CIS 2033 based on

Dekking et al. A Modern Introduction to Probability and Statistics. 2007

Slides by Solomon Feitelson

Instructor Longin Jan Latecki

C4: DISCRETE_RANDOM_VARIABLES

DEFINITION OF A DISCRETE RANDOM VARIABLE

Let Ω be an arbitrary sample space. We can have 2 types of sample spaces:

0) FINITE: $\Omega = \{a_1, a_2, \cdots, a_n\}$

1) INFINITE: $\Omega = a_1, a_2, \cdots$

As Ω is arbitrary, it can assume any arbitrary collection of arbitrary values.

A discrete random variable X is a function $X : \Omega \to \mathbb{R}$.

X receives an arbitrary sample space Ω as input, and maps Ω to a set of Real Numbers $\mathbb{R}$.

DEFINITION OF A PROBABILITY MASS FUNCTION

The Probability Mass Function p of a discrete random variable X is the function $p : \mathbb{R} \to [0,1]$.

p receives a set of Real Numbers $\mathbb{R}$ as input, and maps the set to the inclusive interval $[0,1]$.

We define p as:

$$p(a) = P(\{X = a\}) \mid -\infty < a < \infty.$$

Explanation of the formula:

0) Since Ω is arbitrary, value a assumes an arbitrary range.

1) Set $\{X = a\} \subseteq \Omega$ describes an event or events that occur in Ω, for a particular a.

2) $P(\{X = a\})$ is the probability that a particular a will occur in Ω.

If X assumes a finite number of values $\{a_1, a_2, \cdots, a_n\}$, then:

0) $p(a_i) > 0$.

1) $p(a_1) + p(a_2) + \cdots + p(a_n) = 1$.

2) $p(a) = 0$ for all other values of a.

If X assumes an infinite number of values $a_1, a_2, \cdots$, then:

0) $p(a_i) > 0$.

1) $p(a_1) + p(a_2) + \cdots = 1$.

2) $p(a) = 0$ for all other values of a.

DEFINITION OF A CUMULATIVE/DISTRIBUTION FUNCTION

The Cumulative/Distribution Function F of a random variable X is the function $F:\mathbb{R}\to[0,1]$.

F receives a set of Real Numbers $\mathbb{R}$ as input, and maps the set to the inclusive interval $[0,1]$.

We define F as:

$$F\left(a_w\right) = P\left(\{X \leq a_w\}\right) \mid -\infty < a_w < \infty.$$

Another valid definition of F is:

$$F\left(a_w\right) = \sum_{k\,=\,a_1}^{a_w} \left[p\left(k\right)\right].$$

Explanation of the formulas:

0) Since Ω is arbitrary, a_w assumes an arbitrary range.

1) Set $[\{X \leq a_w\} \subseteq \Omega] \mid [\forall k \in X, [k \leq a_w]]$.

2) $P\left(\{X \leq a_w\}\right)$ is the cumulative probability of all such events $k \in X$.

DEFINITION OF A BERNOULLI DISTRIBUTION

A discrete random variable X has a Bernoulli Distribution with these parameters:

Success Probability $p \,|\, 0 \leq p \leq 1$.

Failure Probability $1-p \,|\, 0 \leq 1-p \leq 1$.

We define the Probability Mass Function p_x of a Bernoulli Distribution as:

$$p_x(1) = P(\{X = 1\}) = p.$$

$$p_x(0) = P(\{X = 0\}) = 1 - p.$$

Explanation of the formulas:

0) A Bernoulli Distribution is suitable to model experiments with only 2 possible outcomes:

$$1 = "success"; 0 = "failure".$$

1) $p_x(1) = P(\{X = 1\})$ is the probability that the event $1 = "success"$ shall occur.

2) $p_x(0) = P(\{X = 0\})$ is the probability that the event $0 = "failure"$ shall occur.

We denote this Bernoulli Distribution as $Ber(p)$.

DEFINITION OF A BINOMIAL DISTRIBUTION

Intuition: The probability of k successful outcomes in n trials.

A discrete random variable X has a Binomial Distribution with these parameters:

$n \mid n = 1, 2, \cdots$

Success Probability $p \mid 0 \leq p \leq 1.$

Failure Probability $1 - p \mid 0 \leq 1 - p \leq 1.$

We define the Probability Mass Function p_x of a Binomial Distribution as:

$$p_x(k) = P(\{X = k\}) = \begin{bmatrix} n \\ k \end{bmatrix} \left[p^k\right] \left[[1-p]^{[n-k]}\right] \mid k = 0, 1, \cdots, n.$$

Explanation of the formula:

0) $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k![n-k]!}$: Combination: from n elements, the number of order-insensitive ways to choose k elements.

1) $\left[p^k\right]$: The probability of a "success," repeated k times.

2) $\left[[1-p]^{[n-k]}\right]$: The probability of a "failure," repeated $[n-k]$ times.

We denote this Binomial Distribution by $Bin(n, p)$.

DEFINITION OF A GEOMETRIC DISTRIBUTION

Intuition: The probability of being successful in kth trial. This means that all previous $k-1$ failed.

A discrete random variable X has a Geometric Distribution with these parameters:

Success Probability $p \mid 0 \leq p \leq 1$. Failure Probability $1-p \mid 0 \leq 1-p \leq 1$.

We define the Probabillity Mass Function p_x of a Geometric Distribution as:

$$p_x(k) = P(\{X = k\}) = \left[[1-p]^{[k-1]}\right] p \mid [k = 1, 2, \cdots].$$

Explanation of the formula:

0) $[k]$: The number of "repetitions" of the experiment, until a "success" occurs.

1) $\left[[1-p]^{[k-1]}\right]$: The probability of a "failure," repeated $[k-1]$ times.

2) $[p]$: The probability of a "success," repeated 1 time, after the execution of $[k-1]$ "failures."