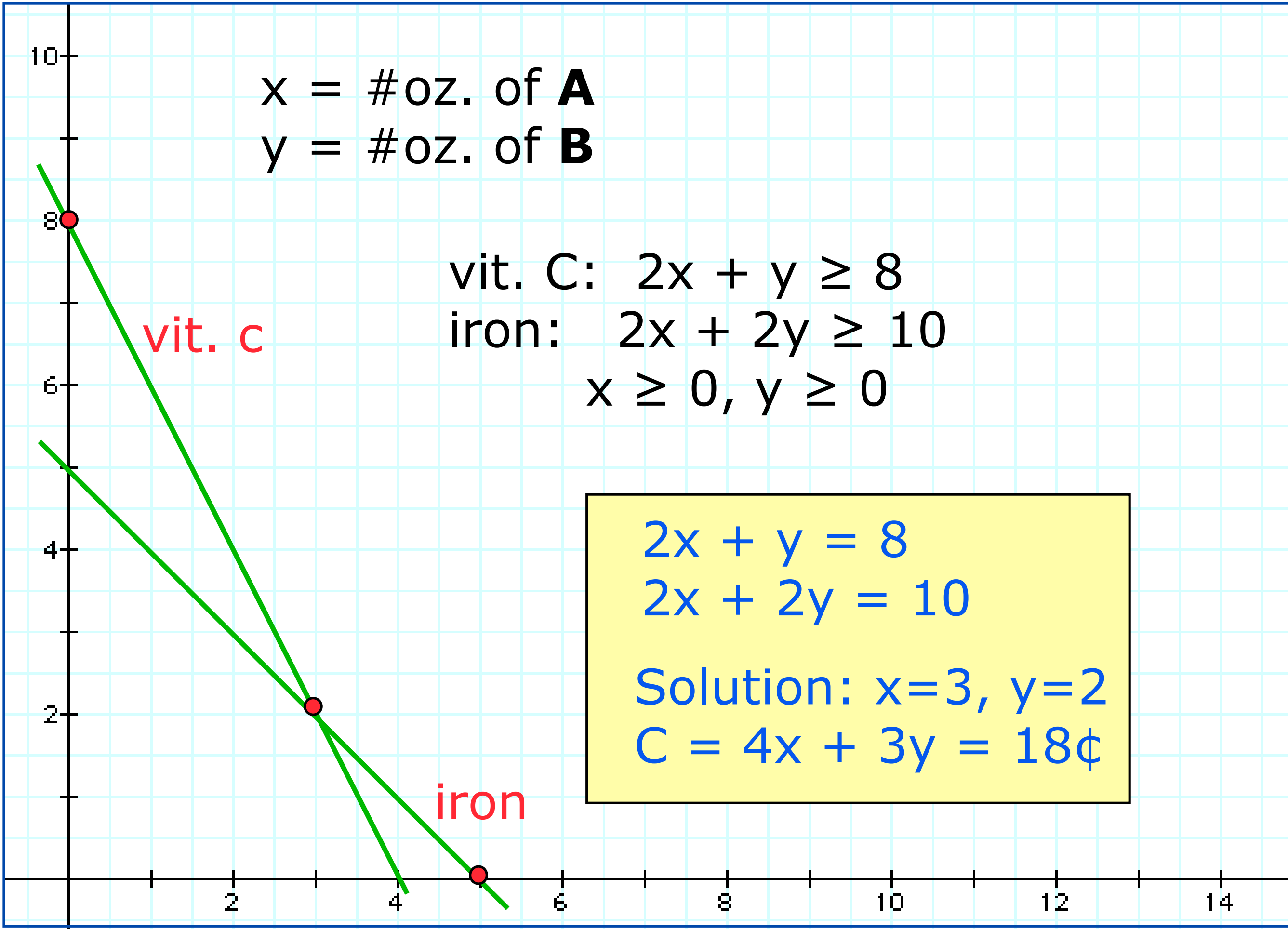

x = #oz. of **A**
y = #oz. of **B**
vit. C: 2x + y ≥ 8
iron: 2x + 2y ≥ 10
x ≥ 0, y ≥ 0
Cost = C = 4x+3y
The 3 blue lines are
48 = 4x+3y
36 = 4x+3y
24 = 4x+3y
vit. c
iron
10
8
6
4
2
2
4
6
8
10
12
14


x = #oz. of **A**
y = #oz. of **B**
The cost will be minimized if the strategy followed is the one corresponding to this corner point
vit. c
iron
Cost = C = 4x+3y
The 3 blue lines are
48 = 4x+3y
36 = 4x+3y
24 = 4x+3y
10
8
6
4
2
2
4
6
8
10
12
14


x = #oz. of **A**
y = #oz. of **B**
vit. C: 2x + y ≥ 8
iron: 2x + 2y ≥ 10
x ≥ 0, y ≥ 0
2x + y = 8
2x + 2y = 10
vit. c
iron
10
8
6
4
2
2
4
6
8
10
12
14

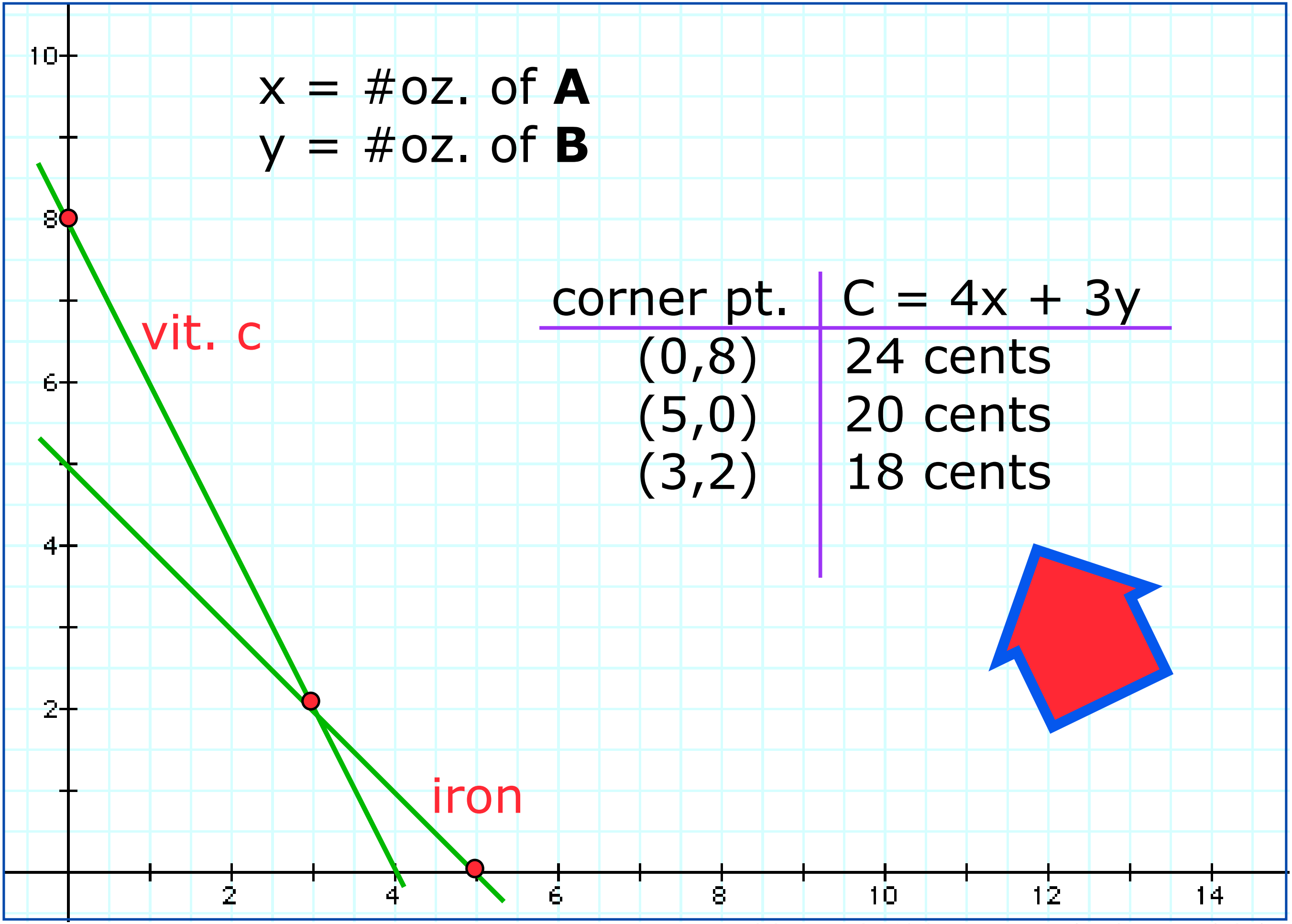
x = #oz. of **A**
y = #oz. of **B**

vit. C: $2x + y \geq 8$
iron: $2x + 2y \geq 10$
$x \geq 0, y \geq 0$



$2x + y = 8$
$2x + 2y = 10$

Solution: $x=3$, $y=2$
$C = 4x + 3y = 18$¢



corner pt.	$C = 4x + 3y$
(0,8)	24 cents
(5,0)	20 cents
(3,2)	18 cents

Example - bicycle factories

A small business makes 3-speed and 10-speed bicycles at two different factories. Factory **A** produces 16 3-speed and 20 10-speed bikes in one day while factory **B** produces 12 3-speed and 20 10-speed bikes daily. It costs $1000/day to operate factory **A** and $800/day to operate factory **B**. An order for 96 3-speed bikes and 140 10-speed bikes has just arrived. How many days should each factory be operated in order to fill this order at a minimum cost? What is the minimum cost?

