CIS 2166

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Info and Links to Linear Algebra Resources

Welcome to the next topic of the course, **Matrix and Linear Algebra**!

- Homework will consist of reading assignments and problem sets, but it will also include watching videos.
- 3Blue1Brown has a YouTube channel called The Essence of Linear Algebra. Watch the first 4 videos before the next class.
- A free textbook, Gregory Hartman, *Fundamentals of Matrix Algebra*, 3rd edition, is available on this site.

A fundamental problem

In general, would like to solve n linear equations with m unknowns. When $n = m$, we have what we'll call "the nice case."

- Row picture (you likely have seen this before – 'lines meeting in a plane')
- $\Rightarrow$ **Column picture** (this may be new to you)
- Matrix form (a compact way of writing down the problem)

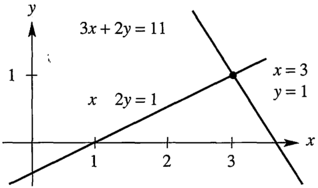
Throughout, we'll emphasize the "geometric viewpoint."

A 2×2 example, the row picture

We'd like to solve this system of equations:

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}$$

This 'row picture'



shows that the lines (rows) intersect at a $\boxed{x = 3, y = 1}$, which is the *unique solution* to this system.

In matrix notation,

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

Or

$$\boxed{\mathbf{A} \cdot \mathbf{x} = \mathbf{b}}$$

with the understanding that

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix},$$

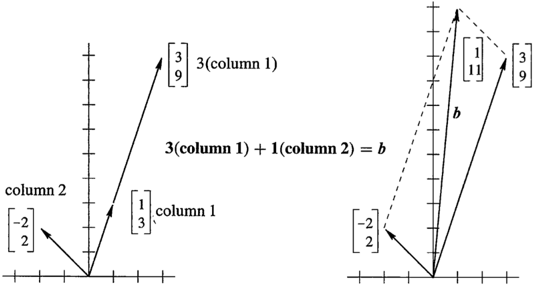
$$\text{and} \ \ \mathbf{b} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

A 2×2 example, the column picture

In the column picture, we interpret the same linear system as a "vector equation."

$$\text{Combination equals } \mathbf{b}: \qquad x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \mathbf{b}.$$

The left hand side is a combination of two column vectors. Goal: find the combination of these vectors (i.e., find x and y) that equals the vector on the right (i.e., $\mathbf{b}$). The answer turns out to be $\boxed{x = 3, y = 1}$.



Taking stock of the algebra involved

Scalar multiplication: $$3 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}.$$

Vector addition: $$\begin{bmatrix} 3 \\ 9 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

The right side of the Figure on the previous slide shows this addition. The resulting vector, $\begin{bmatrix} 1 \\ 11 \end{bmatrix}$, is called the **linear combination** of the columns. The problem was to find the right coefficients $x = 3$ and $y = 1$:

Linear combination: $$3 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

So *linear combination* combines the basic operations – *scalar multiplication* and *vector addition* – into one step.

Summary of oprations

At the heart of linear algebra are two operations on vectors – vector addition and scalar multiplication. Combining these gives the linear combination of the vectors.

Vector: $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ written down as a column of components v_1 and v_2.

Scalar multiplication: $2 \cdot \mathbf{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}$ "stretches" $\mathbf{v}$ 2 times.

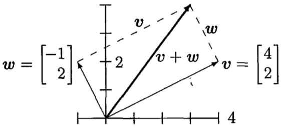
Vector addition: $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$

Linear combination: $c\mathbf{v} + d\mathbf{w} = c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} c + 2d \\ c + 3d \end{bmatrix}$

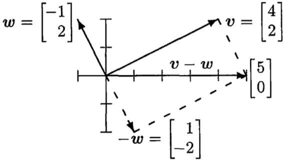
Example: $\mathbf{v} + \mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is the combination with $c = d = 1$

Picturing operations

You can visualize $\mathbf{v} + \mathbf{w}$ using arrows.



$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



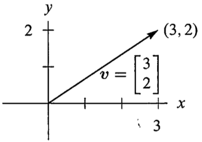
$$v - w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

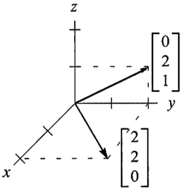
Vector addition (head to tail). At the end of $\mathbf{v}$, place the start of $\mathbf{w}$.

