Compute the eigenvalues and eigenvecors for matrix $\boldsymbol{A}$. How about $\boldsymbol{A}^2$, $\boldsymbol{A}^{-1}$, and $\boldsymbol{A} + 4\boldsymbol{I}$?

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$Ax = \lambda x$

$(A - \lambda I)x = 0$

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$

$= (2-\lambda)^2 - 1 = 4 - 4\lambda + \lambda^2 - 1$

$= \lambda^2 - 4\lambda + 3 = 0$

$\left.\begin{array}{l} \operatorname{tr} A = 4 = \lambda_1 + \lambda_2 \\ \det A = 3 = \lambda_1 \cdot \lambda_2 \end{array}\right\} \Rightarrow \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 3 \end{array}$

$\lambda_1 = 1$

$(A - I)x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \left.\begin{array}{l} x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{array}\right\} \Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$

$(A - 3I)y = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \left.\begin{array}{l} -y_1 - y_2 = 0 \\ -y_1 - y_2 = 0 \end{array}\right\} \Rightarrow y = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$Ax = \lambda x$

$AAx = A\lambda x = \lambda Ax = \lambda\lambda x = \lambda^2 x$

A^2 has the same eigenvectors and λ^2 as eigenvalues

i.e. 1 and 9 are eigenvalues of A^2

$Ax = \lambda x$ we assume $\lambda \neq 0$

$A^{-1}Ax = A^{-1}\lambda x$

$x = \lambda A^{-1}x$

$\frac{1}{\lambda}x = A^{-1}x$

$\Rightarrow$ A^{-1} has the same eigenvectors and $\frac{1}{\lambda}$ eigenvalues,

i.e., 1 and $\frac{1}{3}$ are eigenvalues of A^{-1}

$(A+4I)x = Ax + 4Ix = \lambda x + 4x = (\lambda+4)x$

$A+4I$ has the same eigenvectors and $(\lambda+4)$ eigenvalues, i.e. 5 and 8.