x = # days factory **A** is operated
y = # days factory **B** is operated

x = # days factory **A** is operated
y = # days factory **B** is operated

x = # days factory **A** is operated
y = # days factory **B** is operated

3-speed constraint: $16x + 12y \geq 96$

x = # days factory **A** is operated

y = # days factory **B** is operated

3-speed constraint: $16x + 12y \geq 96$

10-speed constraint: $20x + 20y \geq 140$

$x \geq 0, y \geq 0$

x = # days factory **A** is operated

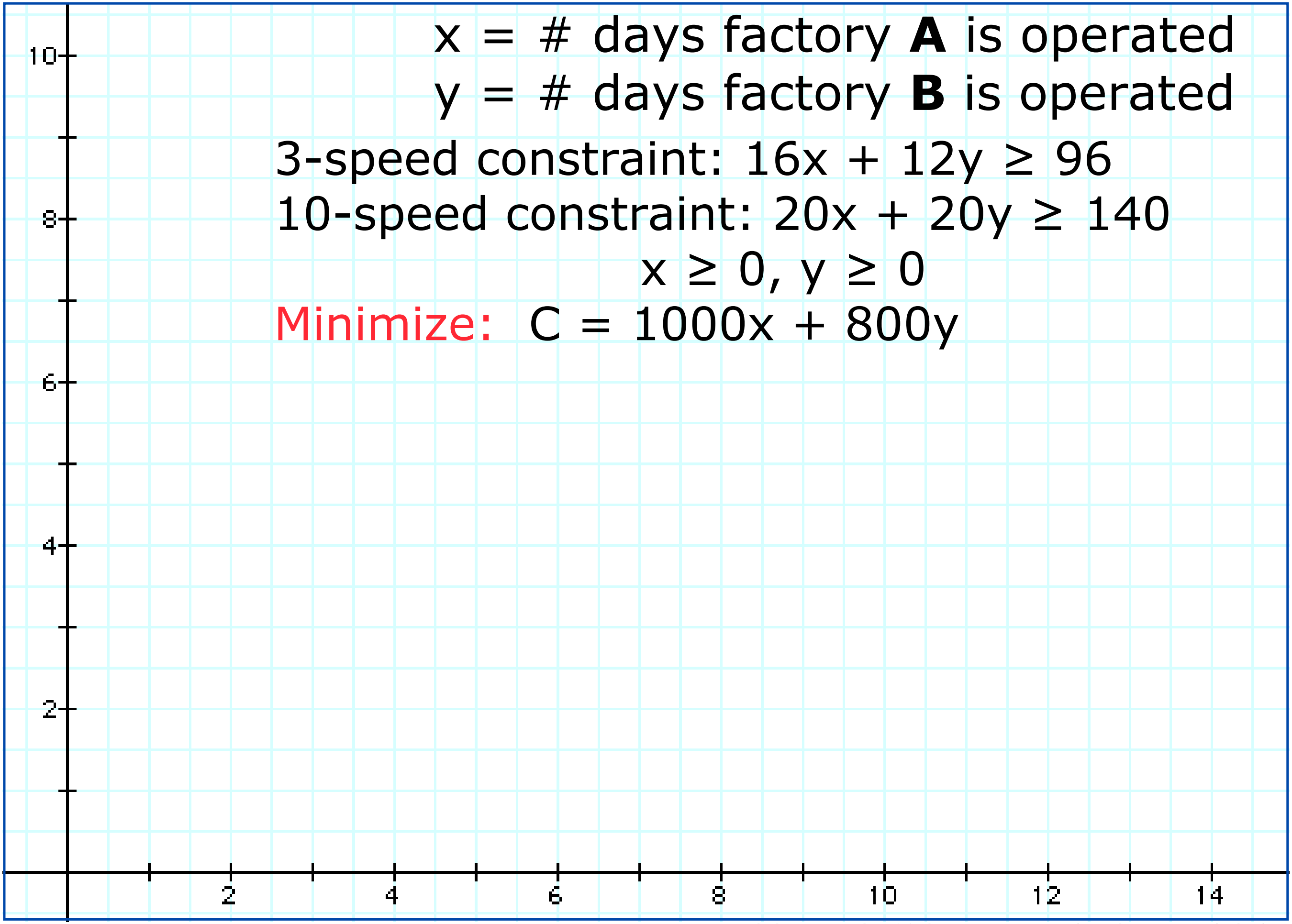
y = # days factory **B** is operated

3-speed constraint: $16x + 12y \geq 96$

10-speed constraint: $20x + 20y \geq 140$

$x \geq 0, y \geq 0$

Minimize: $C = 1000x + 800y$



x = # days factory **A** is operated

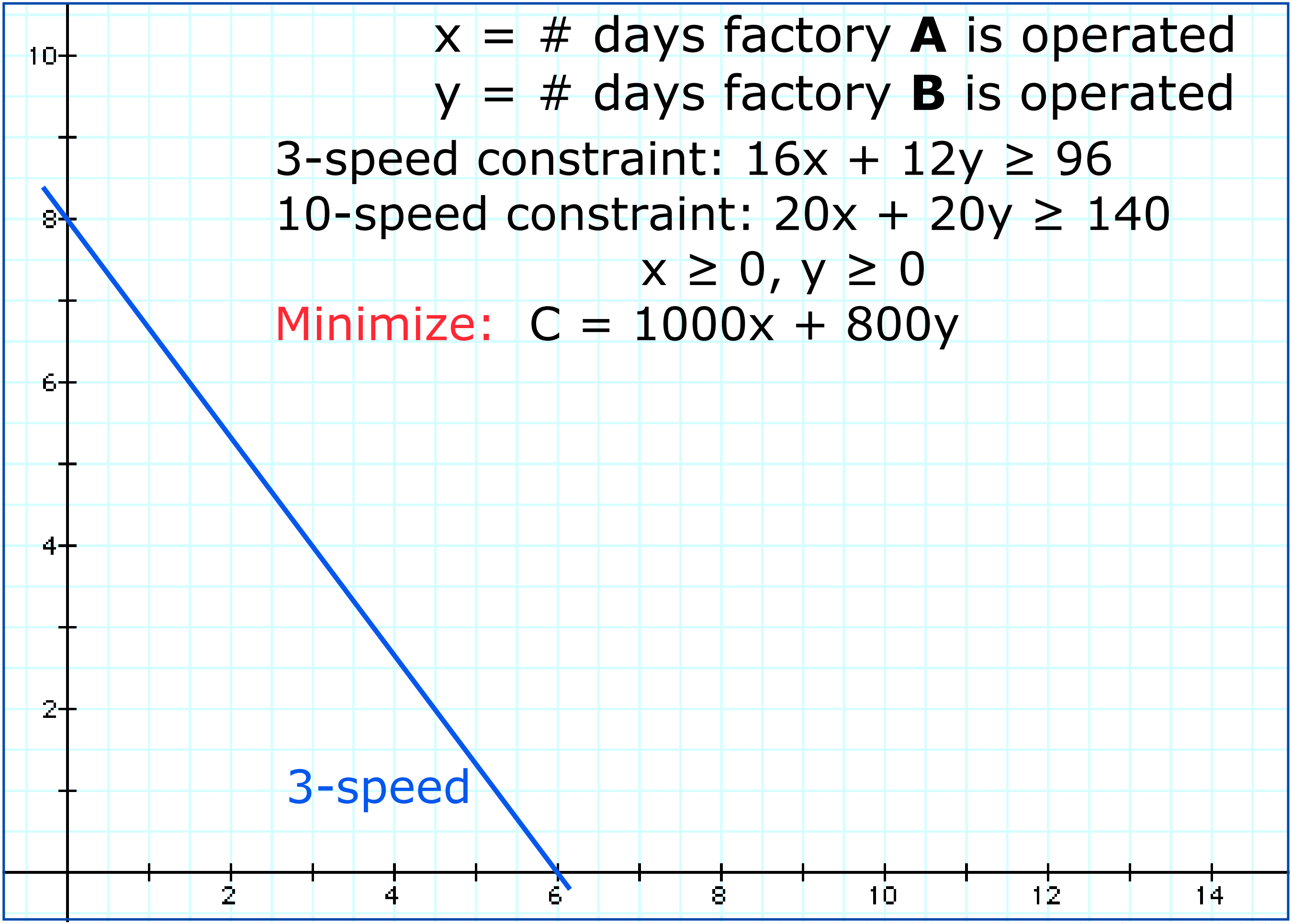
y = # days factory **B** is operated

3-speed constraint: $16x + 12y \geq 96$

