2.5 Inverse Matrices

Suppose \( A \) is a square matrix. We look for an “inverse matrix” \( A^{-1} \) of the same size, such that \( A^{-1} \) times \( A \) equals \( I \). Whatever \( A \) does, \( A^{-1} \) undoes. Their product is the identity matrix—which does nothing to a vector, so \( A^{-1}Ax = x \). But \( A^{-1} \) might not exist.

What a matrix mostly does is to multiply a vector \( x \). Multiplying \( Ax = b \) by \( A^{-1} \) gives \( A^{-1}Ax = A^{-1}b \). This is \( x = A^{-1}b \). The product \( A^{-1}A \) is like multiplying by a number and then dividing by that number. A number has an inverse if it is not zero—matrices are more complicated and more interesting. The matrix \( A^{-1} \) is called “\( A \) inverse.”

**Definition**

The matrix \( A \) is invertible if there exists a matrix \( A^{-1} \) such that

\[
A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.
\]  

Not all matrices have inverses. This is the first question we ask about a square matrix: Is \( A \) invertible? We don’t mean that we immediately calculate \( A^{-1} \). In most problems we never compute it! Here are six “notes” about \( A^{-1} \).

**Note 1** The inverse exists if and only if elimination produces \( n \) pivots (row exchanges are allowed). Elimination solves \( Ax = b \) without explicitly using the matrix \( A^{-1} \).

**Note 2** The matrix \( A \) cannot have two different inverses. Suppose \( BA = I \) and also \( AC = I \). Then \( B = C \), according to this “proof by parentheses”:

\[
B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{or} \quad B = C.
\]

This shows that a left-inverse \( B \) (multiplying from the left) and a right-inverse \( C \) (multiplying \( A \) from the right to give \( AC = I \)) must be the same matrix.

**Note 3** If \( A \) is invertible, the one and only solution to \( Ax = b \) is \( x = A^{-1}b \):

Multiply \( Ax = b \) by \( A^{-1} \). Then \( x = A^{-1}Ax = A^{-1}b \).

**Note 4** (Important) Suppose there is a nonzero vector \( x \) such that \( Ax = 0 \). Then \( A \) cannot have an inverse. No matrix can bring 0 back to \( x \).

If \( A \) is invertible, then \( Ax = 0 \) can only have the zero solution \( x = A^{-1}0 = 0 \).

**Note 5** A 2 by 2 matrix is invertible if and only if \( ad - bc \) is not zero:

\[
2 \text{ by } 2 \text{ Inverse:} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

This number \( ad - bc \) is the determinant of \( A \). A matrix is invertible if its determinant is not zero (Chapter 5). The test for \( n \) pivots is usually decided before the determinant appears.
Chapter 2. Solving Linear Equations

Note 6  A diagonal matrix has an inverse provided no diagonal entries are zero:

If $A = \begin{bmatrix} d_1 & \cdots & \vdots \\ \vdots & \ddots & \vdots \\ d_n & \cdots & d_n \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 1/d_1 & \cdots & \vdots \\ \vdots & \ddots & \vdots \\ 1/d_n & \cdots & 1/d_n \end{bmatrix}$.

Example 1  The 2 by 2 matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is not invertible. It fails the test in Note 5, because $ad - bc$ equals $2 - 2 = 0$. It fails the test in Note 3, because $Ax = 0$ when $x = (2, -1)$. It fails to have two pivots as required by Note 1.

Elimination turns the second row of this matrix $A$ into a zero row.

The Inverse of a Product $AB$

For two nonzero numbers $a$ and $b$, the sum $a + b$ might or might not be invertible. The numbers $a = 3$ and $b = -3$ have inverses $\frac{1}{3}$ and $-\frac{1}{3}$. Their sum $a + b = 0$ has no inverse.

But the product $ab = -9$ does have an inverse, which is $\frac{1}{9}$ times $-\frac{1}{3}$.

For two matrices $A$ and $B$, the situation is similar. It is hard to say much about the invertibility of $A + B$. But the product $AB$ has an inverse, if and only if the two factors $A$ and $B$ are separately invertible (and the same size). The important point is that $A^{-1}$ and $B^{-1}$ come in reverse order:

If $A$ and $B$ are invertible then so is $AB$. The inverse of a product $AB$ is

$$ (AB)^{-1} = B^{-1}A^{-1}. $$

(4)

To see why the order is reversed, multiply $AB$ times $B^{-1}A^{-1}$. Inside that is $BB^{-1} = I$:

