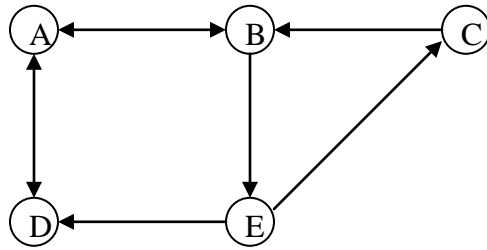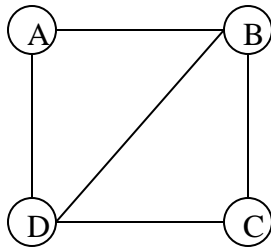**1.** (a) Let a directed graph G_1 be given.

Does each of the following list of vertices form a path in G_1? If yes, determine (by circling) if the path is simple, if it is a circuit, and give its length.

a, b, e, c, b   Yes [simple circuit length ☐] No

a, d, a, d, a   Yes [simple circuit length ☐] No

a, d, e, b, a   Yes [simple circuit length ☐] No

a, b, e, c, b, a   Yes [simple circuit length ☐] No

(b) For the simple graph G_2

Find M^2, where M is the adjacency matrix of G_2

$$M^2 = \begin{Bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{Bmatrix}$$

Find the number of paths from A to D in G_2 of length 2.   ☐

1

2. Use structural induction to show that $l(T) = i(T) + 1$ for a full binary tree T, where $l(T)$ is the number of leaves of T and $i(T)$ is the number of internal vertices of T.

Note: The root r is a leaf of the full binary tree with exactly one vertex r. This tree has no internal vertices. If a full binary tree has more than one vertex, then its root belongs to its internal vertices.

3. Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.
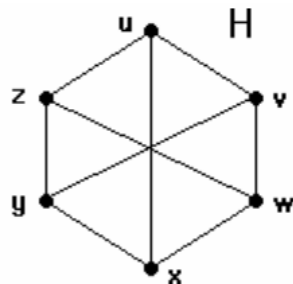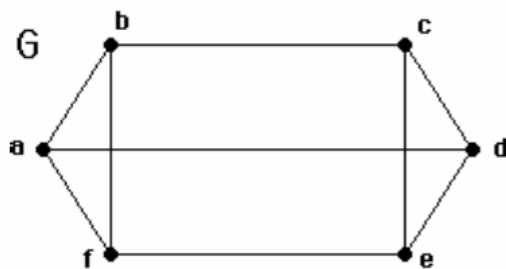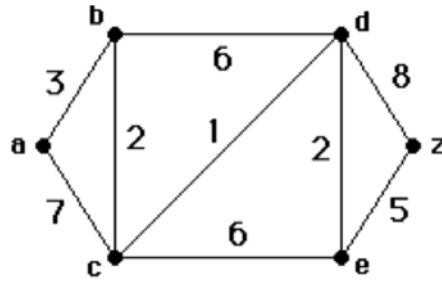

4. Let $a_1 = 2$, $a_2 = 9$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Show that $a_n \leq 3^n$ for all positive integers n.

5. Prove that for all positive integers n the following formula holds

$$\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

6. Let $f(n) = 5n^2 + 2n\log(n) + 3n + 1$. Show that $f(n)$ is $O(n^2)$. Be sure to specify the values of the witnesses C and k.

7. Use Dijkstra's algorithm to find the length of the shortest path between the vertices *a* and *z* in the following weighted graph. Use the table below to log in your computation.

a	*b*	*c*	*d*	*e*	*z*	*S*
0	∞	∞	∞	∞	∞	*a*
X						
X						
X						
X						
X						
X						
X						
X						

Draw a tree representing the shortest distances from *a* to each of the other vertices. Indicate the distance next to each vertex.
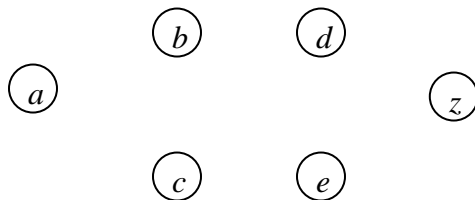
8. How many vertices and how many edges does each of the following graphs have?
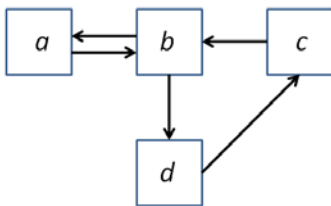
(a) K_5

(b) C_4

(c) W_5

(d) $K_{2,5}$

9. Use powers of the adjacency matrix A of the directed graph given below to find the length of the shortest path from b to c in this graph. Justify your answer and show your work.

10. Describe an algorithm for finding the second largest integer in a sequence of distinct integers. Give a big-O estimate of the number of comparisons used by your algorithm.

11. Give a recursive definition of
a) the set of odd positive integers.
b) the set of positive integer powers of 3.
c) the set of polynomials with integer coefficients.

12. Let S be the subset of the set of ordered pairs of integers defined recursively by
Basis step: (0, 0) $\in$ *S.*
Recursive step: If *(a, b)* $\in$ *S*, then *(a + 2, b + 3)* $\in$ *S* and *(a + 3, b + 2)* $\in$ *S.*
a) List the elements of S produced by the first two applications of the recursive definition.
b) Use structural induction to show that $5 \mid a + b$ when *(a, b)* $\in$ *S.*

13. Write a pseudocode for an algorithm for evaluating a polynomial of degree n,
$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, at $x = c$.
What is big-O estimate of the time complexity of your algorithm (in terms of the number of multiplications and additions used) as a function of n? Explain your answer.

14. Show that $\log(n!)$ is $\Theta(n \cdot \log(n))$.