10-speed constraint: $20x + 20y \geq 140$

$x \geq 0, y \geq 0$

Minimize: $C = 1000x + 800y$



x = # days factory **A** is operated

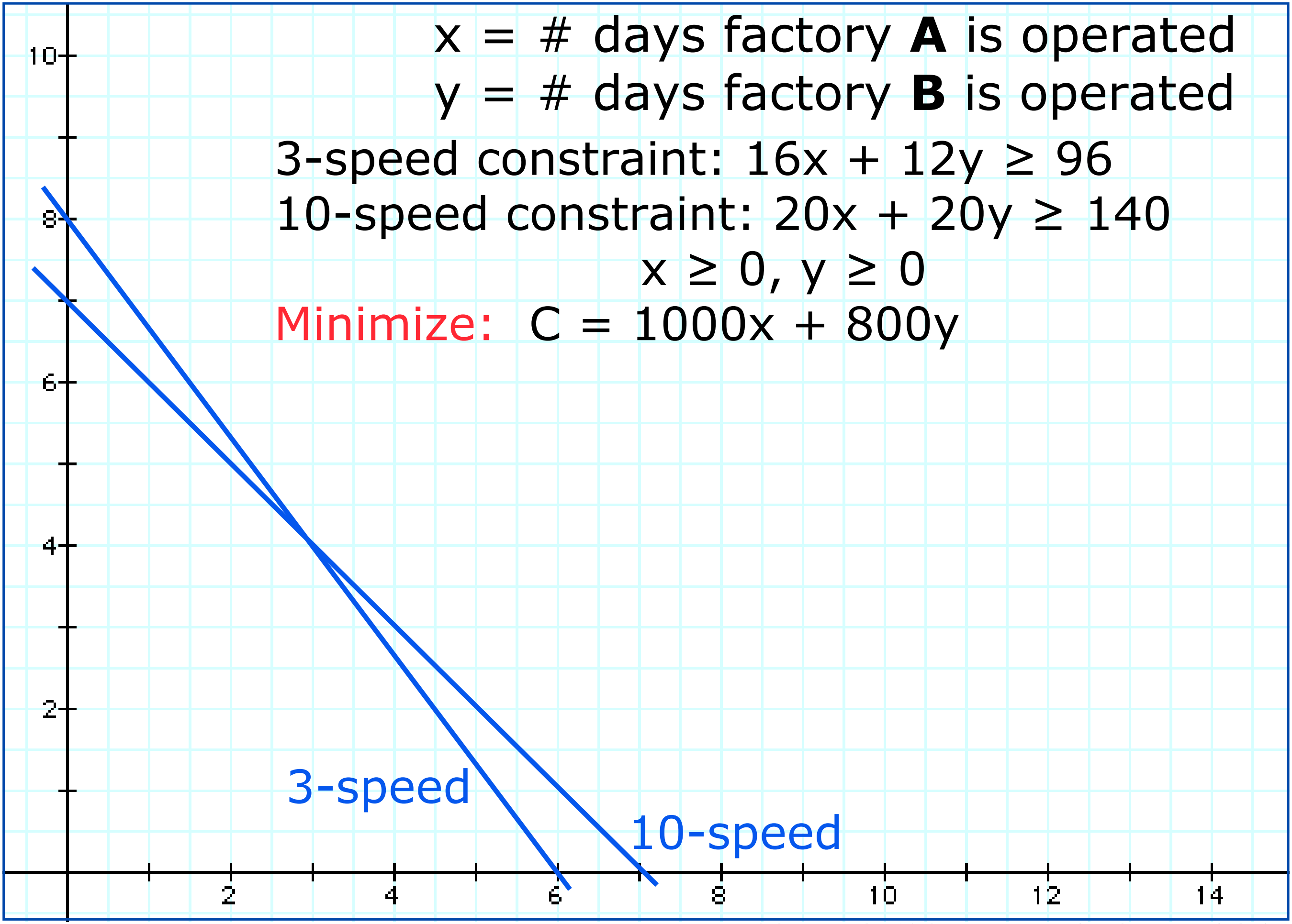
y = # days factory **B** is operated

3-speed constraint: $16x + 12y \geq 96$

10-speed constraint: $20x + 20y \geq 140$

$x \geq 0, y \geq 0$

Minimize: $C = 1000x + 800y$



x = # days factory **A** is operated

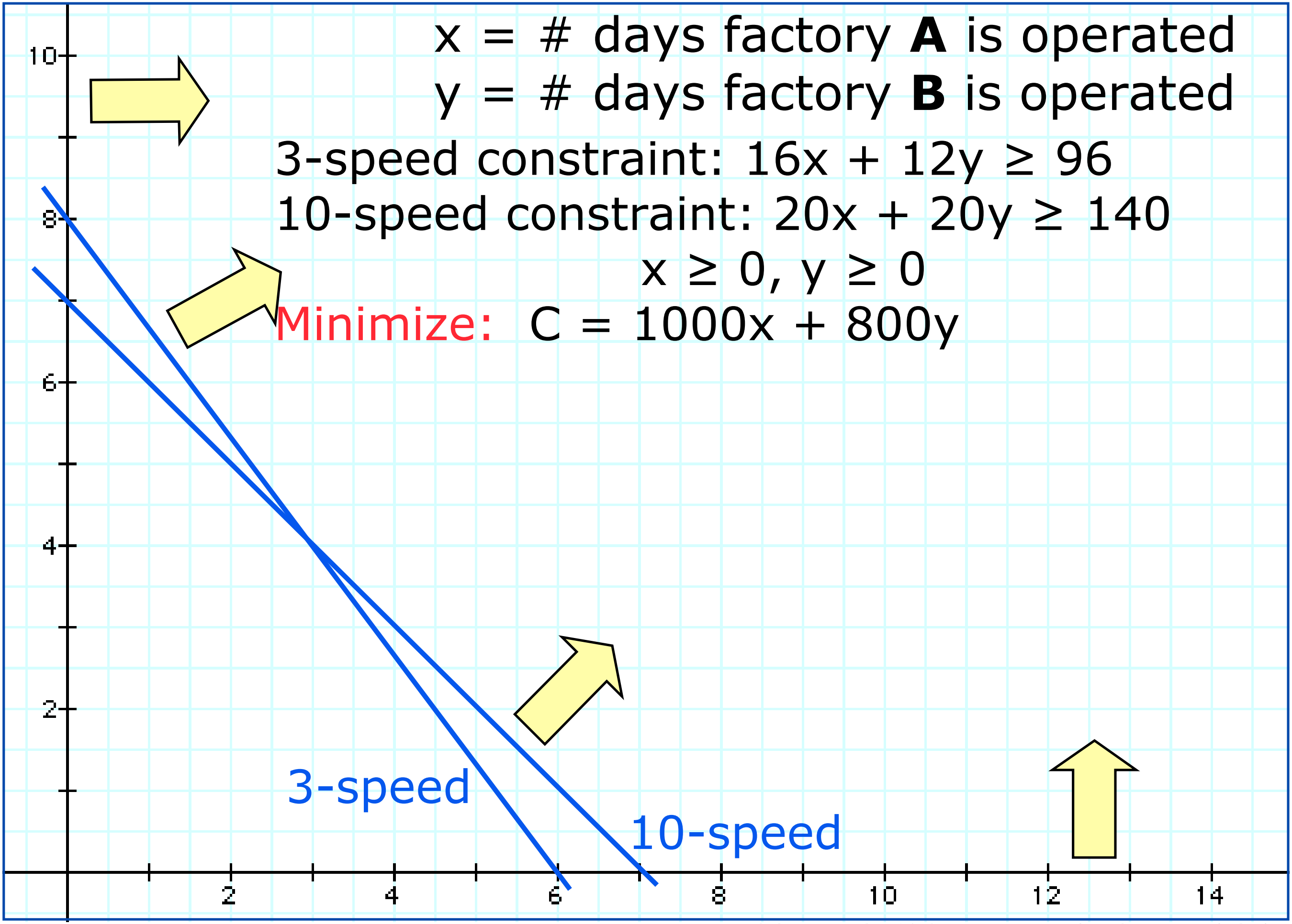
y = # days factory **B** is operated

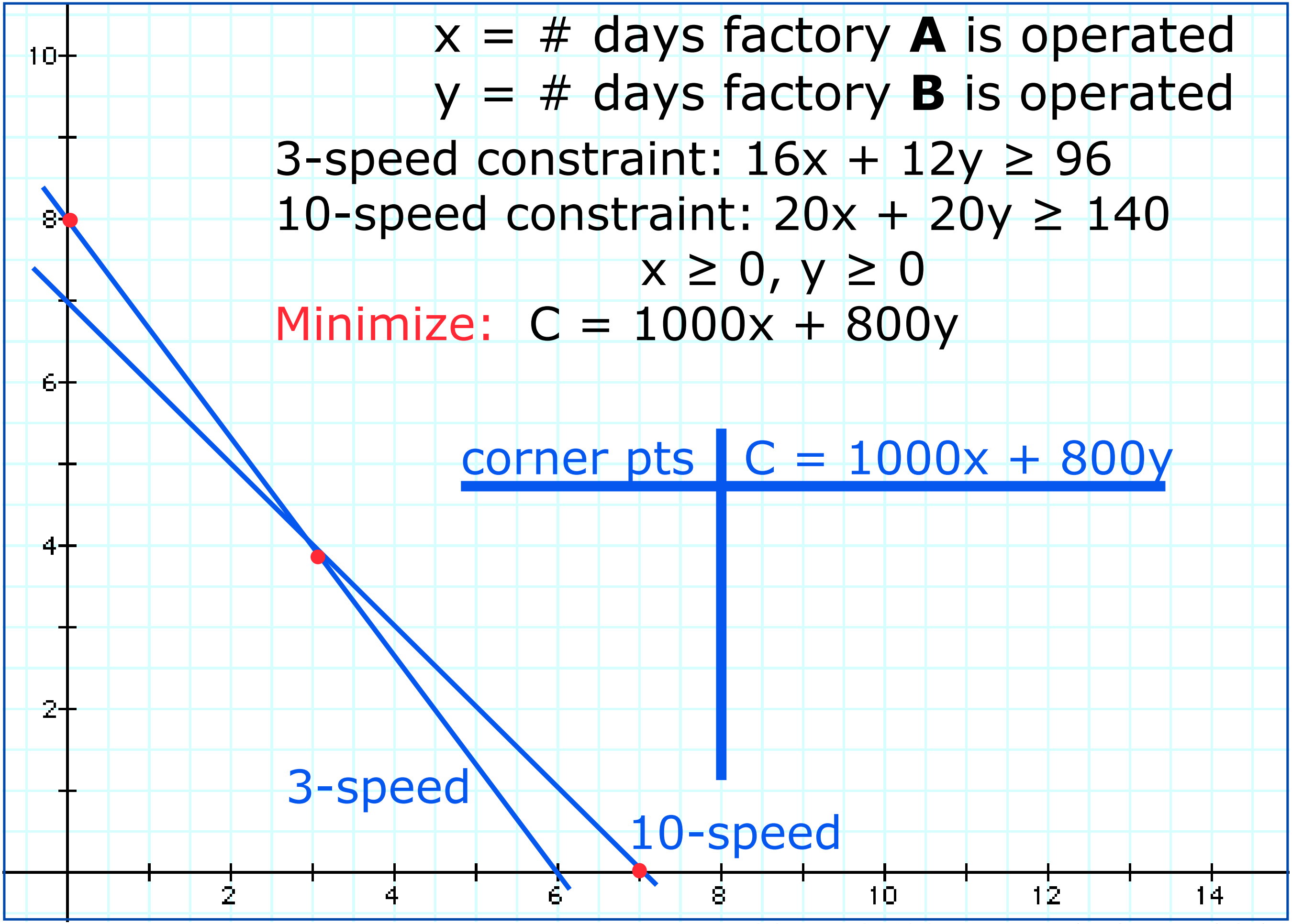
3-speed constraint: $16x + 12y \geq 96$

10-speed constraint: $20x + 20y \geq 140$

$x \geq 0,\ y \geq 0$

Minimize: $C = 1000x + 800y$





x = # days factory **A** is operated
y = # days factory **B** is operated
3-speed constraint: 16x + 12y ≥ 96
10-speed constraint: 20x + 20y ≥ 140
x ≥ 0, y ≥ 0
Minimize: C = 1000x + 800y
corner pts
C = 1000x + 800y
3-speed
10-speed
2
4
6
8
10
12
14