**Inverse of $AB$**  $(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I.$

We moved parentheses to multiply $BB^{-1}$ first. Similarly $B^{-1}A^{-1}$ times $AB$ equals $I$. This illustrates a basic rule of mathematics: Inverses come in reverse order. It is also common sense: If you put on socks and then shoes, the first to be taken off are the socks. The same reverse order applies to three or more matrices:

**Reverse order**  $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$

(5)

Example 2  **Inverse of an elimination matrix.** If $E$ subtracts 5 times row 1 from row 2, then $E^{-1}$ adds 5 times row 1 to row 2:

$$ E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. $$

Multiply $EE^{-1}$ to get the identity matrix $I$. Also multiply $E^{-1}E$ to get $I$. We are adding and subtracting the same 5 times row 1. Whether we add and then subtract (this is $EE^{-1}$) or subtract and then add (this is $E^{-1}E$), we are back at the start.
For square matrices, an inverse on one side is automatically an inverse on the other side. If \( AB = I \) then automatically \( BA = I \). In that case \( B \) is \( A^{-1} \). This is very useful to know but we are not ready to prove it.

**Example 3** Suppose \( F \) subtracts 4 times row 2 from row 3, and \( F^{-1} \) adds it back:

\[
F = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -4 & 1
\end{bmatrix}
\text{ and } \quad F^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 4 & 1
\end{bmatrix}.
\]

Now multiply \( F \) by the matrix \( E \) in Example 2 to find \( FE \). Also multiply \( E^{-1} \) times \( F^{-1} \) to find \( FE \).

\[
FE = \begin{bmatrix}
1 & 0 & 0 \\
-5 & 1 & 0 \\
20 & -4 & 1
\end{bmatrix}
\text{ is inverted by } \quad E^{-1}F^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
5 & 1 & 0 \\
0 & 4 & 1
\end{bmatrix}.
\]

The result is beautiful and correct. The product \( FE \) contains “20” but its inverse doesn’t. \( E \) subtracts 5 times row 1 from row 2. Then \( F \) subtracts 4 times the new row 2 (changed by row 1) from row 3. In this order \( FE \), row 3 feels an effect from row 1. In the order \( E^{-1}F^{-1} \), that effect does not happen. First \( F^{-1} \) adds 4 times row 2 to row 3. After that, \( E^{-1} \) adds 5 times row 1 to row 2. There is no 20, because row 3 doesn’t change again. In this order \( E^{-1}F^{-1} \), row 3 feels no effect from row 1.

In elimination order \( F \) follows \( E \). In reverse order \( E^{-1} \) follows \( F^{-1} \).

\( E^{-1}F^{-1} \) is quick. The multipliers 5, 4 fall into place below the diagonal of 1’s.

In this special multiplication \( E^{-1}F^{-1} \) and \( E^{-1}F^{-1}G^{-1} \) will be useful in the next section. We will explain it again, more completely. In this section our job is \( A^{-1} \), and we expect some serious work to compute it. Here is a way to organize that computation.

### Calculating \( A^{-1} \) by Gauss-Jordan Elimination

I hinted that \( A^{-1} \) might not be explicitly needed. The equation \( Ax = b \) is solved by \( x = A^{-1}b \). But it is not necessary or efficient to compute \( A^{-1} \) and multiply it times \( b \). Elimination goes directly to \( x \). Elimination is also the way to calculate \( A^{-1} \), as we now show. The Gauss-Jordan idea is to solve \( AA^{-1} = I \), finding each column of \( A^{-1} \).

\( A \) multiplies the first column of \( A^{-1} \) (call that \( x_1 \)) to give the first column of \( I \) (call that \( e_1 \)). This is our equation \( Ax_1 = e_1 = (1, 0, 0) \). There will be two more equations. Each of the columns \( x_1, x_2, x_3 \) of \( A^{-1} \) is multiplied by \( A \) to produce a column of \( I \):

\[
3 \text{ columns of } A^{-1} \quad AA^{-1} = A\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix} = \begin{bmatrix}e_1 & e_2 & e_3\end{bmatrix} = I.
\]

To invert a 3 by 3 matrix \( A \), we have to solve three systems of equations: \( Ax_1 = e_1 \) and \( Ax_2 = e_2 = (0, 1, 0) \) and \( Ax_3 = e_3 = (0, 0, 1) \). Gauss-Jordan finds \( A^{-1} \) this way.
The **Gauss-Jordan method** computes $A^{-1}$ by solving *all n equations together*. Usually the “augmented matrix” $[A \ b]$ has one extra column $b$. Now we have three right sides $e_1, e_2, e_3$ (when $A$ is 3 by 3). They are the columns of $I$, so the augmented matrix is really the block matrix $[A \ I]$. I take this chance to invert my favorite matrix $K$, with 2’s on the main diagonal and 1’s next to the 2’s:

$$
\begin{bmatrix}
K & e_1 & e_2 & e_3
\end{bmatrix} =
\begin{bmatrix}
2 & -1 & 0 & 1 & 0 & 0 \\
-1 & 2 & -1 & 0 & 1 & 0 \\
0 & -1 & 2 & 0 & 0 & 1
\end{bmatrix}
$$

