1.3 Matrices

This section is based on two carefully chosen examples. They both start with three vectors. I will take their combinations using *matrices*. The three vectors in the first example are $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$:

First example $$\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Their linear combinations in three-dimensional space are $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$:

Combinations $$c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}. \tag{1}$$

Now something important: *Rewrite that combination using a matrix.* The vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ go into the columns of the matrix A. That matrix "multiplies" a vector:

Same combination is now A times $\boldsymbol{x}$ $$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}. \tag{2}$$

The numbers c, d, e are the components of a vector $\boldsymbol{x}$. The matrix A times the vector $\boldsymbol{x}$ is the same as the combination $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ of the three columns:

Matrix times vector $$A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}. \tag{3}$$

This is more than a definition of $A\boldsymbol{x}$, because the rewriting brings a crucial change in viewpoint. At first, the numbers c, d, e were multiplying the vectors. Now the matrix is multiplying those numbers. **The matrix A acts on the vector $\boldsymbol{x}$.** The result $A\boldsymbol{x}$ is a combination $\boldsymbol{b}$ of the columns of A.

To see that action, I will write x_1, x_2, x_3 instead of c, d, e. I will write b_1, b_2, b_3 for the components of $A\boldsymbol{x}$. With new letters we see

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2 - x_1} \\ \boldsymbol{x_3 - x_2} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \boldsymbol{b}. \tag{4}$$

The input is $\boldsymbol{x}$ and the output is $\boldsymbol{b} = A\boldsymbol{x}$. This A is a "**difference matrix**" because $\boldsymbol{b}$ contains differences of the input vector $\boldsymbol{x}$. The top difference is $x_1 - x_0 = x_1 - 0$.

Here is an example to show differences of numbers (squares in $\boldsymbol{x}$, odd numbers in $\boldsymbol{b}$):

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \text{squares} \qquad A\boldsymbol{x} = \begin{bmatrix} 1-0 \\ 4-1 \\ 9-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \boldsymbol{b}\,. \tag{5}$$

That pattern would continue for a 4 by 4 difference matrix. The next square would be $x_4 = 16$. The next difference would be $x_4 - x_3 = 16 - 9 = 7$ (this is the next odd number). The matrix finds all the differences at once.

Important Note. You may already have learned about multiplying $A\boldsymbol{x}$, a matrix times a vector. Probably it was explained differently, using the rows instead of the columns. The usual way takes the dot product of each row with $\boldsymbol{x}$:

Dot products with rows

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,0,0) \cdot (x_1,x_2,x_3) \\ (-1,1,0) \cdot (x_1,x_2,x_3) \\ (0,-1,1) \cdot (x_1,x_2,x_3) \end{bmatrix}.$$

Those dot products are the same x_1 and $x_2 - x_1$ and $x_3 - x_2$ that we wrote in equation (4). The new way is to work with $A\boldsymbol{x}$ *a column at a time*. Linear combinations are the key to linear algebra, and the output $A\boldsymbol{x}$ is a linear combination of the **columns** of A.

With numbers, you can multiply $A\boldsymbol{x}$ either way (I admit to using rows). With letters, columns are the good way. Chapter 2 will repeat these rules of matrix multiplication, and explain the underlying ideas. There we will multiply matrices both ways.

Linear Equations

One more change in viewpoint is crucial. Up to now, the numbers x_1, x_2, x_3 were known (called c, d, e at first). The right hand side $\boldsymbol{b}$ was not known. We found that vector of differences by multiplying $A\boldsymbol{x}$. **Now we think of $\boldsymbol{b}$ as known and we look for $\boldsymbol{x}$.**

Old question: Compute the linear combination $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w}$ to find $\boldsymbol{b}$.
New question: Which combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ produces a particular vector $\boldsymbol{b}$?