x = # days factory **A** is operated

y = # days factory **B** is operated

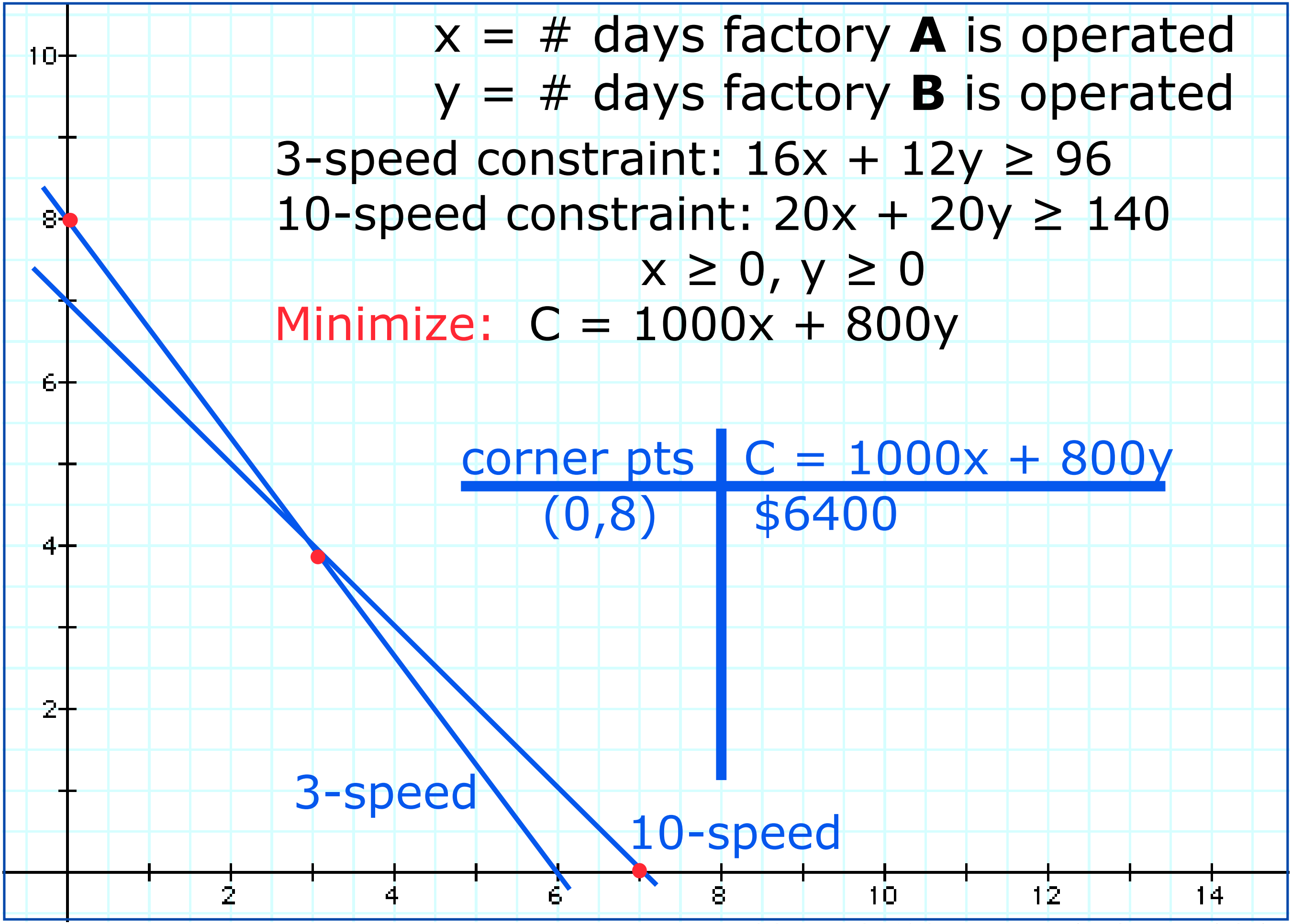
3-speed constraint: $16x + 12y \geq 96$

10-speed constraint: $20x + 20y \geq 140$

$x \geq 0,\ y \geq 0$

Minimize: $C = 1000x + 800y$

corner pts	C = 1000x + 800y
(0,8)	$6400



x = # days factory **A** is operated

y = # days factory **B** is operated

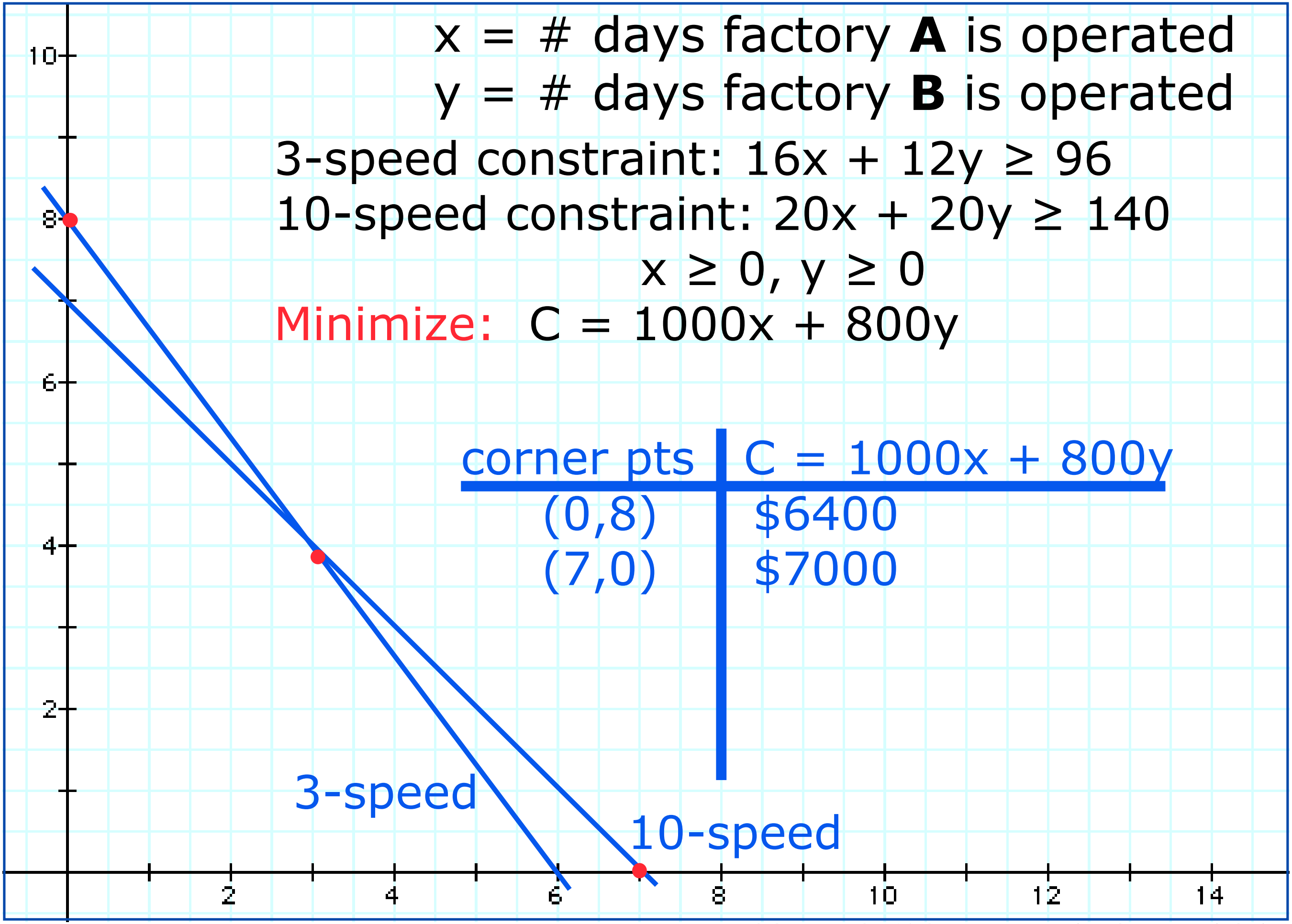
3-speed constraint: $16x + 12y \geq 96$

10-speed constraint: $20x + 20y \geq 140$

$x \geq 0,\ y \geq 0$

Minimize: $C = 1000x + 800y$

corner pts	C = 1000x + 800y
(0,8)	\$6400
(7,0)	\$7000



x = # days factory **A** is operated

y = # days factory **B** is operated

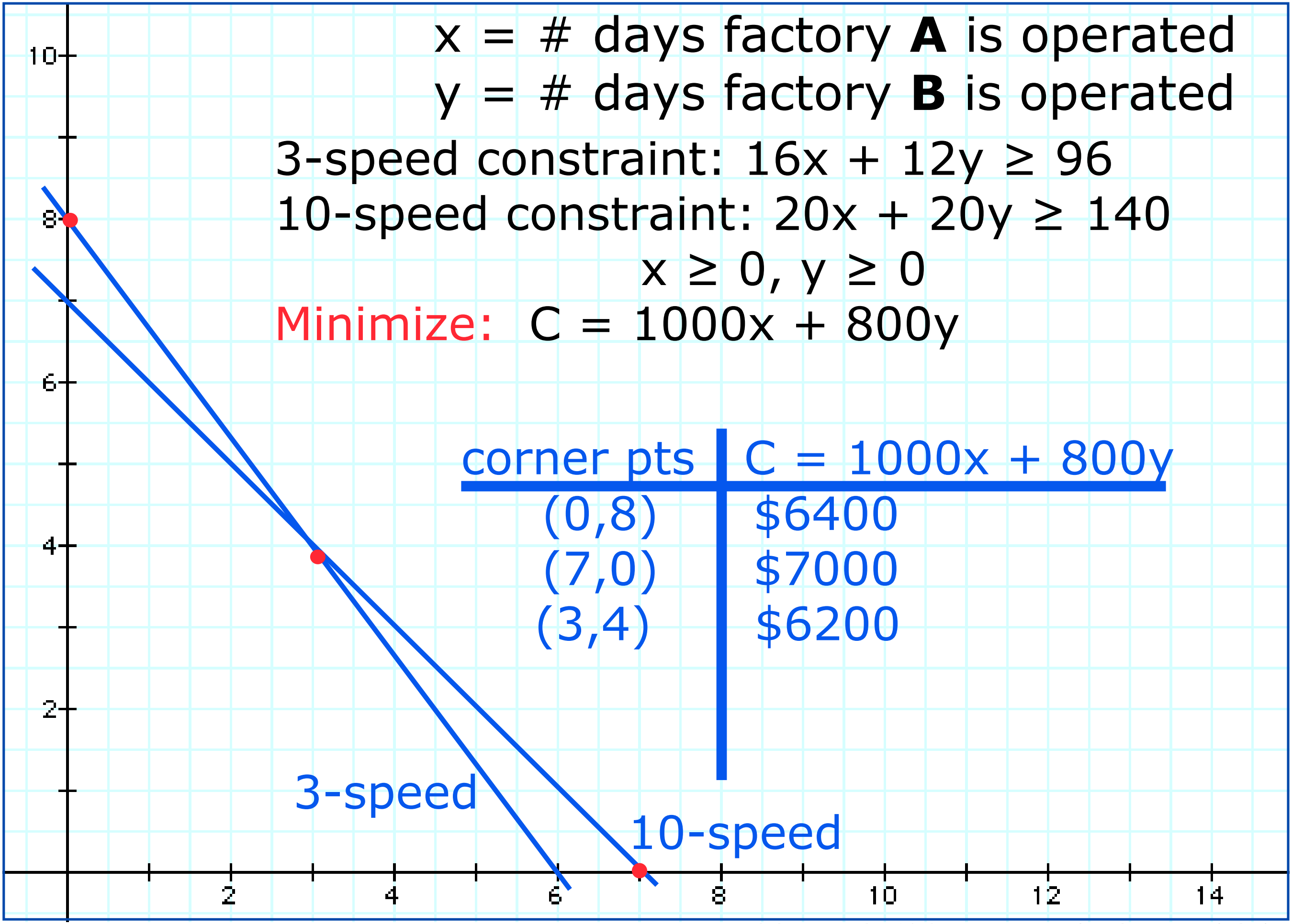
3-speed constraint: $16x + 12y \geq 96$

10-speed constraint: $20x + 20y \geq 140$

$x \geq 0,\ y \geq 0$

Minimize: $C = 1000x + 800y$

corner pts	C = 1000x + 800y
(0,8)	\$6400
(7,0)	\$7000
(3,4)	\$6200



Example - ski manufacturing

Michigan Polar Products makes downhill and cross-country skis. A pair of downhill skis requires 2 man-hours for cutting, 1 man-hour for shaping and 3 man-hours for finishing while a pair of cross-country skis requires 2 man-hours for cutting, 2 man-hours for shaping and 1 man-hour for finishing. Each day the company has available 140 man-hours for cutting, 120 man-hours for shaping and 150 man-hours for finishing. How many pairs of each type of ski should the company manufacture each day in order to maximize profit if a pair of downhill skis yields a profit of \$10 and a pair of cross-country skis yields a profit of \$8?