Multiplying $\mathbf{v}$ by a scalar c 'stretches' $\mathbf{v}$ c times. For example, $3 \cdot \mathbf{v}$ is 3 times longer than $\mathbf{v}$ and is pointing in the same direction as $\mathbf{v}$, and $-1 \cdot \mathbf{w}$ is as long as $\mathbf{w}$ and is pointing in the opposite direction from $\mathbf{w}$ as shown in the Figure on the right.

Picturing operations

Things are not very different in 3 dimensions. For example, add the two vectors in the Figure on the right and draw the result.





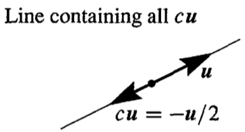
Vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ correspond to points (x, y) and (x, y, z).

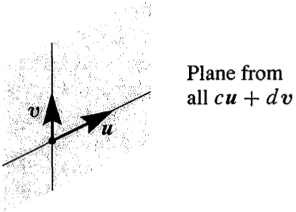
In what follows, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ will sometimes be written as $\mathbf{v} = (v_1, v_2, v_3)$ (this is still the same *column* vector, written this way mainly to save space).

Important questions

1. What is the picture of *all* combinations $c\mathbf{u}$?
2. What is the picture of *all* combinations $c\mathbf{u} + d\mathbf{v}$?
3. What is the picture of *all* combinations $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$?

1. The combinations $c\mathbf{u}$ fill a **line**.
2. The combinations $c\mathbf{u} + d\mathbf{v}$ fill a **plane**.
3. The combinations $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$ fill a **three-dimensional-space**.





A 3-D example

With 3 variables, you can still 'visualize' the process of finding a solution.

$$\mathbf{A}\mathbf{x} = \mathbf{b} \qquad \begin{array}{rcl} x + 2y + 3z & = & 6 \\ 2x + 5y + 2z & = & 4 \\ 6x - 3y + z & = & 2 \end{array}$$

Here

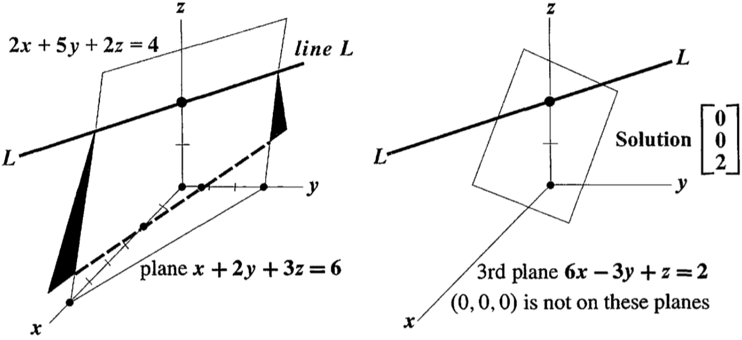
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$$

For the column picture, look for the linear combination of **A**'s columns that lead to vector **b**.

$$x \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ -6 \end{bmatrix} + z \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

The next two slides show both the row and the column pictures.

A 3-D example, the row picture



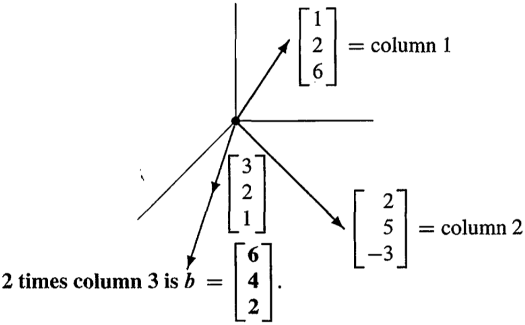
The row picture: Two planes meet in a line, three planes at a point.

A 3-D example, the column picture

The column picture starts by combining the column vectors of **A** :

$$x\begin{bmatrix}1\\2\\6\end{bmatrix}+y\begin{bmatrix}2\\5\\-6\end{bmatrix}+z\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}$$

The unknowns are the coefficients x, y, z. We want to multiply the three column vectors by the correct numbers x, y, z to produce $\mathbf{b} = (6, 4, 2)$.



What's to come

Here's what to expect in the next couple of lectures:

- We'll develop an algorithm for finding solutions to a general system of m linear equations with n unknowns.
- The human imagination hits some severe constraints when attempting to visualize solutions to systems with more than three variables. We are going to have to rely on algebra, but we'll see that the geometric insights prove to be invaluable.
- To get there, we are going to have to spend a bit of time (not much) with some mundane stuff, like operations on matrices. The point of doing this will become apparent when we discuss applications, where the elegance of linear algebra comes to life, and our efforts pay off handsomely.