This is the inverse problem—to find the input $\boldsymbol{x}$ that gives the desired output $\boldsymbol{b} = A\boldsymbol{x}$. You have seen this before, as a system of linear equations for x_1, x_2, x_3. The right hand sides of the equations are b_1, b_2, b_3. We can solve that system to find x_1, x_2, x_3:

$$A\boldsymbol{x} = \boldsymbol{b} \qquad \begin{aligned} x_1 \qquad\quad &= b_1 \\ -x_1 + x_2 \quad &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned} \qquad \textbf{Solution} \qquad \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3. \end{aligned} \tag{6}$$

Let me admit right away—most linear systems are not so easy to solve. In this example, the first equation decided $x_1 = b_1$. Then the second equation produced $x_2 = b_1 + b_2$. *The equations could be solved in order* (top to bottom) *because the matrix A was selected to be* **lower triangular**.

Look at two specific choices 0, 0, 0 and 1, 3, 5 of the right sides b_1, b_2, b_3:

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ gives } \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \text{ gives } \boldsymbol{x} = \begin{bmatrix} 1 \\ 1+3 \\ 1+3+5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}.$$

The first solution (all zeros) is more important than it looks. In words: *If the output is* $\boldsymbol{b} = \boldsymbol{0}$, *then the input must be* $\boldsymbol{x} = \boldsymbol{0}$. That statement is true for this matrix A. It is not true for all matrices. Our second example will show (for a different matrix C) how we can have $C\boldsymbol{x} = \boldsymbol{0}$ when $C \neq 0$ and $\boldsymbol{x} \neq \boldsymbol{0}$.

This matrix A is "**invertible**". From $\boldsymbol{b}$ we can recover $\boldsymbol{x}$.

The Inverse Matrix

Let me repeat the solution $\boldsymbol{x}$ in equation (6). A sum matrix will appear!

$$A\boldsymbol{x} = \boldsymbol{b} \text{ is solved by } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad (7)$$

If the differences of the x's are the b's, the sums of the b's are the x's. That was true for the odd numbers $\boldsymbol{b} = (1, 3, 5)$ and the squares $\boldsymbol{x} = (1, 4, 9)$. It is true for all vectors. **The sum matrix** S **in equation** (7) **is the inverse of the difference matrix** A.

Example: The differences of $\boldsymbol{x} = (1, 2, 3)$ are $\boldsymbol{b} = (1, 1, 1)$. So $\boldsymbol{b} = A\boldsymbol{x}$ and $\boldsymbol{x} = S\boldsymbol{b}$:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} \text{ and } S\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}$$

Equation (7) for the solution vector $\boldsymbol{x} = (x_1, x_2, x_3)$ tells us two important facts:

1. For every $\boldsymbol{b}$ there is one solution to $A\boldsymbol{x} = \boldsymbol{b}$. **2.** A matrix S produces $\boldsymbol{x} = S\boldsymbol{b}$.

The next chapters ask about other equations $A\boldsymbol{x} = \boldsymbol{b}$. Is there a solution? How is it computed? In linear algebra, the notation for the "inverse matrix" is A^{-1}:

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{is solved by} \quad \boldsymbol{x} = A^{-1}\boldsymbol{b} = S\boldsymbol{b}.$$

Note on calculus. Let me connect these special matrices A and S to calculus. The vector $\boldsymbol{x}$ changes to a function $x(t)$. The differences $A\boldsymbol{x}$ become the *derivative* $dx/dt = b(t)$. In the inverse direction, the sum $S\boldsymbol{b}$ becomes the *integral* of $b(t)$. The Fundamental Theorem of Calculus says that ***integration S is the inverse of differentiation A***.

$$\boldsymbol{Ax = b} \textbf{\textit{ and }} \boldsymbol{x = Sb} \qquad \frac{dx}{dt} = b \text{ and } x(t) = \int_0^t b. \qquad (8)$$

The derivative of distance traveled (x) is the velocity (b). The integral of $b(t)$ is the distance $x(t)$. Instead of adding $+C$, I measured the distance from $x(0) = 0$. In the same way, the differences started at $x_0 = 0$. This zero start makes the pattern complete, when we write $x_1 - x_0$ for the first component of $A\boldsymbol{x}$ (we just wrote x_1).