x = # pairs of downhill skis

y = # pairs of cross country skis

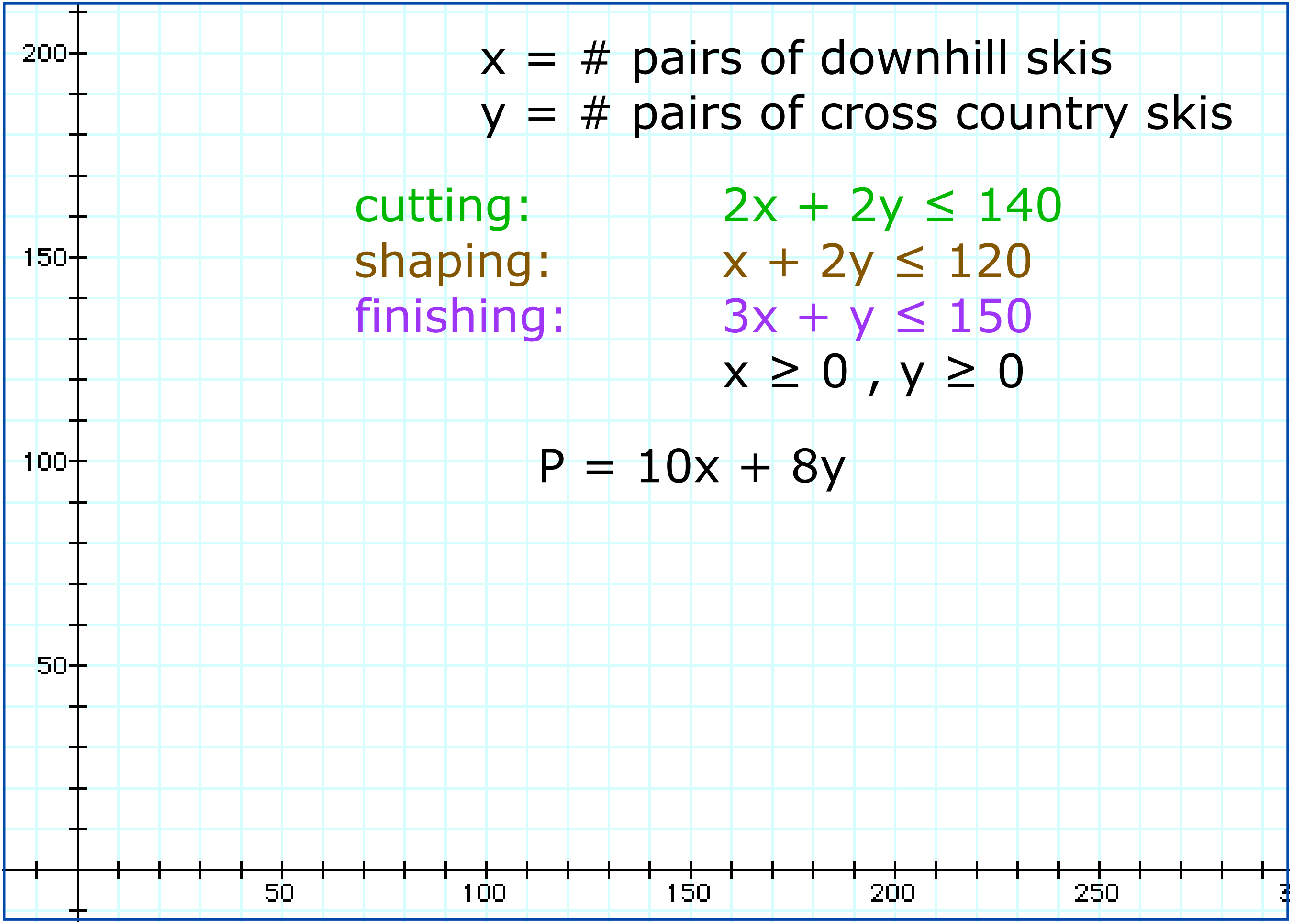
cutting: $2x + 2y \leq 140$

shaping: $x + 2y \leq 120$

finishing: $3x + y \leq 150$

$x \geq 0$, $y \geq 0$

$$P = 10x + 8y$$



x = # pairs of downhill skis
y = # pairs of cross country skis

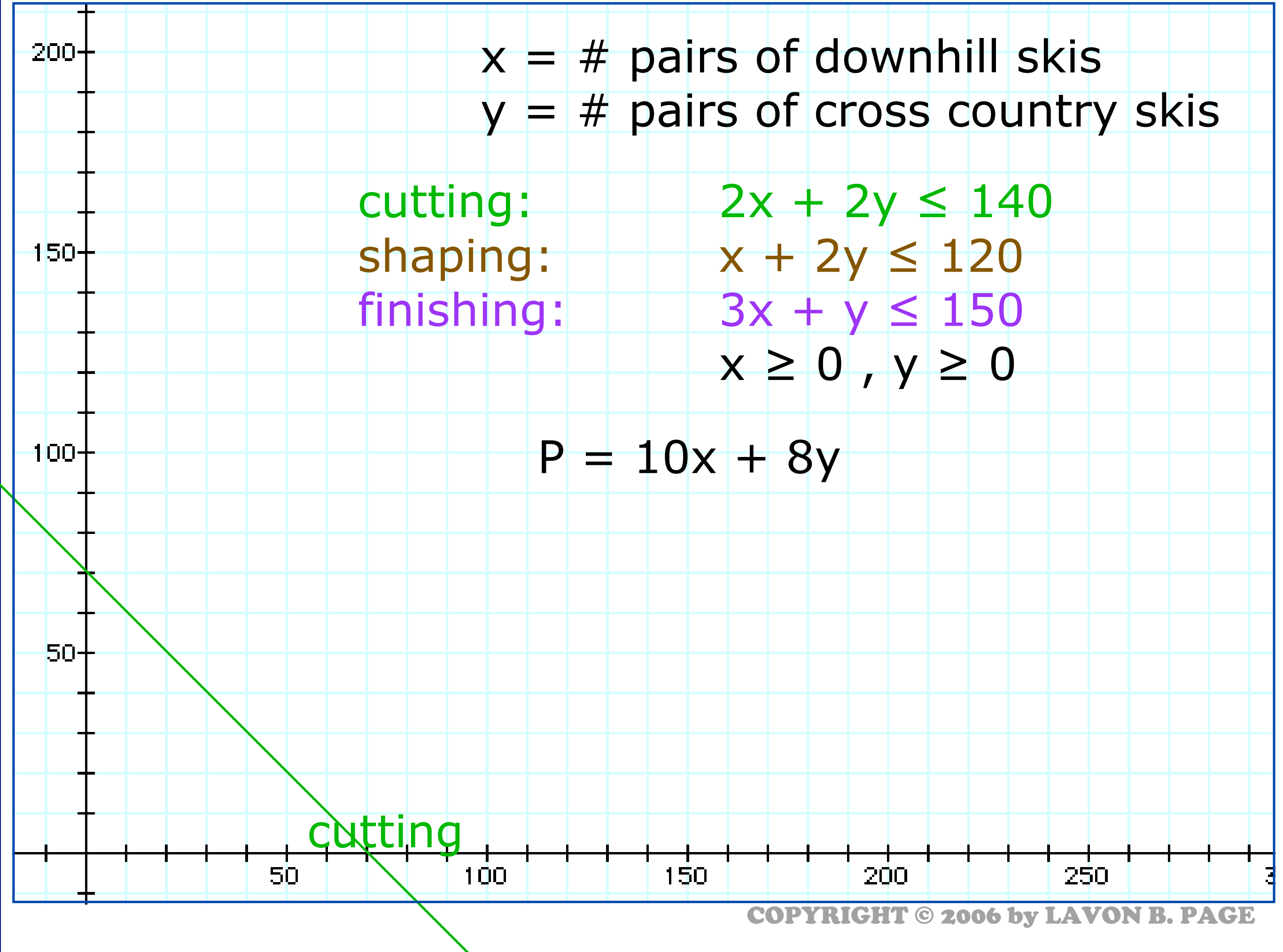
cutting: $2x + 2y \le 140$

shaping: $x + 2y \le 120$

finishing: $3x + y \le 150$

$x \ge 0 \,,\, y \ge 0$

$P = 10x + 8y$



x = # pairs of downhill skis

y = # pairs of cross country skis

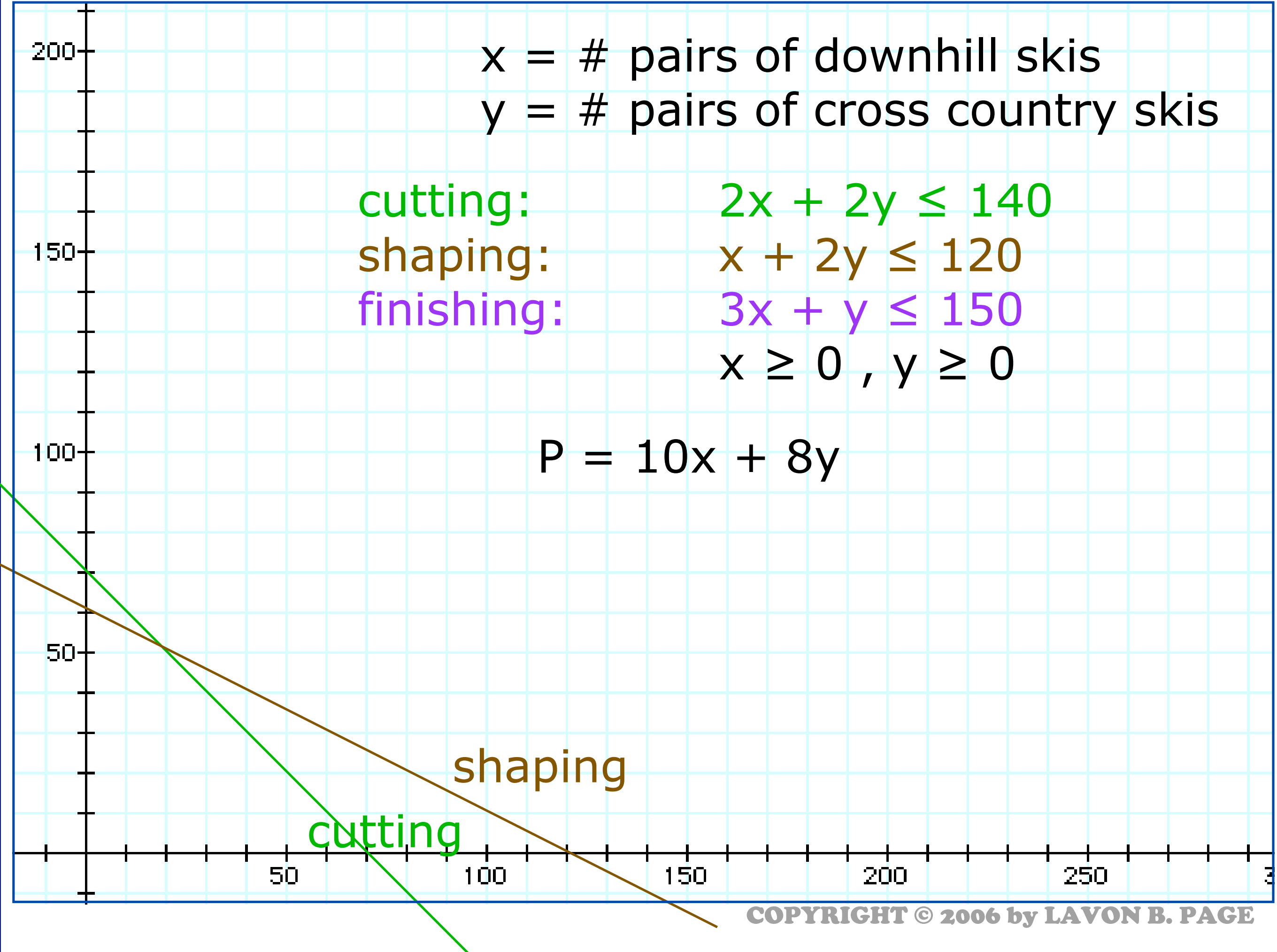
cutting: $2x + 2y \le 140$

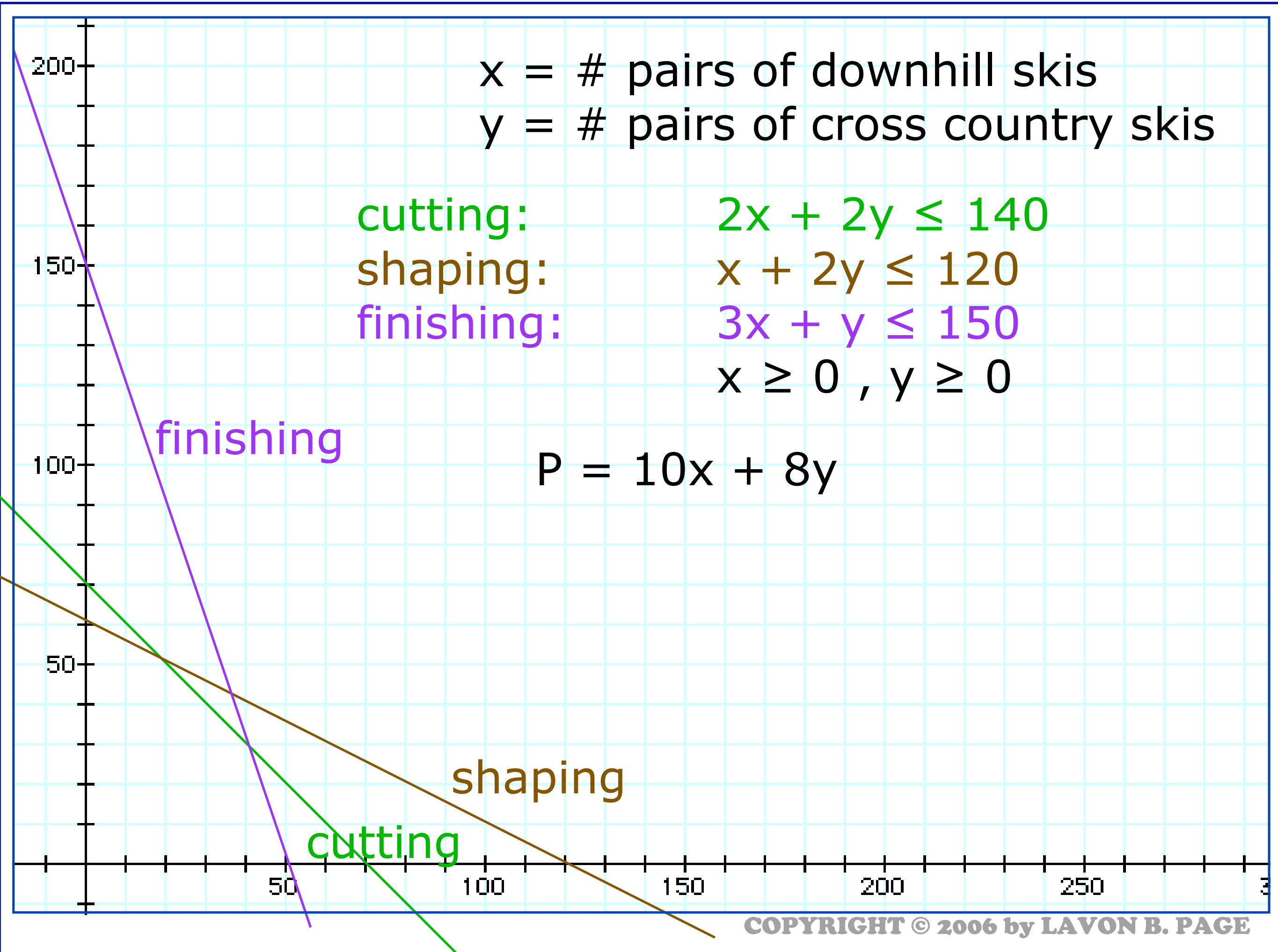
shaping: $x + 2y \le 120$