**Start Gauss-Jordan on $K$**

$$
\begin{align*}
\rightarrow\ &
\begin{bmatrix}
2 & -1 & 0 & 1 & 0 & 0 \\
0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\
0 & -1 & 2 & 0 & 0 & 1
\end{bmatrix} \quad \text{(} \frac{1}{2} \text{ row 1 + row 2)} \\
\rightarrow\ &
\begin{bmatrix}
2 & -1 & 0 & 1 & 0 & 0 \\
0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\
0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1
\end{bmatrix} \quad \text{(} \frac{3}{4} \text{ row 2 + row 3)}
\end{align*}
$$

We are halfway to $K^{-1}$. The matrix in the first three columns is $U$ (upper triangular). The pivots $2, \frac{3}{2}, \frac{4}{3}$ are on its diagonal. Gauss would finish by back substitution. The contribution of Jordan is *to continue with elimination*. He goes all the way to the “**reduced echelon form**”. Rows are added to rows above them, to produce *zeros above the pivots*:

\begin{align*}
\text{(Zero above third pivot)} & \quad \rightarrow\ &
\begin{bmatrix}
2 & -1 & 0 & 1 & 0 & 0 \\
0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \\
0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1
\end{bmatrix} \quad \text{(} \frac{3}{4} \text{ row 3 + row 2)} \\
\text{(Zero above second pivot)} & \quad \rightarrow\ &
\begin{bmatrix}
2 & 0 & 0 & \frac{3}{4} & 1 & \frac{1}{4} \\
0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \\
0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1
\end{bmatrix} \quad \text{(} \frac{3}{2} \text{ row 2 + row 1)}
\end{align*}

The last Gauss-Jordan step is to divide each row by its pivot. The new pivots are 1. We have reached $I$ in the first half of the matrix, because $K$ is invertible. The **three columns of $K^{-1}$ are in the second half of $[I \ K^{-1}]$**:

\[
\begin{bmatrix}
1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\
0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\
0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4}
\end{bmatrix}
\]

$$
\begin{bmatrix}
I & x_1 & x_2 & x_3
\end{bmatrix} = [I \ K^{-1}].
$$

Starting from the 3 by 6 matrix $[K \ I]$, we ended with $[I \ K^{-1}]$. Here is the whole Gauss-Jordan process on one line for any invertible matrix $A$:

**Gauss-Jordan**

\[
\text{Multiply } [A \ I] \text{ by } A^{-1} \text{ to get } [I \ A^{-1}].
\]
The elimination steps create the inverse matrix while changing \( A \) to \( I \). For large matrices, we probably don’t want \( A^{-1} \) at all. But for small matrices, it can be very worthwhile to know the inverse. We add three observations about this particular \( K^{-1} \) because it is an important example. We introduce the words symmetric, tridiagonal, and determinant:

1. \( K \) is symmetric across its main diagonal. So is \( K^{-1} \).
2. \( K \) is tridiagonal (only three nonzero diagonals). But \( K^{-1} \) is a dense matrix with no zeros. That is another reason we don’t often compute inverse matrices. The inverse of a band matrix is generally a dense matrix.
3. The product of pivots is \( 2(\frac{1}{2})\left(\frac{2}{3}\right) = 4 \). This number 4 is the determinant of \( K \).

\[
K^{-1} \text{ involves division by the determinant } \quad K^{-1} = \begin{bmatrix}
\frac{3}{4} & 2 & 1 \\
2 & 4 & 2 \\
1 & 2 & 3
\end{bmatrix}.
\]

This is why an invertible matrix cannot have a zero determinant.

