Example 1. Solve the following system by using the Gauss-Jordan elimination method. Conclude that the system of equations is inconsistent, i.e., it has no solutions.

$$\begin{cases} x + 2y - 3z = 2 \\ 6x + 3y - 9z = 6 \\ 7x + 14y - 21z = 13 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right] \xrightarrow{-7R_1 + R_3 \to R_3} \sim \left[\begin{array}{cccc} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$$0x + 0y + 0z = -1$$

$$0 = -1 \quad ↯$$

Example 2. Solve the following system by using the Gauss-Jordan elimination method. Conclude that there are infinitely many solutions for this system.

$$\begin{cases} 4y + z = 2 \\ 2x + 6y - 2z = 3 \\ 4x + 8y - 5z = 4 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \sim \begin{bmatrix} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{bmatrix} \xrightarrow{-2R_1 + R_3 \to R_3} \sim \begin{bmatrix} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{bmatrix} \xrightarrow{R_2 + R_3 \to R_3} \sim$$

$$\begin{bmatrix} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \to R_1} \sim \begin{bmatrix} 1 & 3 & -1 & \frac{3}{2} \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 + R_1 \to R_1;} \sim \begin{bmatrix} 1 & 7 & 0 & \frac{7}{2} \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 7 & 0 & 7/2 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7/4 & 0 \\ 0 & 1 & 1/4 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x, y are dependent variables, z is a free variable

$$\begin{aligned} x - \tfrac{7}{4} z &= 0 \\ y + \tfrac{1}{4} z &= \tfrac{1}{2} \end{aligned} \Rightarrow \begin{aligned} x &= \tfrac{7}{4} z \\ y &= -\tfrac{1}{4} z + \tfrac{1}{2} \end{aligned}$$

z is free

vector solution

$$\begin{bmatrix} \frac{7}{4} z \\ -\frac{1}{4} z + \frac{1}{2} \\ z \end{bmatrix} = \begin{bmatrix} \frac{7}{4} z \\ -\frac{1}{4} z \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix} = z \begin{bmatrix} 7/4 \\ -\frac{1}{4} \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix}$$

a straight line in $\mathbb{R}^3$