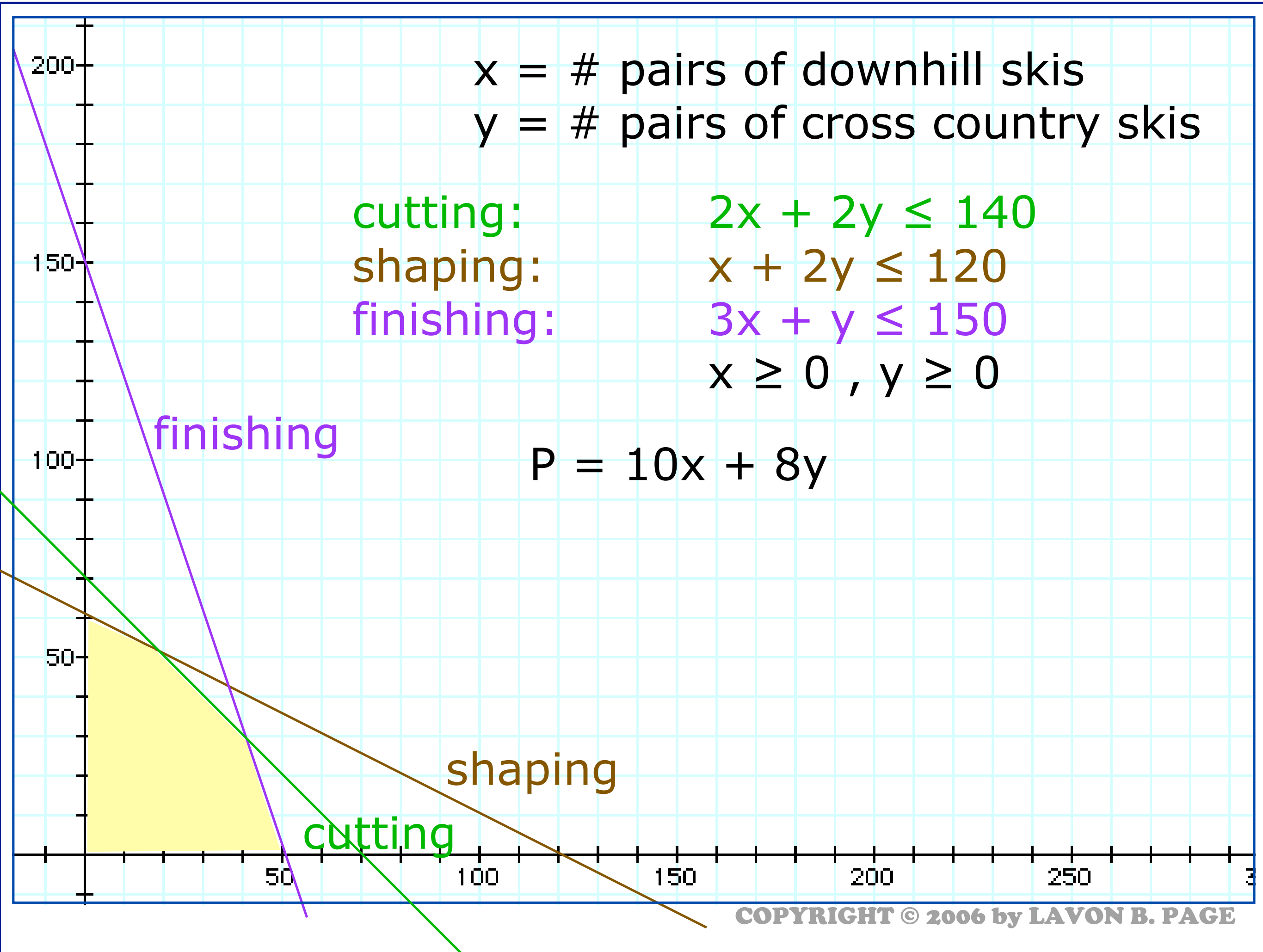
finishing: $3x + y \le 150$

$x \ge 0 \, , \, y \ge 0$

$P = 10x + 8y$




x = # pairs of downhill skis
y = # pairs of cross country skis
cutting: 2x + 2y ≤ 140
shaping: x + 2y ≤ 120
finishing: 3x + y ≤ 150
x ≥ 0 , y ≥ 0
P = 10x + 8y
finishing
shaping
cutting
200
150
100
50
50
100
150
200
250


x = # pairs of downhill skis
y = # pairs of cross country skis
cutting: 2x + 2y ≤ 140
shaping: x + 2y ≤ 120
finishing: 3x + y ≤ 150
x ≥ 0 , y ≥ 0
P = 10x + 8y
finishing
shaping
cutting
200
150
100
50
50
100
150
200
250
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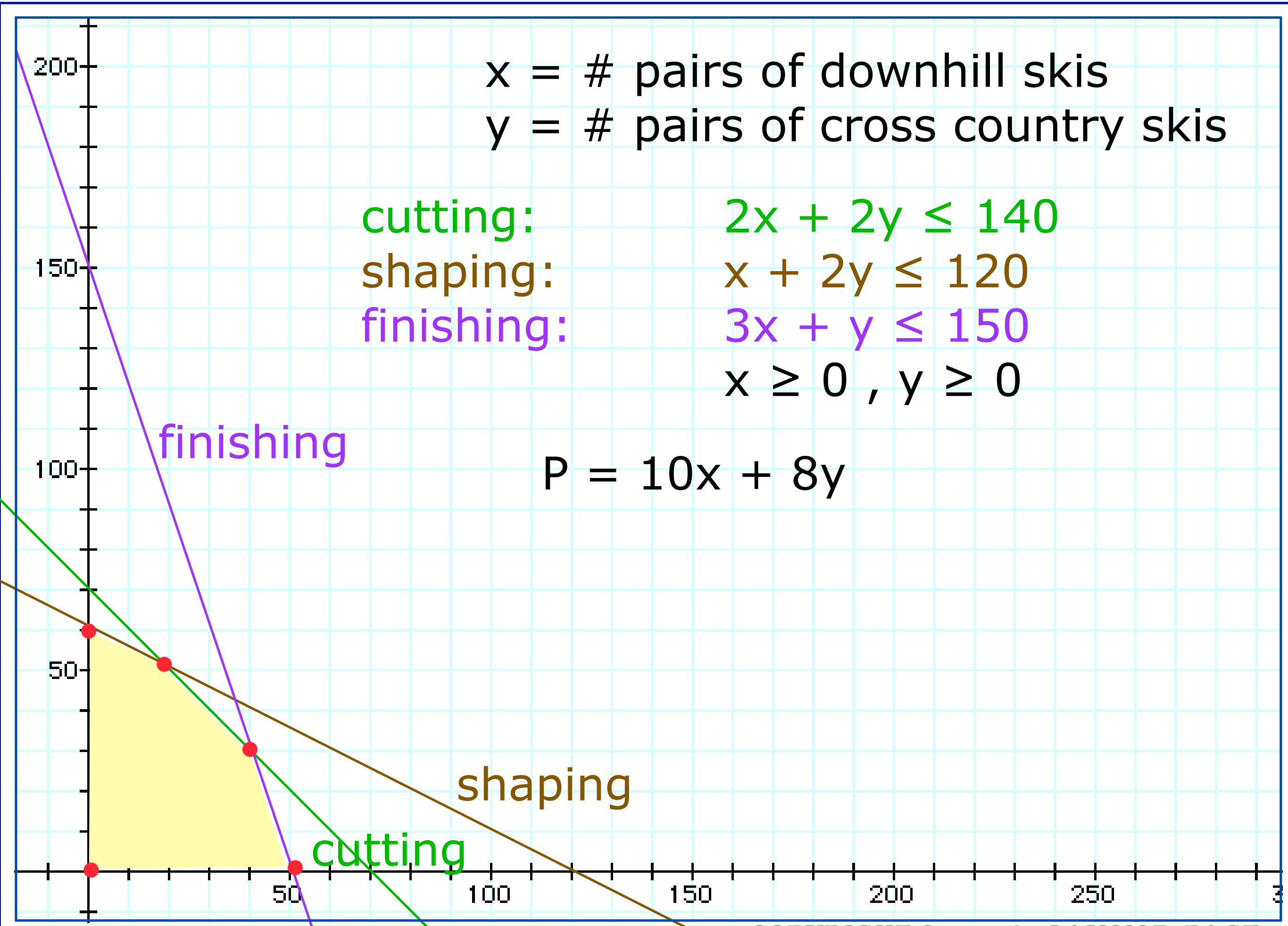