**Example 4** Find \( A^{-1} \) by Gauss-Jordan elimination starting from \( A = \begin{bmatrix} \frac{3}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{7}{3} \end{bmatrix} \). There are two row operations and then a division to put 1’s in the pivots:

\[
\begin{bmatrix}
2 & 3 & 1 & 0 \\
4 & 7 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
2 & 3 & 1 & 0 \\
0 & 1 & -2 & 1
\end{bmatrix} \quad \text{(this is } \begin{bmatrix} U & L^{-1} \end{bmatrix})
\]

\[
\rightarrow \begin{bmatrix}
2 & 0 & 7 & -3 \\
0 & 1 & -2 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & \frac{7}{2} & -\frac{3}{2} \\
0 & 1 & -2 & 1
\end{bmatrix} \quad \text{(this is } \begin{bmatrix} I & A^{-1} \end{bmatrix}).
\]

That \( A^{-1} \) involves division by the determinant \( ad - bc = 2 \cdot 7 - 3 \cdot 4 = 2 \). The code for \( X = \text{inverse}(A) \) can use \texttt{rref}, the “row reduced echelon form” from Chapter 3:

\[
I = \text{eye}(n); \quad \text{\% Define the } n \text{ by } n \text{ identity matrix}
R = \text{rref}([A I]); \quad \text{\% Eliminate on the augmented matrix } [A I]
X = R(:, n + 1 : n + n) \quad \text{\% Pick } A^{-1} \text{ from the last } n \text{ columns of } R
\]

\( A \) must be invertible, or elimination cannot reduce it to \( I \) (in the left half of \( R \)). Gauss-Jordan shows why \( A^{-1} \) is expensive. We must solve \( n \) equations for its \( n \) columns.

**To solve** \( A x = b \) **without** \( A^{-1} \), **we deal with one column** \( b \) **to find one column** \( x \).

In defense of \( A^{-1} \), we want to say that its cost is not \( n \) times the cost of one system \( A x = b \). Surprisingly, the cost for \( n \) columns is only multiplied by 3. This saving is because the \( n \) equations \( A x_i = e_i \) all involve the same matrix \( A \). Working with the right sides is relatively cheap, because elimination only has to be done once on \( A \).

The complete \( A^{-1} \) needs \( n^3 \) elimination steps, where a single \( x \) needs \( n^3 / 3 \). The next section calculates these costs.
Singular versus Invertible

We come back to the central question. Which matrices have inverses? The start of this section proposed the pivot test: \( A^{-1} \) exists exactly when \( A \) has a full set of \( n \) pivots. (Row exchanges are allowed.) Now we can prove that by Gauss-Jordan elimination:

1. With \( n \) pivots, elimination solves all the equations \( Ax_i = e_i \). The columns \( x_i \) go into \( A^{-1} \). Then \( AA^{-1} = I \) and \( A^{-1} \) is at least a right-inverse.

2. Elimination is really a sequence of multiplications by \( E \)'s and \( P \)'s and \( D^{-1} \):

\[
(D^{-1} \cdots E \cdots P \cdots E)A = I. \tag{9}
\]

\( D^{-1} \) divides by the pivots. The matrices \( E \) produce zeros below and above the pivots. \( P \) will exchange rows if needed (see Section 2.7). The product matrix in equation (9) is evidently a left-inverse. With \( n \) pivots we have reached \( A^{-1}A = I \).

The right-inverse equals the left-inverse. That was Note 2 at the start of this section. So a square matrix with a full set of pivots will always have a two-sided inverse. Reasoning in reverse will now show that \( A \) must have \( n \) pivots if \( AC = I \). (Then we deduce that \( C \) is also a left-inverse and \( CA = I \).) Here is one route to those conclusions:

1. If \( A \) doesn’t have \( n \) pivots, elimination will lead to a zero row.
2. Those elimination steps are taken by an invertible \( M \). So a row of \( MA \) is zero.
3. If \( AC = I \) had been possible, then \( MAC = M \). The zero row of \( MA \) times \( C \), gives a zero row of \( M \) itself.
4. An invertible matrix \( M \) can’t have a zero row! \( A \) must have \( n \) pivots if \( AC = I \).

That argument took four steps, but the outcome is short and important.

Elimination gives a complete test for invertibility of a square matrix. \( A^{-1} \) exists (and Gauss-Jordan finds it) exactly when \( A \) has \( n \) pivots. The argument above shows more:

If \( AC = I \) then \( CA = I \) and \( C = A^{-1} \)

Example 5 If \( L \) is lower triangular with 1’s on the diagonal, so is \( L^{-1} \).

A triangular matrix is invertible if and only if no diagonal entries are zero.

Here \( L \) has 1’s so \( L^{-1} \) also has 1’s. Use the Gauss-Jordan method to construct \( L^{-1} \). Start by subtracting multiples of pivot rows from rows below. Normally this gets us halfway to the inverse, but for \( L \) it gets us all the way. \( L^{-1} \) appears on the right when \( I \) appears on the left. Notice how \( L^{-1} \) contains 11, from 3 times 5 minus 4.
2.5 Inverse Matrices

Gauss-Jordan on triangular $L$

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
3 & 1 & 0 & 0 & 1 & 0 \\
4 & 5 & 1 & 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -3 & 1 & 0 \\
0 & 5 & 1 & 1 & -4 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -3 & 1 & 0 \\
0 & 0 & 1 & 11 & -5 & 1
\end{bmatrix}
\]

$L$ goes to $I$ by a product of elimination matrices $E_{32}E_{31}E_{21}$. So that product is $L^{-1}$.

