Section 8.3
Representing Graphs and Graph Isomorphism

We wish to be able to determine when two graphs are identical except perhaps for the labeling of the vertices.

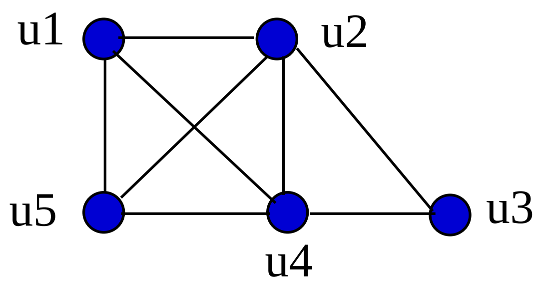
We derive some alternate representations which are extensions of connection matrices we have seen before.

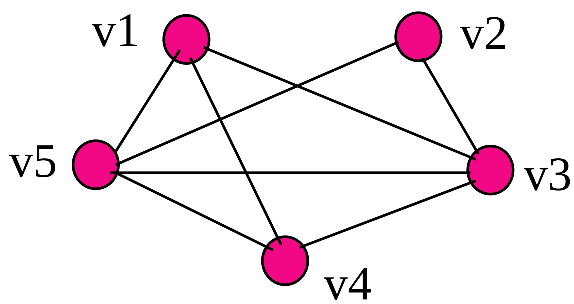
Adjacency Matrices

A_{ij} = 1 if there is an edge from vertex i to vertex j
= 0 else

Examples:

G1 and G2:





$$G1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$G2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

In pseudographs, A_{ij} = number of edges from vertex i to vertex j.

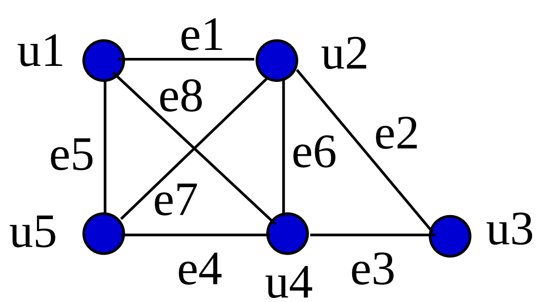
Incidence Matrices

A_{ij} = 1 if edge j is incident with vertex i
= 0 else

Note: this method requires labeling of edges. Only 2 1's per column.

Examples:

G1:

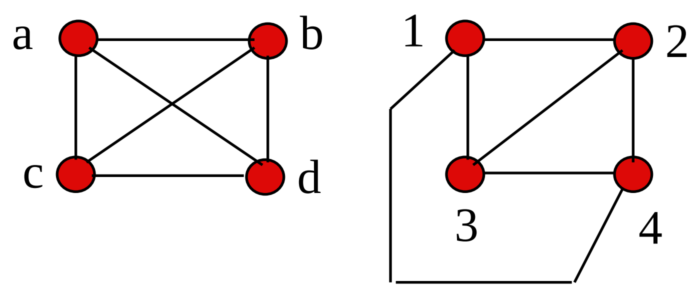


$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Isomorphism

The two graphs below are really the same graph.

One is drawn so that no edges intersect (planar).



We say these graphs are *isomorphic*.

Definition: Let *G1 = (V1, E1)* and *G2 = (V2, E2)* be simple graphs. The graphs *G1* and *G2* are *isomorphic* iff

- There exists a bijection *f*: *V1 -> V2*
- if for all *v1* and *v2* in *V1*,

v1 and *v2* are adjacent in *G1*

iff

f(v1) and *f(v2)* are adjacent in *G2*,.

Note: This is a hard problem to solve. Normally to prove existence we must construct the isomorphism.