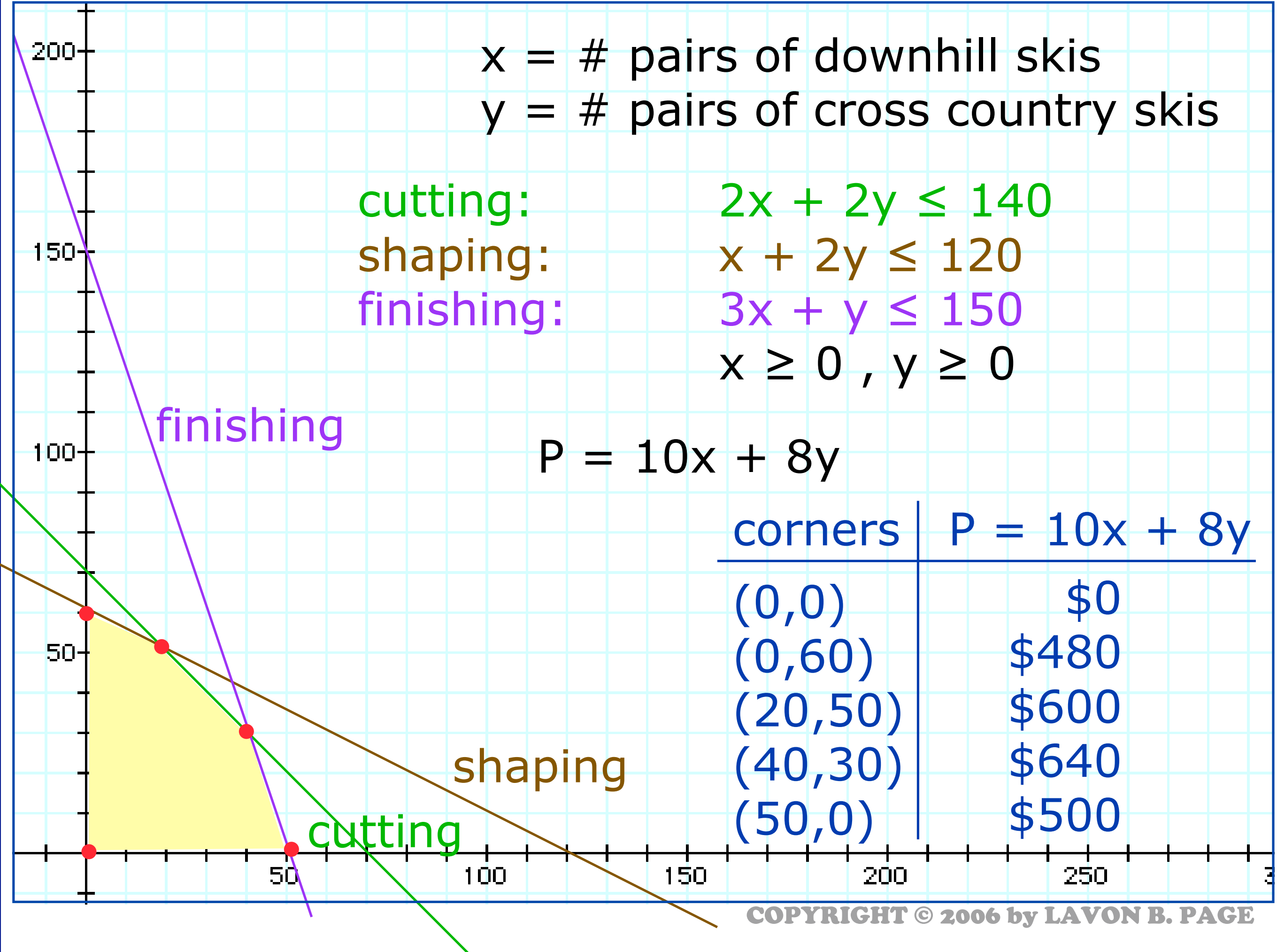
x = # pairs of downhill skis
y = # pairs of cross country skis
cutting: 2x + 2y ≤ 140
shaping: x + 2y ≤ 120
finishing: 3x + y ≤ 150
x ≥ 0 , y ≥ 0
P = 10x + 8y
finishing
shaping
cutting
200
150
100
50
50
100
150
200
250

x = # pairs of downhill skis
y = # pairs of cross country skis

cutting: $2x + 2y \leq 140$
shaping: $x + 2y \leq 120$
finishing: $3x + y \leq 150$
$x \geq 0 \ , \ y \geq 0$

$P = 10x + 8y$

corners	$P = 10x + 8y$
(0,0)	\$0
(0,60)	\$480
(20,50)	\$600
(40,30)	\$640
(50,0)	\$500



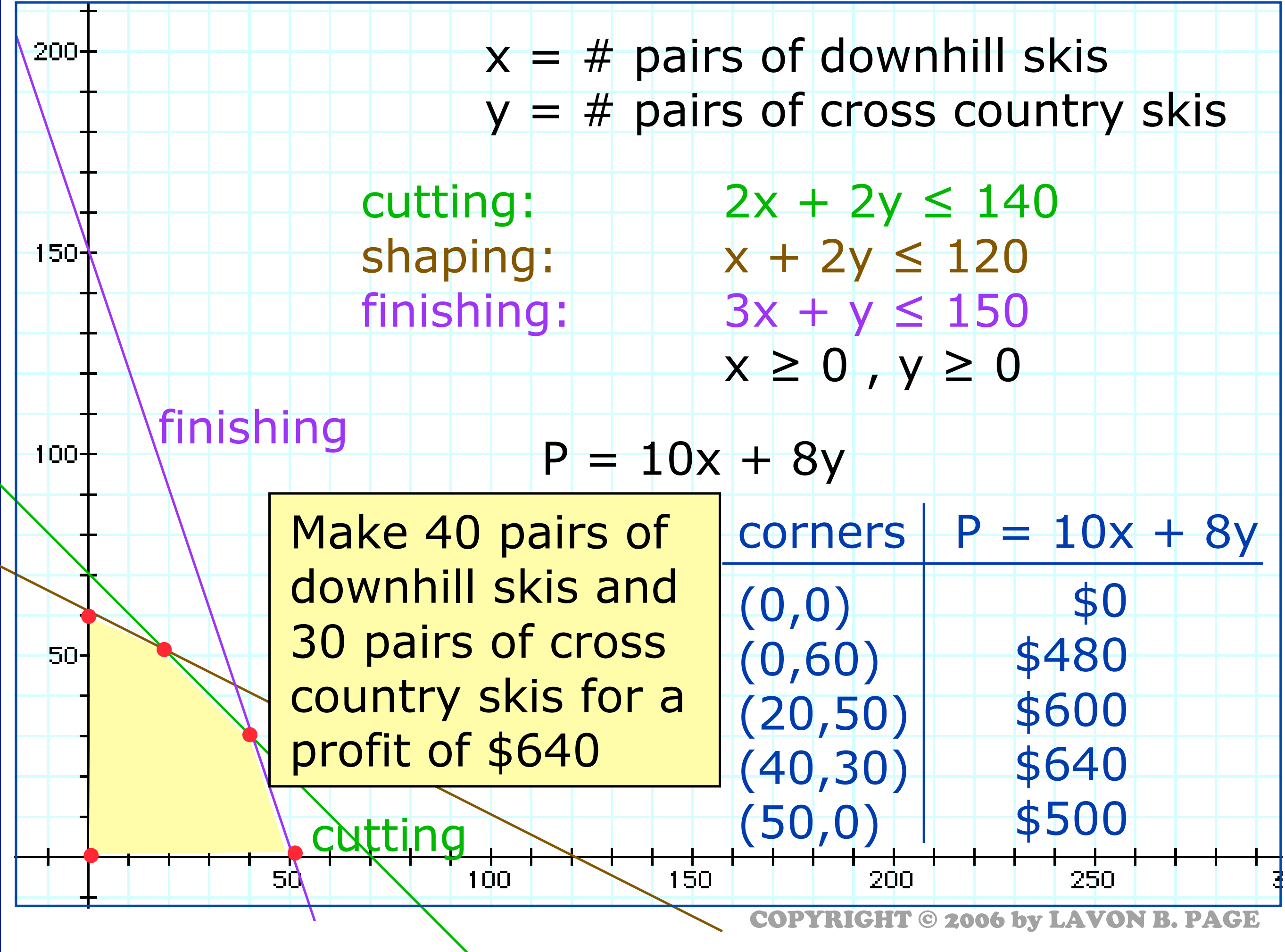
x = # pairs of downhill skis
y = # pairs of cross country skis

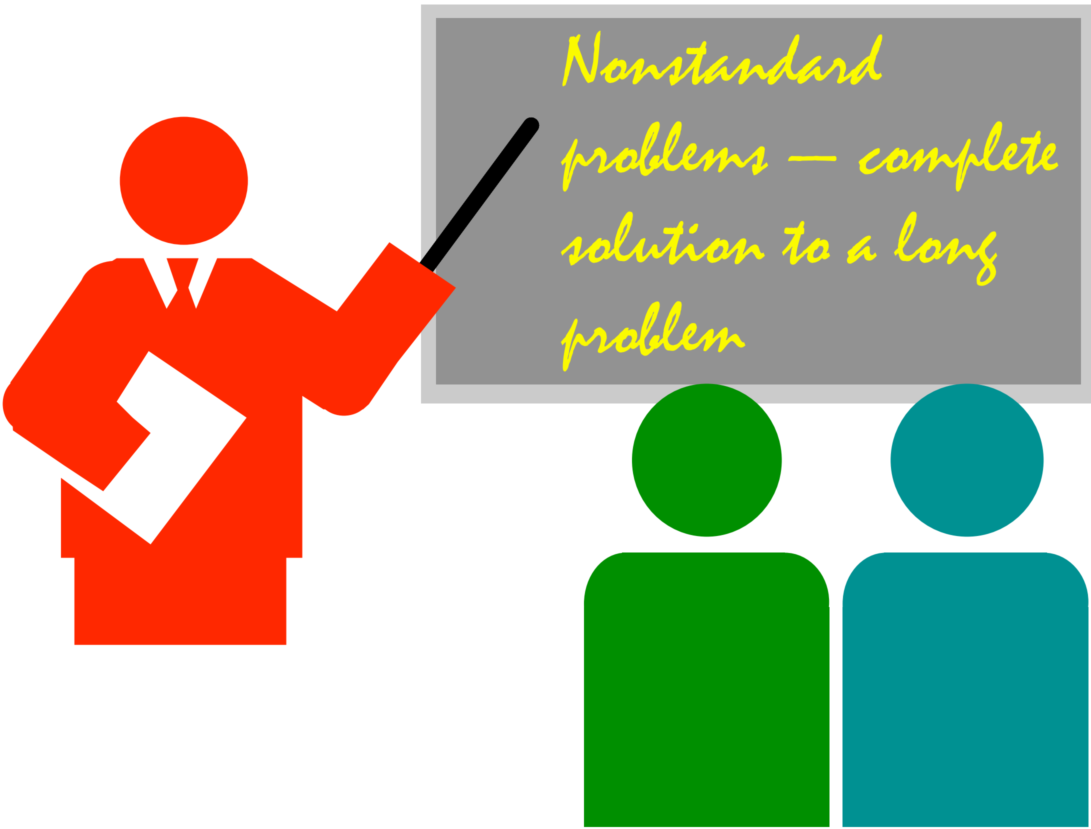
cutting: $2x + 2y \le 140$
shaping: $x + 2y \le 120$
finishing: $3x + y \le 150$
$x \ge 0 \,,\, y \ge 0$