Notice another analogy with calculus. The differences of squares $0, 1, 4, 9$ are odd numbers $1, 3, 5$. The derivative of $x(t) = t^2$ is $2t$. A perfect analogy would have produced the even numbers $b = 2, 4, 6$ at times $t = 1, 2, 3$. But differences are not the same as derivatives, and our matrix A produces not $2t$ but $2t - 1$ (these one-sided "backward differences" are centered at $t - \frac{1}{2}$):

$$x(t) - x(t-1) = t^2 - (t-1)^2 = t^2 - (t^2 - 2t + 1) = 2t - 1. \tag{9}$$

The Problem Set will follow up to show that "forward differences" produce $2t + 1$. A better choice (not always seen in calculus courses) is a **centered difference** that uses $x(t+1) - x(t-1)$. Divide Δx by the distance Δt from $t - 1$ to $t + 1$, which is 2:

$$\textbf{Centered difference of } \boldsymbol{x(t) = t^2} \qquad \frac{(t+1)^2 - (t-1)^2}{2} = 2t \quad \text{exactly.} \tag{10}$$

Difference matrices are great. Centered is best. Our second example is *not invertible*.

Cyclic Differences

This example keeps the same columns $\boldsymbol{u}$ and $\boldsymbol{v}$ but changes $\boldsymbol{w}$ to a new vector $\boldsymbol{w}^*$:

$$\textbf{Second example} \qquad \boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \boldsymbol{w}^* = \begin{bmatrix} \mathbf{-1} \\ 0 \\ 1 \end{bmatrix}.$$

Now the linear combinations of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}^*$ lead to a **cyclic difference matrix** C:

$$\textbf{Cyclic} \qquad C\boldsymbol{x} = \begin{bmatrix} 1 & 0 & \mathbf{-1} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}. \tag{11}$$

This matrix C is not triangular. It is not so simple to solve for $\boldsymbol{x}$ when we are given $\boldsymbol{b}$. Actually it is impossible to find *the* solution to $C\boldsymbol{x} = \boldsymbol{b}$, because the three equations either have **infinitely many solutions** or else **no solution**:

$$\begin{array}{l} \boldsymbol{C x = 0} \\ \textbf{Infinitely} \\ \textbf{many } \boldsymbol{x} \end{array} \quad \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ is solved by all vectors } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix}. \tag{12}$$

Every constant vector (c, c, c) has zero differences when we go cyclically. This undetermined constant c is like the $+C$ that we add to integrals. The cyclic differences have $x_1 - x_3$ in the first component, instead of starting from $x_0 = 0$.

The other very likely possibility for $Cx = b$ is **no solution** at all:

$$C\boldsymbol{x} = \boldsymbol{b} \qquad \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \qquad \begin{array}{l} \text{Left sides add to } 0 \\ \text{Right sides add to } 9 \\ \textit{No solution } x_1, x_2, x_3 \end{array} \tag{13}$$

Look at this example geometrically. No combination of $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}^*$ will produce the vector $\boldsymbol{b} = (1, 3, 5)$. The combinations don't fill the whole three-dimensional space. The right sides must have $b_1 + b_2 + b_3 = 0$ to allow a solution to $C\boldsymbol{x} = \boldsymbol{b}$, because the left sides $x_1 - x_3$, $x_2 - x_1$, and $x_3 - x_2$ always add to zero.

Put that in different words. **All linear combinations** $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w}^* = \boldsymbol{b}$ **lie on the plane given by** $b_1 + b_2 + b_3 = 0$. This subject is suddenly connecting algebra with geometry. Linear combinations can fill all of space, or only a plane. We need a picture to show the crucial difference between $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ (the first example) and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$.