All pivots are 1’s (a full set). $L^{-1}$ is lower triangular, with the strange entry “11”.

That 11 does not appear to spoil 3, 4, 5 in the good order $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = L$.

### REVIEW OF THE KEY IDEAS

1. The inverse matrix gives $AA^{-1} = I$ and $A^{-1}A = I$.

2. $A$ is invertible if and only if it has $n$ pivots (row exchanges allowed).

3. If $Ax = 0$ for a nonzero vector $x$, then $A$ has no inverse.

4. The inverse of $AB$ is the reverse product $B^{-1}A^{-1}$. And $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

5. The Gauss-Jordan method solves $AA^{-1} = I$ to find the $n$ columns of $A^{-1}$. The augmented matrix $[A \ I]$ is row-reduced to $[I \ A^{-1}]$.

### WORKED EXAMPLES

2.5 A The inverse of a triangular difference matrix $A$ is a triangular sum matrix $S$:

\[
\begin{bmatrix}
1 & 0 & 0 | 1 & 0 & 0 \\
0 & -1 & 1 | 0 & 0 & 0 \\
0 & 1 & 0 | 1 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 | 1 & 0 & 0 \\
0 & 1 & 0 | 1 & 1 & 0 \\
0 & -1 & 1 | 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 | 1 & 0 & 0 \\
0 & 1 & 0 | 1 & 1 & 0 \\
0 & 0 & 1 | 1 & 1 & 1
\end{bmatrix}
= [I \ A^{-1}] = [I \ \text{sum matrix}]
\]

If I change $a_{13}$ to $-1$, then all rows of $A$ add to zero. The equation $Ax = 0$ will now have the nonzero solution $x = (1, 1, 1)$. A clear signal: *This new $A$ can’t be inverted.*
2.5 B Three of these matrices are invertible, and three are singular. Find the inverse when it exists. Give reasons for noninvertibility (zero determinant, too few pivots, nonzero solution to $Ax = 0$) for the other three. The matrices are in the order $A, B, C, D, S, E$:

$$
\begin{bmatrix}
4 & 3 \\
8 & 6
\end{bmatrix}
\begin{bmatrix}
4 & 3 \\
8 & 7
\end{bmatrix}
\begin{bmatrix}
6 & 6 \\
6 & 0
\end{bmatrix}
\begin{bmatrix}
6 & 6 \\
6 & 6
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
$$

Solution

$$ B^{-1} = \frac{1}{4} \begin{bmatrix} 7 & -3 \\ -8 & 4 \end{bmatrix} \quad C^{-1} = \frac{1}{36} \begin{bmatrix} 0 & 6 \\ 6 & -6 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} $$

$A$ is not invertible because its determinant is $4 \cdot 6 - 3 \cdot 8 = 24 - 24 = 0$. $D$ is not invertible because there is only one pivot; the second row becomes zero when the first row is subtracted. $E$ is not invertible because a combination of the columns (the second column minus the first column) is zero—in other words $Ex = 0$ has the solution $x = (-1, 1, 0)$.

Of course all three reasons for noninvertibility would apply to each of $A, D, E$.

2.5 C Apply the Gauss-Jordan method to invert this triangular “Pascal matrix” $L$. You see Pascal’s triangle—adding each entry to the entry on its left gives the entry below. The entries of $L$ are “binomial coefficients”. The next row would be $1, 4, 6, 4, 1$.

$$
\begin{bmatrix}
1 & 3 & 3 & 1 \\
1 & 2 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
= \text{abs(pascal (4,1))}
$$

Solution Gauss-Jordan starts with $[L \ I]$ and produces zeros by subtracting row 1:

$$
\left[ \begin{array}{cc|ccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 1 & 0 \\
1 & 3 & 3 & 1 & 0 & 0 & 1
\end{array} \right]
\rightarrow
\left[ \begin{array}{cc|ccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 3 & 3 & 1 & -1 & 0 & 0 & 1
\end{array} \right].
$$

The next stage creates zeros below the second pivot, using multipliers 2 and 3. Then the last stage subtracts 3 times the new row 3 from the new row 4:

$$
\rightarrow
\left[ \begin{array}{cc|ccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 3 & 1 & 2 & -3 & 0 & 1
\end{array} \right]
\rightarrow
\left[ \begin{array}{cc|ccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 3 & -3 & 1
\end{array} \right]
= [I \ L^{-1}].
$$

All the pivots were 1! So we didn’t need to divide rows by pivots to get $I$. The inverse matrix $L^{-1}$ looks like $L$ itself, except odd-numbered diagonals have minus signs.

