1. (a) Let a directed graph $G_1$ be given.

Does each of the following list of vertices form a path in $G_1$? If yes, determine (by circling) if the path is simple, if it is a circuit, and give its length.

- $a, b, e, c, b$  
  Yes [ simple  circuit   length ]   No

- $a, d, a, d, a$  
  Yes [ simple  circuit   length ]   No

- $a, d, e, b, a$  
  Yes [ simple  circuit   length ]   No

- $a, b, e, c, b, a$  
  Yes [ simple  circuit   length ]   No

(b) For the simple graph $G_2$

![Graph $G_2$](image)

Find $M^2$, where $M$ is the adjacency matrix of $G_2$

$$M^2 = \begin{pmatrix}
\text{elements}
\end{pmatrix}$$

Find the number of paths from A to D in $G_2$ of length 2.
2. Let $f(n) = 1 + 2 + 3 + \ldots + n$. Show that $f(n)$ is $O(n^2)$. Be sure to specify the values of the witnesses $C$ and $k$. 
3. Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

![Graph G and H](image)

4. Show that a vertex c in the connected simple graph G is a cut vertex if and only if there are vertices u and v, both different from c, such that every path between u and v passes through c.
5. 
(a) Is there an Euler circuit in the following graph? If so, find such a circuit. If not, explain why no such circuit exists.
(b) Is there a Hamilton circuit in the following graph? If so, find such a circuit. If not, prove why no such circuit exists.
6. Let \( f(n) = 2n^2 + 5\log(n) + 8n + 7 \). Show that \( f(n) \) is \( O(n^2) \). Be sure to specify the values of the witnesses \( C \) and \( k \).

7. Use Dijkstra’s algorithm to find the length of the shortest path between the vertices \( a \) and \( z \) in the following weighted graph.
8. How many vertices and how many edges does each of the following graphs have?
(a) $K_5$

(b) $C_4$

(c) $W_5$

(d) $K_{2,5}$
9. Describe an algorithm for finding the second largest integer in a sequence of distinct integers. Give a big-O estimate of the number of comparisons used by your algorithm.

10. Give a recursive algorithm for finding the string $w^i$, the concatenation of $i$ copies of $w$, when $w$ is a bit string.
11. Give a recursive definition of
   a) the set of odd positive integers.
   b) the set of positive integer powers of 3.
   c) the set of polynomials with integer coefficients.

12. Let $S$ be the subset of the set of ordered pairs of integers defined recursively by

   **Basis step**: $(0, 0) \in S$.

   **Recursive step**: If $(a, b) \in S$, then $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$.

   a) List the elements of $S$ produced by the first two applications of the recursive definition.
   b) Use structural induction to show that $5 \mid a + b$ when $(a, b) \in S$. 
13. Write a pseudocode for an algorithm for evaluating a polynomial of degree $n$,
$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, at $x = c$.
What is big-O estimate of the time complexity of your algorithm (in terms of the number of
multiplications and additions used) as a function of $n$? Explain your answer.

14. Show that $\log(n!)$ is $\Theta(n \cdot \log(n))$. 
15. Prove that \(1! + 2! + \ldots + n! = (n + 1)! - 1\) whenever \(n\) is a positive integer.

16. Prove that \(1 + 3 + \ldots + (2n - 1) = n^2\) whenever \(n\) is a positive integer.