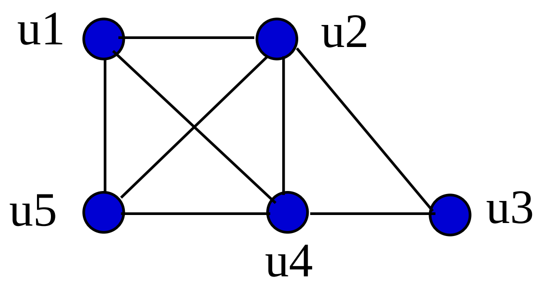
Invariants - things that *G1* and *G1* must have in common to be isomorphic:

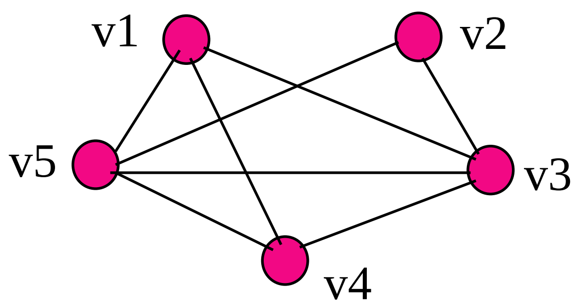
- the same number of vertices
- the same number of edges
- degrees of corresponding vertices are the same.
- if one is bipartite, the other must be
- if one is complete, the other must be
- if one is a wheel, the other must be

etc.

Example:

Determine if the following two graphs G1 and G2 are isomorphic.





Solution:

Check . . .

- They have the same number of vertices = 5

- They have the same number of edges = 8

- They have the same number of vertices with the same degrees: 2, 3, 3, 4, 4.

- Now we try to construct the isomorphism f using the degrees of vertices to help us.

  - deg(u3) = deg(v2) = 2 so

$$f(u3) = v2$$

is our only choice.

• deg(u1) = deg(u5) = deg(v1) = deg(v4) = 3 so we must have either

i) f(u1) = v1 and f(u5) = v4

or

ii) f(u1) = v4 and f(u5) = v1

Perhaps either choice will work.

• Finally since deg(u2) = deg(u4) = deg(v3) = deg(v5) = 4 we must have either

i) f(u2) = v3 and f(u4) = v5

or

ii) f(u2) = v5 and f(u4) = v3.

We first try the relabeling using i) in each case to get the function

$$3 \rightarrow 2, 1 \rightarrow 1, 5 \rightarrow 4, 2 \rightarrow 3, 4 \rightarrow 5$$

• permute the rows and columns of the adjacency matrix of *G1* using the above map to see if we get the adjacency matrix of *G2*.

or

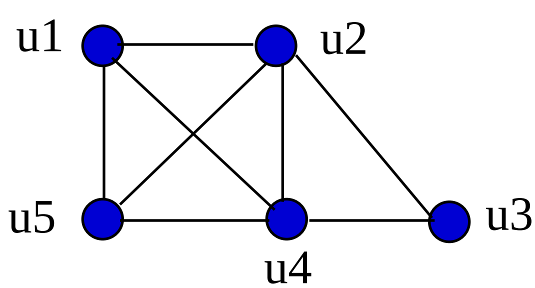
• change the labels of the graph *G2* to produce the graph *G2** according to the above permutation and recalculate the adjacency matrix.

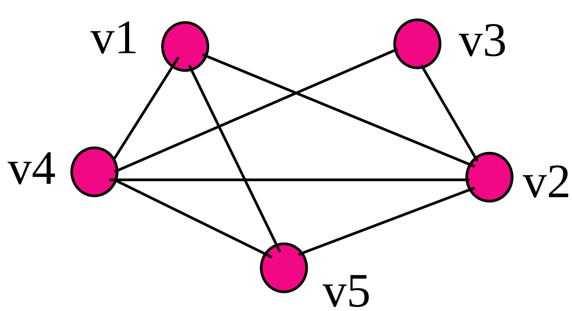
Recall:

$$G1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$G2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The new labeling of *G2*, *G2**, becomes





The new adjacency matrix becomes:

$$G2^* = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

which is the same adjacency matrix as for G1. Hence we have found an isomorphism!

Observation: Doing these by hand is a bummer!

Questions:

- suppose we had tried the relabelings implied by cases ii) instead?

- what is the worst case complexity?

Ex. 35, 41, p. 565