The same pattern continues to $n$ by $n$ Pascal matrices, $L^{-1}$ has “alternating diagonals”.

2.5. Inverse Matrices

Problem Set 2.5

1. Find the inverses (directly or from the 2 by 2 formula) of \( A, B, C \):

\[
A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.
\]

2. For these “permutation matrices” find \( P^{-1} \) by trial and error (with 1’s and 0’s):

\[
P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.
\]

3. Solve for the first column \((x, y)\) and second column \((t, z)\) of \( A^{-1} \):

\[
\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

4. Show that \( \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix} \) is not invertible by trying to solve \( AA^{-1} = I \) for column 1 of \( A^{-1} \):

\[
\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{(For a different } A, \text{ could column 1 of } A^{-1} \text{ be possible to find but not column 2?)}
\]

5. Find an upper triangular \( U \) (not diagonal) with \( U^2 = I \) which gives \( U = U^{-1} \).

6. (a) If \( A \) is invertible and \( AB = AC \), prove quickly that \( B = C \).

(b) If \( A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \), find two different matrices such that \( AB = AC \).

7. (Important) If \( A \) has row 1 + row 2 = row 3, show that \( A \) is not invertible:

(a) Explain why \( Ax = (1, 0, 0) \) cannot have a solution.

(b) Which right sides \((b_1, b_2, b_3)\) might allow a solution to \( Ax = b \)?

(c) What happens to row 3 in elimination?

8. If \( A \) has column 1 + column 2 = column 3, show that \( A \) is not invertible:

(a) Find a nonzero solution \( x \) to \( Ax = 0 \). The matrix is 3 by 3.

(b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot.

9. Suppose \( A \) is invertible and you exchange its first two rows to reach \( B \). Is the new matrix \( B \) invertible and how would you find \( B^{-1} \) from \( A^{-1} \)?

10. Find the inverses (in any legal way) of

\[
A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.
\]
11. (a) Find invertible matrices $A$ and $B$ such that $A + B$ is not invertible.
(b) Find singular matrices $A$ and $B$ such that $A + B$ is invertible.

12. If the product $C = AB$ is invertible ($A$ and $B$ are square), then $A$ itself is invertible.
Find a formula for $A^{-1}$ that involves $C^{-1}$ and $B$.

13. If the product $M = ABC$ of three square matrices is invertible, then $B$ is invertible.
(Do $A$ and $C$.) Find a formula for $B^{-1}$ that involves $M^{-1}$ and $A$ and $C$.

14. If you add row 1 of $A$ to row 2 to get $B$, how do you find $B^{-1}$ from $A^{-1}$?

15. Notice the order. The inverse of $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} A$ is ______.

16. Prove that a matrix with a column of zeros cannot have an inverse.

17. (a) What matrix $E$ has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.
(b) What single matrix $L$ has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

18. If $B$ is the inverse of $A^2$, show that $AB$ is the inverse of $A$.

19. Find the numbers $a$ and $b$ that give the inverse of $5 \times \text{eye}(4) - \text{ones}(4,4)$:
\[
\begin{bmatrix}
4 & -1 & -1 & -1 \\
-1 & 4 & -1 & -1 \\
-1 & -1 & 4 & -1 \\
-1 & -1 & -1 & 4
\end{bmatrix}^{-1} = \begin{bmatrix}
a & b & b & b \\
b & a & b & b \\
b & b & a & b \\
b & b & b & a
\end{bmatrix}.
\]

What are $a$ and $b$ in the inverse of $6 \times \text{eye}(5) - \text{ones}(5,5)$?