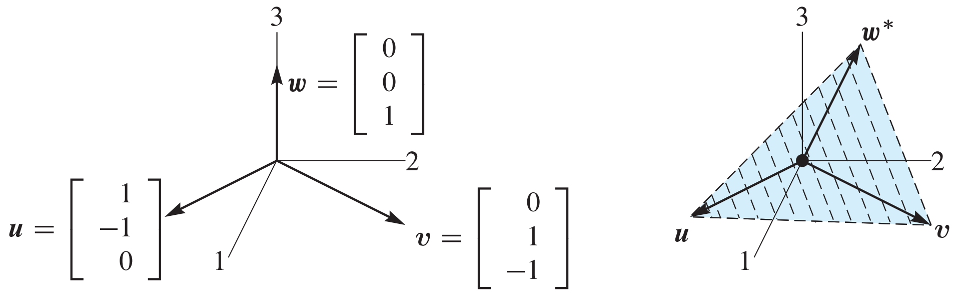

Figure 1.10: Independent vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Dependent vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ in a plane.

Independence and Dependence

Figure 1.10 shows those column vectors, first of the matrix A and then of C. The first two columns $\boldsymbol{u}$ and $\boldsymbol{v}$ are the same in both pictures. If we only look at the combinations of those two vectors, we will get a two-dimensional plane. **The key question is whether the third vector is in that plane**:

Independence $\quad \boldsymbol{w}$ is not in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$.
Dependence $\quad \boldsymbol{w}^*$ *is* in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$.

The important point is that the new vector $\boldsymbol{w}^*$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}^* = 0 \qquad \boldsymbol{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -\boldsymbol{u} - \boldsymbol{v}. \tag{14}$$

All three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ have components adding to zero. Then all their combinations will have $b_1 + b_2 + b_3 = 0$ (as we saw above, by adding the three equations). This is the equation for the plane containing all combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$. By including $\boldsymbol{w}^*$ we get *no new vectors* because $\boldsymbol{w}^*$ is already on that plane.

The original $\boldsymbol{w} = (0, 0, 1)$ is not on the plane: $0 + 0 + 1 \neq 0$. The combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ fill the whole three-dimensional space. We know this already, because the solution $\boldsymbol{x} = S\boldsymbol{b}$ in equation (6) gave the right combination to produce any $\boldsymbol{b}$.

The two matrices A and C, with third columns $\boldsymbol{w}$ and $\boldsymbol{w}^*$, allowed me to mention two key words of linear algebra: independence and dependence. The first half of the course will develop these ideas much further—I am happy if you see them early in the two examples:

$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are **independent**. No combination except $0\boldsymbol{u} + 0\boldsymbol{v} + 0\boldsymbol{w} = \boldsymbol{0}$ gives $\boldsymbol{b} = \boldsymbol{0}$.

$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ are **dependent**. Other combinations (specifically $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}^*$) give $\boldsymbol{b} = \boldsymbol{0}$.

You can picture this in three dimensions. The three vectors lie in a plane or they don't. Chapter 2 has n vectors in n-dimensional space. *Independence or dependence* is the key point. The vectors go into the columns of an n by n matrix:

Independent columns: $A\boldsymbol{x} = \boldsymbol{0}$ has one solution. A is an **invertible matrix**.

Dependent columns: $A\boldsymbol{x} = \boldsymbol{0}$ has many solutions. A is a **singular matrix**.

Eventually we will have n vectors in m-dimensional space. The matrix A with those n columns is now *rectangular* (m by n). Understanding $A\boldsymbol{x} = \boldsymbol{b}$ is the problem of Chapter 3.

■ REVIEW OF THE KEY IDEAS ■

1. **Matrix times vector**: $A\boldsymbol{x}$ = combination of the columns of A.
2. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$, when A is an invertible matrix.
3. The difference matrix A is inverted by the sum matrix $S = A^{-1}$.
4. The cyclic matrix C has no inverse. Its three columns lie in the same plane. Those dependent columns add to the zero vector. $C\boldsymbol{x} = \boldsymbol{0}$ has many solutions.
5. This section is looking ahead to key ideas, not fully explained yet.

