1. (a) Let a directed graph $G_1$ be given.

Does each of the following list of vertices form a path in $G_1$? If yes, determine (by circling) if the path is simple, if it is a circuit, and give its length.

- a, b, e, c, b  
  Yes [ simple  circuit   length ]  No
- a, d, a, d, a  
  Yes [ simple  circuit   length ]  No
- a, d, e, b, a  
  Yes [ simple  circuit   length ]  No
- a, b, e, c, b, a  
  Yes [ simple  circuit   length ]  No

(b) For the simple graph $G_2$

Find $M_2$, where $M$ is the adjacency matrix of $G_2$

$$M^2 = \begin{pmatrix} \end{pmatrix}$$

Find the number of paths from A to D in $G_2$ of length 2.
2. Provide a pseudo code of an algorithm for finding a closest pair of numbers in a set of \( n \) real distinct numbers and give a worst-case estimate of the number of comparisons.
3. Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

4. Let \( a_1 = 2, a_2 = 9, \) and \( a_n = 2a_{n-1} + 3a_{n-2} \) for \( n \geq 3 \). Show using induction that \( a_n \leq 3^n \) for all positive integers \( n \).
5. Use mathematical induction to show that \[ \sum_{j=0}^{n} (j + 1) = \frac{(n+1)(n+2)}{2} \] whenever \( n \) is a nonnegative integer.

6. Let \( f(n) = 2n\log(n^2+5) + 3n + 1 \). What is big-O estimate of \( f(n) \)? Be sure to specify the values of the witnesses \( C \) and \( k \).
7. Use Dijkstra’s algorithm to find the length of the shortest path between the vertices $a$ and $z$ in the following weighted graph. Use the table below to log in your computation.

![Weighted Graph](image)

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$z$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$a$</td>
</tr>
<tr>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
</tr>
</tbody>
</table>

Draw a tree representing the shortest distances from $a$ to each of the other vertices. Indicate the distance next to each vertex.

![Tree Representation](image)
8. How many vertices and how many edges does each of the following graphs have?
(a) $K_5$

(b) $C_4$

(c) $W_5$

(d) $K_{2,5}$

9. Write a pseudocode for an algorithm for evaluating a polynomial of degree $n,$
$p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0,$ at $x = c.$
What is big-O estimate of the time complexity of your algorithm (in terms of the number of multiplications and additions used) as a function of $n$? Explain your answer.
10. Let $S$ be the subset of the set of ordered pairs of integers defined recursively by

*Basis step:* $(0, 0) \in S.$

*Recursive step:* If $(a, b) \in S$, then $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$.

**a)** List the elements of $S$ produced by the first two applications of the recursive definition.

**b)** Use structural induction to show that $5 \mid a + b$ when $(a, b) \in S$. 
11. For which values of $n$ do these graphs have an Euler circuit?
   a) $K_n$ b) $C_n$ c) $W_n$ d) $Q_n$

12. What is the effect in the time required to solve a problem when you double the size of the input from $n$ to $2n$? Express your answer in the simplest form possible, either as a ratio or a difference. Explain the meaning of your answer.
   a) $\log n$
   b) $100n$
   c) $n^2$
13. Use mathematical induction to prove that every postage of n cents greater than 5 cents can be formed from 3-cent and 4-cent stamps.

14. Give a recursive algorithm for finding the maximum of a finite set of integers, the recursion should make use of the fact that the maximum of n integers is the larger of the last integer in the list and the maximum of the first \( n - 1 \) integers in the list.