# Section 7.4
# Closures of Relations

**Definition:** The *closure* of a relation $R$ with respect to property P is the relation obtained by adding the minimum number of ordered pairs to $R$ to obtain property P.

In terms of the digraph representation of $R$

• To find the reflexive closure - add loops.

• To find the symmetric closure - add arcs in the opposite direction.

• To find the transitive closure - if there is a path from $a$ to $b$, add an arc from $a$ to $b$.

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Note: Reflexive and symmetric closures are easy. Transitive closures can be very complicated.

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**Definition:** Let A be a set and let $\Delta = \{<x, x> \mid x \text{ in } A\}$. $\Delta$ is called the *diagonal relation* on $A$ (sometimes called the *equality* relation $E$).

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Note that $D$ is the smallest (has the fewest number of ordered pairs) relation which is reflexive on $A$.

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## Reflexive Closure

**Theorem:** Let $R$ be a relation on A. The *reflexive closure* of $R$, denoted r($R$), is $R \quad \Delta$.

• Add loops to all vertices on the digraph representation of $R$.

• Put 1's on the diagonal of the connection matrix of $R$.

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## *Symmetric Closure*

**Definition:** Let $R$ be a relation on A. Then $R^{-1}$ or the *inverse* of $R$ is the relation $R^{-1} = \{< y,x > | < x,y > \quad R\}$

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Note: to get $R^{-1}$

• reverse all the arcs in the digraph representation of $R$

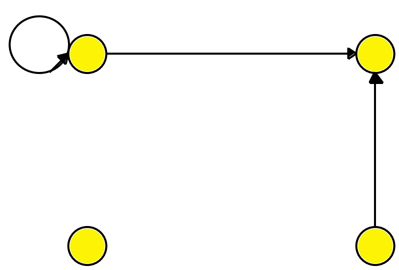
• take the transpose $M^T$ of the connection matrix $M$ of $R$.

Note: This relation is sometimes denoted as $R^T$ or $R^c$ and called the *converse* of $R$

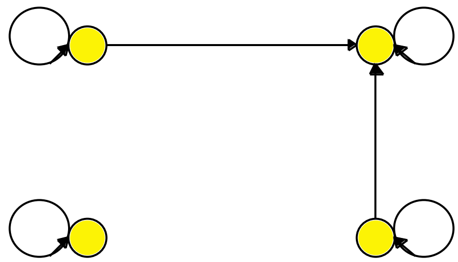
The composition of the relation with its inverse does not necessarily produce the diagonal relation (recall that the composition of a bijective <u>function</u> with its inverse is the identity).

**Theorem:** Let $R$ be a relation on A. The *symmetric closure* of $R$, denoted s($R$), is the relation $R \quad R^{-1}$.

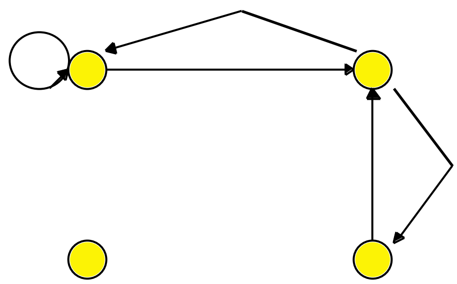
Examples:



$R$



r($R$)



s($R$)

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Examples:

- If $A = Z$, then r( $\neq$ ) = $Z$ x $Z$
- If $A = Z^{+}$, then s( < ) = $\neq$.

  What is the (infinite) connection matrix of s(<)?
- If $A = Z$, then s($\leq$) = ?

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**Theorem:** Let $R_1$ and $R_2$ be relations from $A$ to $B$. Then

- $(R^{-1})^{-1} = R$
- $(R_1 \quad R_2)^{-1} = R_1^{-1} \quad R_2^{-1}$
- $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$
- $(A \times B)^{-1} = B \times A$
- $\quad^{-1} =$
- $\overline{R}^{-1} = \overline{R^{-1}}$
- $(R_1 - R_2)^{-1} = R_1^{-1} - R_2^{-1}$
- If $A = B$, then $(R_1R_2)^{-1} = R_2^{-1} R_1^{-1}$
- If $R_1 \quad R_2$ then $R_1^{-1} \quad R_2^{-1}$

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**Theorem:** $R$ is symmetric iff $R = R^{-1}$

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## Paths

**Definition:** A *path* of *length n* in a digraph G is a sequence of edges $\langle x_0, x_1\rangle\langle x_1, x_2\rangle \ldots \langle x_{n-1}, x_n\rangle$.

The terminal vertex of the previous arc matches with the initial vertex of the following arc.

If $x_0 = x_n$ the path is called a *cycle* or *circuit*. Similarly for relations.

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**Theorem:** Let R be a relation on A. There is a path of length $n$ from $a$ to $b$ iff $\langle a, b \rangle \quad R^n$.

Proof: (by induction)

• *Basis*: An arc from $a$ to $b$ is a path of length 1 which is in $R^1 = R$. Hence the assertion is true for $n = 1$.

• *Induction Hypothesis*: Assume the assertion is true for $n$.

Show it must be true for $n+1$.

There is a path of length n+1 from a to b iff there is an x in A such that there is a path of length 1 from $a$ to $x$ and a path of length $n$ from $x$ to $b$.