■ WORKED EXAMPLES ■

1.3 A Change the southwest entry a_{31} of A (row 3, column 1) to $a_{31} = \mathbf{1}$:

$$A\boldsymbol{x} = \boldsymbol{b} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \mathbf{1} & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ \boldsymbol{x_1} - x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Find the solution $\boldsymbol{x}$ for any $\boldsymbol{b}$. From $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ read off the inverse matrix A^{-1}.

Solution Solve the (linear triangular) system $A\boldsymbol{x} = \boldsymbol{b}$ from top to bottom:

$$\begin{array}{ll} \text{first} & x_1 = b_1 \\ \text{then} & x_2 = b_1 + b_2 \\ \text{then} & x_3 = \phantom{b_1 +{}} b_2 + b_3 \end{array} \qquad \text{This says that } \boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

This is good practice to see the columns of the inverse matrix multiplying b_1, b_2, and b_3. The first column of A^{-1} is the solution for $\boldsymbol{b} = (1,0,0)$. The second column is the solution for $\boldsymbol{b} = (0,1,0)$. The third column $\boldsymbol{x}$ of A^{-1} is the solution for $A\boldsymbol{x} = \boldsymbol{b} = (0,0,1)$.

The three columns of A are still independent. They don't lie in a plane. The combinations of those three columns, using the right weights x_1, x_2, x_3, can produce any three-dimensional vector $\boldsymbol{b} = (b_1, b_2, b_3)$. Those weights come from $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

1.3 B This E is an **elimination matrix**. E has a subtraction, E^{-1} has an addition.

$$E\boldsymbol{x} = \boldsymbol{b} \qquad \begin{bmatrix} \mathbf{1} & 0 \\ -\boldsymbol{\ell} & \mathbf{1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad\qquad E = \begin{bmatrix} \mathbf{1} & 0 \\ -\boldsymbol{\ell} & \mathbf{1} \end{bmatrix}$$

The first equation is $x_1 = b_1$. The second equation is $x_2 - \ell x_1 = b_2$. The inverse will *add* $\ell x_1 = \ell b_1$, because the elimination matrix *subtracted* ℓx_1 :

$$\boldsymbol{x} = E^{-1}\boldsymbol{b} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ \ell b_1 + b_2 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 \\ \boldsymbol{\ell} & \mathbf{1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} \mathbf{1} & 0 \\ \boldsymbol{\ell} & \mathbf{1} \end{bmatrix}$$

1.3 C Change C from a cyclic difference to a **centered difference** producing $x_3 - x_1$:

$$C\boldsymbol{x} = \boldsymbol{b} \qquad \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - 0 \\ x_3 - x_1 \\ 0 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{15}$$

Show that $C\boldsymbol{x} = \boldsymbol{b}$ can only be solved when $b_1 + b_3 = 0$. That is a plane of vectors $\boldsymbol{b}$ in three-dimensional space. Each column of C is in the plane, the matrix has no inverse. So this plane contains all combinations of those columns (which are all the vectors $C\boldsymbol{x}$).

Solution The first component of $\boldsymbol{b} = C\boldsymbol{x}$ is x_2, and the last component of $\boldsymbol{b}$ is $-x_2$. So we always have $b_1 + b_3 = 0$, for every choice of $\boldsymbol{x}$.

If you draw the column vectors in C, the first and third columns fall on the same line. In fact (column 1) $= -$(column 3). So the three columns will lie in a plane, and C is *not* an invertible matrix. We cannot solve $C\boldsymbol{x} = \boldsymbol{b}$ unless $b_1 + b_3 = 0$.

I included the zeros so you could see that this matrix produces "centered differences". Row i of $C\boldsymbol{x}$ is x_{i+1} (*right of center*) minus x_{i-1} (*left of center*). Here is the 4 by 4 centered difference matrix:

$$C\boldsymbol{x} = \boldsymbol{b} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - 0 \\ x_3 - x_1 \\ x_4 - x_2 \\ 0 - x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{16}$$

Surprisingly this matrix is now invertible! The first and last rows give x_2 and x_3. Then the middle rows give x_1 and x_4. It is possible to write down the inverse matrix C^{-1}. But 5 by 5 will be singular (*not invertible*) again . . .