20. Show that $A = 4 \times \text{eye}(4) - \text{ones}(4,4)$ is not invertible: Multiply $A \times \text{ones}(4,1)$.

21. There are sixteen 2 by 2 matrices whose entries are 1’s and 0’s. How many of them are invertible?

**Questions 22–28 are about the Gauss-Jordan method for calculating $A^{-1}$.**

22. Change $I$ into $A^{-1}$ as you reduce $A$ to $I$ (by row operations):
\[
\begin{bmatrix}
A & I
\end{bmatrix} = \begin{bmatrix}
1 & 3 & 1 & 0 \\
2 & 7 & 0 & 1
\end{bmatrix} \text{ and } \begin{bmatrix}
A & I
\end{bmatrix} = \begin{bmatrix}
1 & 4 & 1 & 0 \\
3 & 9 & 0 & 1
\end{bmatrix}
\]

23. Follow the 3 by 3 text example but with plus signs in $A$. Eliminate above and below
the pivots to reduce $[A \ I]$ to $[I \ A^{-1}]$:
\[
\begin{bmatrix}
A & I
\end{bmatrix} = \begin{bmatrix}
2 & 1 & 0 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 & 1 & 0 \\
0 & 1 & 2 & 0 & 0 & 1
\end{bmatrix}.
\]
Use Gauss-Jordan elimination on $\begin{bmatrix} U & I \end{bmatrix}$ to find the upper triangular $U^{-1}$:

$$
UU^{-1} = I = \begin{bmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$

25. Find $A^{-1}$ and $B^{-1}$ (if they exist) by elimination on $\begin{bmatrix} A & I \end{bmatrix}$ and $\begin{bmatrix} B & I \end{bmatrix}$:

$$
A = \begin{bmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{bmatrix}.
$$

26. What three matrices $E_{21}$ and $E_{12}$ and $D^{-1}$ reduce $A = \begin{bmatrix} 1 & 2 \\ 1 & 6 \end{bmatrix}$ to the identity matrix?

Multiply $D^{-1}E_{12}E_{21}$ to find $A^{-1}$.

27. Invert these matrices $A$ by the Gauss-Jordan method starting with $\begin{bmatrix} A & I \end{bmatrix}$:

$$
\begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 3 \\
0 & 0 & 1
\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 3
\end{bmatrix}.
$$

28. Exchange rows and continue with Gauss-Jordan to find $A^{-1}$:

$$
\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix}
0 & 2 & 1 & 0 \\
2 & 2 & 0 & 1
\end{bmatrix}.
$$

29. True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.

(b) Every matrix with 1’s down the main diagonal is invertible.

(c) If $A$ is invertible then $A^{-1}$ and $A^2$ are invertible.

30. For which three numbers $c$ is this matrix not invertible, and why not?

$$
A = \begin{bmatrix}
2 & c & c \\
c & c & c \\
8 & 7 & c
\end{bmatrix}.
$$

31. Prove that $A$ is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or $A^{-1}$):

$$
A = \begin{bmatrix}
a & b & b \\
a & a & b \\
a & a & a
\end{bmatrix}.$$
This matrix has a remarkable inverse. Find $A^{-1}$ by elimination on $[A \quad I]$. Extend to a 5 by 5 “alternating matrix” and guess its inverse; then multiply to confirm.

\[
\begin{bmatrix}
1 & -1 & 1 & -1 \\
0 & 1 & -1 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Invert $A$ and solve $Ax = (1, 1, 1, 1, 1)$.

Suppose the matrices $P$ and $Q$ have the same rows as $I$ but in any order. They are “permutation matrices”. Show that $P - Q$ is singular by solving $(P - Q)x = 0$.

Find and check the inverses (assuming they exist) of these block matrices:

\[
\begin{bmatrix}
I & 0 \\
C & I
\end{bmatrix}
\begin{bmatrix}
A & 0 \\
C & D
\end{bmatrix}
\begin{bmatrix}
0 & I \\
I & D
\end{bmatrix}.
\]

Could a 4 by 4 matrix $A$ be invertible if every row contains the numbers 0, 1, 2, 3 in some order? What if every row of $B$ contains 0, 1, 2, -3 in some order?

In the Worked Example 2.5 C, the triangular Pascal matrix $L$ has an inverse with “alternating diagonals”. Check that this $L^{-1}$ is $DLD$, where the diagonal matrix $D$ has alternating entries $1, -1, 1, -1$. Then $LDLD = I$, so what is the inverse of $LD = \text{pascal}(4,1)$?

The Hilbert matrices have $H_{ij} = 1/(i + j + 1)$. Ask MATLAB for the exact 6 by 6 inverse $\text{invhilb}(6)$. Then ask it to compute $\text{inv}(\text{hilb}(6))$. How can these be different, when the computer never makes mistakes?

(a) Use $\text{inv}(P)$ to invert MATLAB’s 4 by 4 symmetric matrix $P = \text{pascal}(4)$.

(b) Create Pascal’s lower triangular $L = \text{abs}(\text{pascal}(4,1))$ and test $P = LL^T$.

If $A = \text{ones}(4)$ and $b = \text{rand}(4,1)$, how does MATLAB tell you that $Ax = b$ has no solution? For the special $b = \text{ones}(4,1)$, which solution to $Ax = b$ is found by $A \setminus b$?