From the Induction Hypothesis,

$$\langle a, x \rangle \quad R$$

and since $\langle x , b \rangle$ is a path of length n,

$$\langle x, b \rangle \quad R^n.$$

If

$$\langle a, x \rangle \quad R$$

and

$$<x, b> \quad R^n,$$

then

$$<a, b> \quad R^n \circ R = R^{n+1}$$

by the inductive definition of the powers of R.

Q. E. D.

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## Useful Results for Transitive Closure

**Theorem:**

If $A \quad B$ and $C \quad B$, then $A \quad C \quad B$.

**Theorem:**

If $R \quad S$ and $T \quad U$ then $R \circ T \quad S \circ U$.

**Corollary:**

If $R \quad S$ then $R^n \quad S^n$

**Theorem:**

If $R$ is transitive then so is $R^n$

Trick proof: Show $(R^n)^2 = (R^2)^n \quad R^n$

**Theorem:** If $R^k = R^j$ for some $j > k$, then $R^{j+m} = R^n$ for some $n \leq j$.

We don't get any new relations beyond $R^j$.

As soon as you get a power of $R$ that is the same as one you had before, STOP.

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## Transitive Closure

Recall that the transitive closure of a relation $R$, $t(R)$, is the smallest transitive relation containing $R$.

Also recall

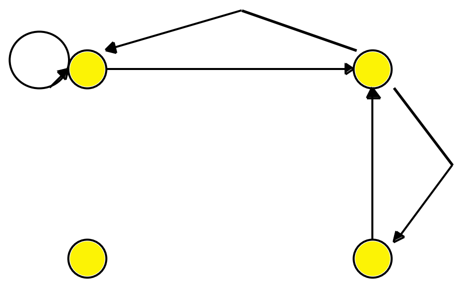
> $R$ is transitive iff $R^n$ is contained in $R$ for all $n$

Hence, if there is a path from $x$ to $y$ then there must be an arc from $x$ to $y$, or $<x, y>$ is in $R$.

Example:

- If $A = Z$ and $R = \{< i, i+1>\}$ then $t(R) = <$

- Suppose $R$: is the following:



*What is t(R)*?

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**Definition:** The *connectivity* relation or the *star closure* of the relation $R$, denoted $R^*$, is the set of ordered pairs $\langle a, b \rangle$ such that there is a path (in $R$) from $a$ to $b$:

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

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Examples:

• Let $A = Z$ and $R = \{\langle i, i+1 \rangle\}$. $R^* = <$.

• Let $A$ = the set of people, $R = \{\langle x, y \rangle \mid$ person $x$ is a parent of person $y\}$. $R^* = ?$

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**Theorem:** $t(R) = R^*$.

Proof:

Note: this is not the same proof as in the text.

We must show that $R^*$

1) is a transitive relation

2) contains $R$

3) is the smallest transitive relation which contains $R$

Proof:

Part 2):

Easy from the definition of $R^*$.

Part 1):

Suppose $\langle x, y \rangle$ and $\langle y, z \rangle$ are in $R^*$.

Show $\langle x, z \rangle$ is in $R^*$.

By definition of $R^*$, $\langle x, y \rangle$ is in $R^m$ for some $m$ and $\langle y, z \rangle$ is in $R^n$ for some $n$.

Then $\langle x, z \rangle$ is in $R^n R^m = R^{m+n}$ which is contained in $R^*$. Hence, $R^*$ must be transitive.

Part 3):

Now suppose $S$ is any transitive relation that contains $R$.

We must show $S$ contains $R^*$ to show $R^*$ is the <u>smallest</u> such relation.

$R$ $S$ so $R^2$ $S^2$ $S$ since $S$ is transitive

Therefore $R^n$ $S^n$ $S$ for all $n$. (why?)

Hence $S$ must contain $R^*$ since it must also contain the union of all the powers of $R$.

Q. E. D.

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In fact, we need only consider paths of length n or less.

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**Theorem:** If $|A| = n$, then any path of length $> n$ must contain a cycle.

Proof:

If we write down a list of more than n vertices representing a path in $R$, some vertex must appear at least twice in the list (by the Pigeon Hole Principle).

Thus $R^k$ for $k > n$ doesn't contain any arcs that don't already appear in the first n powers of $R$.

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**Corollary:** If | A | = n, then t(R) = R* = R    $R^2$    . . .  $R^n$

**Corollary:** We can find the connection matrix of t(R) by computing the join of the first n powers of the connection matrix of R.

Powerful Algorithm!

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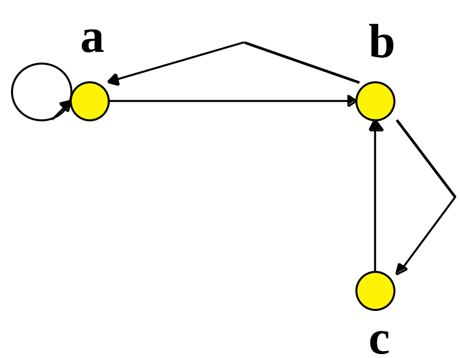
Example:




Do the following in class:

R^2:

R^3:

R^4:

R^5:
•
•
•

t(R) = R*:

So that you don’t get bored, here are some problems to discuss on your next blind date:

1) Do the closure operations commute?

- Does st(R) = ts(R)?
- Does rt(R) = tr(R)?
- Does rs(R) = sr(R)?

2) Do the closure operations distribute

- Over the set operations?
- Over inverse?
- Over complement?
- Over set inclusion?

Examples:

- Does $t(R1 - R2) = t(R1) - t(R2)$?
- Does $r(R^{-1}) = [r(R)]^{-1}$?