Problem Set 1.3

√ **1** Find the linear combination $2s_1 + 3s_2 + 4s_3 = b$. Then write b as a matrix-vector multiplication Sx. Compute the dot products (row of S) $\cdot x$:

$$s_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad s_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad s_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ go into the columns of } S\,.$$

2 Solve these equations $Sy = b$ with s_1, s_2, s_3 in the columns of S:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}.$$

The sum of the first n odd numbers is ______.

3 Solve these three equations for y_1, y_2, y_3 in terms of B_1, B_2, B_3:

$$Sy = B \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

Write the solution y as a matrix $A = S^{-1}$ times the vector B. Are the columns of S independent or dependent?

√ **4** Find a combination $x_1 w_1 + x_2 w_2 + x_3 w_3$ that gives the zero vector:

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are (independent) (dependent). The three vectors lie in a ______. The matrix W with those columns is *not invertible*.

5 The rows of that matrix W produce three vectors (I write them as columns):

$$r_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \qquad r_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \qquad r_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

Linear algebra says that these vectors must also lie in a plane. There must be many combinations with $y_1 r_1 + y_2 r_2 + y_3 r_3 = \mathbf{0}$. Find two sets of y's.

√ **6** Which values of c give dependent columns (combination equals zero)?

$$\begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & c \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$$

7 If the columns combine into $A\boldsymbol{x} = \boldsymbol{0}$ then each row has $\boldsymbol{r} \cdot \boldsymbol{x} = 0$:

$$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \textbf{By rows} \begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{x} \\ \boldsymbol{r}_2 \cdot \boldsymbol{x} \\ \boldsymbol{r}_3 \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The three rows also lie in a plane. Why is that plane perpendicular to $\boldsymbol{x}$?

8 Moving to a 4 by 4 difference equation $A\boldsymbol{x} = \boldsymbol{b}$, find the four components x_1, x_2, x_3, x_4. Then write this solution as $\boldsymbol{x} = S\boldsymbol{b}$ to find the inverse matrix $S = A^{-1}$:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \boldsymbol{b}.$$

9 What is the *cyclic* 4 by 4 difference matrix C? It will have 1 and -1 in each row. Find all solutions $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ to $C\boldsymbol{x} = \boldsymbol{0}$. The four columns of C lie in a "three-dimensional hyperplane" inside four-dimensional space.

10 A *forward* difference matrix Δ is *upper* triangular:

$$\Delta\boldsymbol{z} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_2 - z_1 \\ z_3 - z_2 \\ 0 - z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \boldsymbol{b}.$$

Find z_1, z_2, z_3 from b_1, b_2, b_3. What is the inverse matrix in $\boldsymbol{z} = \Delta^{-1}\boldsymbol{b}$?

11 Show that the forward differences $(t+1)^2 - t^2$ are $2t+1 = $ *odd numbers*. As in calculus, the difference $(t+1)^n - t^n$ will begin with the derivative of t^n, which is _____ .

12 The last lines of the Worked Example say that the 4 by 4 centered difference matrix in (16) *is* invertible. Solve $C\boldsymbol{x} = (b_1, b_2, b_3, b_4)$ to find its inverse in $\boldsymbol{x} = C^{-1}\boldsymbol{b}$.

Challenge Problems

13 The very last words say that the 5 by 5 centered difference matrix *is not* invertible. Write down the 5 equations $C\boldsymbol{x} = \boldsymbol{b}$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero? (The 5 columns lie on a "4-dimensional hyperplane" in 5-dimensional space.)

14 If (a, b) is a multiple of (c, d) with $abcd \neq 0$, *show that* (a, c) *is a multiple of* (b, d). This is surprisingly important; two columns are falling on one line. You could use numbers first to see how a, b, c, d are related. The question will lead to:

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent columns when it has dependent rows.