$P = 10x + 8y$

Make 40 pairs of downhill skis and 30 pairs of cross country skis for a profit of $640

corners	P = 10x + 8y
(0,0)	$0
(0,60)	$480
(20,50)	$600
(40,30)	$640
(50,0)	$500




Nonstandard problems — complete solution to a long problem

Vegetarian Diet

Vern decides to adopt a vegetarian diet consisting of fruits, grains and vegetables. His minimum daily requirements are 14 units of protein, 16 units of carbohydrates, and 12 units of fiber. Suppose a serving of fruits can supply him with 1 unit of protein, 2 units of carbohydrates and 1 unit of fiber while a serving of grains provides 3 units of protein, 2 units of carbohydrates, and 3 units of fiber. A serving of vegetables provides 4 units of protein, 3 units of carbohydrates and 2 units of fiber. If fruit costs 30¢ per serving, grains cost 60¢ per serving and vegetables cost 70¢ per serving, how many servings of each type of food should he eat per day in order to satisfy his daily food requirements at minimum cost?

x = # servings fruit
y = # servings grain
z = # servings veggies

x = # servings fruit

y = # servings grain

z = # servings veggies

<u>Constraints:</u>

$x + 3y + 4z \geq 14$ (protein)

$2x + 2y + 3z \geq 16$ (carbos)

$x + 3y + 2z \geq 12$ (fiber)

$x \geq 0, y \geq 0, z \geq 0$

x = # servings fruit

y = # servings grain

z = # servings veggies

<u>Constraints:</u>

$x + 3y + 4z \geq 14$ (protein)

$2x + 2y + 3z \geq 16$ (carbos)

$x + 3y + 2z \geq 12$ (fiber)

$x \geq 0,\ y \geq 0,\ z \geq 0$

minimize $C = 30x + 60y + 70z$

$x + 3y + 4z \geq 14$ (protein)
$2x + 2y + 3z \geq 16$ (carbos)
$x + 3y + 2z \geq 12$ (fiber)
$x \geq 0,\ y \geq 0,\ z \geq 0$
minimize $C = 30x + 60y + 70z$

$-x - 3y - 4z \leq -14$
$-2x - 2y - 3z \leq -16$
$-x - 3y - 2z \leq -12$
maximize $D = -30x - 60y - 70z$

$$-x - 3y - 4z \leq -14$$
$$-2x - 2y - 3z \leq -16$$
$$-x - 3y - 2z \leq -12$$
maximize $D = -30x - 60y - 70z$

$$-x - 3y - 4z + u = -14$$
$$-2x - 2y - 3z + v = -16$$
$$-x - 3y - 2z + w = -12$$
$$30x + 60y + 70z + D = 0$$

$$-x - 3y - 4z + u = -14$$
$$-2x - 2y - 3z + v = -16$$
$$-x - 3y - 2z + w = -12$$
$$30x + 60y + 70z + D = 0$$

$$\begin{array}{ccccccc|c} x & y & z & u & v & w & D & \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} -1 & -3 & -4 & 1 & 0 & 0 & 0 & -14 \\ -2 & -2 & -3 & 0 & 1 & 0 & 0 & -16 \\ -1 & -3 & -2 & 0 & 0 & 1 & 0 & -12 \\ 30 & 60 & 70 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$
\begin{array}{c} \\ \end{array}
\begin{array}{ccccccc|c}
x & y & z & u & v & w & D & \\
-1 & -3 & \textcircled{-4} & 1 & 0 & 0 & 0 & -14 \\
\textcircled{-2} & -2 & -3 & 0 & 1 & 0 & 0 & -16 \\
-1 & \textcircled{-3} & -2 & 0 & 0 & 1 & 0 & -12 \\
30 & 60 & 70 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

Legal choices for 1st pivot element

$$
\begin{array}{ccccccc|c}
x & y & z & u & v & w & D & \\
-1 & -3 & -4 & 1 & 0 & 0 & 0 & -14 \\
-2 & -2 & -3 & 0 & 1 & 0 & 0 & -16 \\
-1 & -3 & -2 & 0 & 0 & 1 & 0 & -12 \\
30 & 60 & 70 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

We'll use this one

$$
\begin{array}{ccccccc|c}
x & y & z & u & v & w & D & \\
\end{array}
$$

$$
\left[\begin{array}{ccccccc|c}
-1 & -3 & -4 & 1 & 0 & 0 & 0 & -14 \\
1 & 1 & \frac{3}{2} & 0 & \frac{-1}{2} & 0 & 0 & 8 \\
-1 & -3 & -2 & 0 & 0 & 1 & 0 & -12 \\
30 & 60 & 70 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
$$

$$
\begin{array}{ccccccc|c}
x & y & z & u & v & w & D & \\
0 & -2 & \frac{-5}{2} & 1 & \frac{-1}{2} & 0 & 0 & -6 \\
1 & 1 & \frac{3}{2} & 0 & \frac{-1}{2} & 0 & 0 & 8 \\
0 & -2 & \frac{-1}{2} & 0 & \frac{-1}{2} & 1 & 0 & -4 \\
0 & 30 & 25 & 0 & 15 & 0 & 1 & -240
\end{array}
$$

Next pivot element?

$$
\begin{array}{ccccccc|c}
x & y & z & u & v & w & D & \\
0 & -2 & \frac{-5}{2} & 1 & \frac{-1}{2} & 0 & 0 & -6 \\
1 & 1 & \frac{3}{2} & 0 & \frac{-1}{2} & 0 & 0 & 8 \\
0 & -2 & \frac{-1}{2} & 0 & \frac{-1}{2} & 1 & 0 & -4 \\
0 & 30 & 25 & 0 & 15 & 0 & 1 & -240
\end{array}
$$

Possible pivot element?

$$
\begin{array}{ccccccc|c}
x & y & z & u & v & w & D & \\
0 & -2 & \frac{-5}{2} & 1 & \frac{-1}{2} & 0 & 0 & -6 \\
1 & 1 & \frac{3}{2} & 0 & \frac{-1}{2} & 0 & 0 & 8 \\
0 & -2 & \frac{-1}{2} & 0 & \frac{-1}{2} & 1 & 0 & -4 \\
0 & 30 & 25 & 0 & 15 & 0 & 1 & -240
\end{array}
$$

We'll use this one.

$$
\begin{array}{ccccccc}
x & y & z & u & v & w & D
\end{array}
$$

$$
\left[\begin{array}{ccccccc|c}
0 & 0 & -2 & 1 & 0 & -1 & 0 & -2 \\
1 & 3 & 2 & 0 & 0 & -1 & 0 & 12 \\
0 & 4 & 1 & 0 & 1 & -2 & 0 & 8 \\
0 & -30 & 10 & 0 & 0 & 30 & 1 & -360
\end{array}\right]
$$

Here's where we are after the next complete step. The 5th column has been put into unit form. We're at x=12, y=0, z=0, but the -2 in the far right column tells us that we are not yet in the solution region.

$$
\begin{array}{ccccccc|c}
x & y & z & u & v & w & D & \\
0 & 0 & -2 & 1 & 0 & \textcircled{-1} & 0 & -2 \\
1 & 3 & 2 & 0 & 0 & -1 & 0 & 12 \\
0 & 4 & 1 & 0 & 1 & -2 & 0 & 8 \\
0 & -30 & 10 & 0 & 0 & 30 & 1 & -360
\end{array}
$$

Choose next pivot element.
We'll use the -1 in column 6.

$$
\begin{array}{c}
\begin{matrix} x & y & z & u & v & w & D \end{matrix} \\
\left[\begin{array}{ccccccc|c}
0 & 0 & 2 & -1 & 0 & 1 & 0 & 2 \\
1 & 3 & 4 & -1 & 0 & 0 & 0 & 14 \\
0 & 4 & 5 & -2 & 1 & 0 & 0 & 12 \\
0 & -30 & -50 & 30 & 0 & 0 & 1 & -420
\end{array}\right]
\end{array}
$$

When column 6 is put into unit form, this is the tableau. We're at the point x=14, y=0, z=0. There are no negative number above the -420 in the last column, and this tells us that we finally are in the solution region.

x	y	z	u	v	w	D	
0	0	2	-1	0	1	0	2
1	3	4	-1	0	0	0	14
0	4	5	-2	1	0	0	12
0	-30	-50	30	0	0	1	-420

The problem now becomes like a standard problem. We choose our next pivot element by taking the pivot column to be the one with the most negative number in the bottom row (excluding the bottom right number -420).

$$
\begin{array}{ccccccc}
x & y & z & u & v & w & D
\end{array}
$$

$$
\left[\begin{array}{ccccccc|c}
0 & 0 & 1 & \frac{-1}{2} & 0 & \frac{1}{2} & 0 & 1 \\
1 & 3 & 0 & 1 & 0 & -2 & 0 & 10 \\
0 & 4 & 0 & \frac{1}{2} & 1 & \frac{-5}{2} & 0 & 7 \\
0 & -30 & 0 & 5 & 0 & 25 & 1 & -370
\end{array}\right]
$$

After this step, x=10, y=0, z=1. We're in the solution region, but we're not at the optimal solution.

$$
\begin{array}{ccccccc}
x & y & z & u & v & w & D
\end{array}
$$

$$
\left[\begin{array}{ccccccc|c}
0 & 0 & 1 & \frac{-1}{2} & 0 & \frac{1}{2} & 0 & 1 \\
1 & 3 & 0 & 1 & 0 & -2 & 0 & 10 \\
0 & \textcircled{4} & 0 & \frac{1}{2} & 1 & \frac{-5}{2} & 0 & 7 \\
0 & -30 & 0 & 5 & 0 & 25 & 1 & -370
\end{array}\right]
$$

Here's our next pivot element.

$$
\begin{array}{ccccccc|c}
x & y & z & u & v & w & D & \\
0 & 0 & 1 & \frac{-1}{2} & 0 & \frac{1}{2} & 0 & 1 \\
1 & 0 & 0 & \frac{5}{8} & \frac{-3}{4} & \frac{-1}{8} & 0 & \frac{19}{4} \\
0 & 1 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{-5}{8} & 0 & \frac{7}{4} \\
0 & 0 & 0 & \frac{35}{4} & \frac{15}{2} & \frac{25}{4} & 1 & \frac{-635}{2}
\end{array}
$$

Finished! z=1, x=4.75, y=1.75, D=-317.5

$$
\begin{array}{ccccccc|c}
x & y & z & u & v & w & D & \\
0 & 0 & 1 & \frac{-1}{2} & 0 & \frac{1}{2} & 0 & 1 \\
1 & 0 & 0 & \frac{5}{8} & \frac{-3}{4} & \frac{-1}{8} & 0 & \frac{19}{4} \\
0 & 1 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{-5}{8} & 0 & \frac{7}{4} \\
0 & 0 & 0 & \frac{35}{4} & \frac{15}{2} & \frac{25}{4} & 1 & \frac{-635}{2}
\end{array}
$$

Finished! z=1, x=4.75, y=1.75, D=-317.5
Minimum cost is C = $3.175 obtained by eating 4.75 servings of fruit, 1.75 servings of grains, and 1 serving of vegetables.