Challenge Problems

(Recommended) $A$ is a 4 by 4 matrix with 1’s on the diagonal and $-a, -b, -c$ on the diagonal above. Find $A^{-1}$ for this bidiagonal matrix.

Suppose $E_1, E_2, E_3$ are 4 by 4 identity matrices, except $E_1$ has $a, b, c$ in column 1 and $E_2$ has $d, e$ in column 2 and $E_3$ has $f$ in column 3 (below the 1’s). Multiply $L = E_1E_2E_3$ to show that all these nonzeros are copied into $L$.

$E_1E_2E_3$ is in the opposite order from elimination (because $E_3$ is acting first). But $E_1E_2E_3 = L$ is in the correct order to invert elimination and recover $A$. 
2.5. Inverse Matrices

Direct multiplications 1–4 give \( MM^{-1} = I \), and I would recommend doing #3. \( M^{-1} \) shows the change in \( A^{-1} \) (useful to know) when a matrix is subtracted from \( A \):

1. \( M = I - uv \) and \( M^{-1} = I + uv/(1 - vu) \) (rank 1 change in \( I \))
2. \( M = A - uv \) and \( M^{-1} = A^{-1} + A^{-1}uvA^{-1}/(1 - vA^{-1}u) \)
3. \( M = I - UV \) and \( M^{-1} = I_n + U(I_m - VU)^{-1}V \)
4. \( M = A - UW^{-1}V \) and \( M^{-1} = A^{-1} + A^{-1}U(W - VA^{-1}U)^{-1}VA^{-1} \)

The Woodbury-Morrison formula 4 is the “matrix inversion lemma” in engineering. The Kalman filter for solving block tridiagonal systems uses formula 4 at each step. The four matrices \( M^{-1} \) are in diagonal blocks when inverting these block matrices (\( v = 1 \) by \( n \), \( u \) is \( n \) by \( 1 \), \( V = m \) by \( n \), \( U \) is \( n \) by \( m \)).

\[
\begin{bmatrix}
I & u \\
v & 1
\end{bmatrix}
\begin{bmatrix}
A & u \\
v & 1
\end{bmatrix}
\begin{bmatrix}
I_n & U \\
V & I_m
\end{bmatrix}
\begin{bmatrix}
A & U \\
V & W
\end{bmatrix}
\]

Second difference matrices have beautiful inverses if they start with \( T_{11} = 1 \) (instead of \( K_{11} = 2 \)). Here is the 3 by 3 tridiagonal matrix \( T \) and its inverse:

\[
T_{11} = 1 \quad T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\]

One approach is Gauss-Jordan elimination on \( [T \quad I] \). That seems too mechanical. I would rather write \( T \) as the product of first differences \( L \) times \( U \). The inverses of \( L \) and \( U \) in Worked Example 2.5 A are sum matrices, so here are \( T \) and \( T^{-1} \):

\[
LU = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \quad U^{-1}L^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\]

**Question.** (4 by 4) What are the pivots of \( T \)? What is its 4 by 4 inverse? The reverse order \( UL \) gives what matrix \( T^* \)? What is the inverse of \( T^* \)?

Here are two more difference matrices, both important. But are they invertible?

**Cyclic** \( C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \) \( \ \quad \text{Free ends} \ F = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \).

One test is elimination—the fourth pivot fails. Another test is the determinant, we don’t want that. The best way is much faster, and independent of matrix size:

**Produce** \( x \not= 0 \) so that \( Cx = 0 \). Do the same for \( Fx = 0 \). Not invertible.

Show how both equations \( Cx = b \) and \( Fx = b \) lead to \( 0 = b_1 + b_2 + \cdots + b_n \). There is no solution for other \( b \).
Chapter 2. Solving Linear Equations

45 *Elimination for a 2 by 2 block matrix:* When you multiply the first block row by $CA^{-1}$ and subtract from the second row, the “Schur complement” $S$ appears:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix},$$

where $A$ and $D$ are square and $S = D - CA^{-1}B$.

Multiply on the right to subtract $A^{-1}B$ times block column 1 from block column 2.

$$\begin{bmatrix} A & B \\ 0 & S \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = ?$$

Find $S$ for $\begin{bmatrix} A & B \\ C & I \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$.

*The block pivots are $A$ and $S$. If they are invertible, so is $[A \ B : C \ D]$.*

46 How does the identity $A(I + BA) = (I + AB)A$ connect the inverses of $I + BA$ and $I + AB$? Those are both invertible or both singular: not obvious.