

Section 7.1 Relations and Their Properties

Definition: A *binary relation* R from a set A to a set B is a subset $R \subseteq A \times B$.

Note: there are no constraints on relations as there are on functions.

We have a common graphical representation of relations:

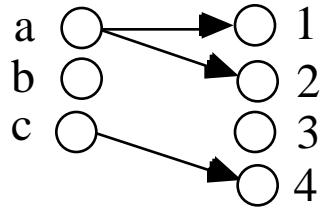
Definition: A *Directed graph* or a *Digraph* D from A to B is a collection of *vertices* $V \subseteq A \cup B$ and a collection of *edges* $R \subseteq A \times B$. If there is an ordered pair $e = \langle x, y \rangle$ in R then there is an *arc* or *edge* from x to y in D . The elements x and y are called the *initial* and *terminal* vertices of the edge e .

Examples:

- Let $A = \{ a, b, c \}$
- $B = \{ 1, 2, 3, 4 \}$
- R is defined by the ordered pairs or edges

$$\{ \langle a, 1 \rangle, \langle a, 2 \rangle, \langle c, 4 \rangle \}$$

can be represented by the digraph D :

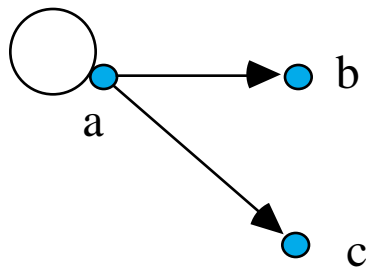


Definition: A binary relation R on a set A is a subset of $A \times A$ or a relation from A to A .

Example:

- $A = \{a, b, c\}$
- $R = \{ \langle a, a \rangle, \langle a, b \rangle, \langle a, c \rangle \}$.

Then a digraph representation of R is:



Note: An arc of the form $\langle x, x \rangle$ on a digraph is called a *loop*.

Question: How many binary relations are there on a set A ?

Special Properties of Binary Relations

Given:

- A Universe U
- A binary relation R on a subset A of U

Definition: R is *reflexive* iff

$$\forall x [x \in U \rightarrow \langle x, x \rangle \in R]$$

Note: if $U = \emptyset$ then the implication is true vacuously

The void relation on a void Universe is reflexive!

Note: If U is not void then all vertices in a reflexive relation must have loops!

Definition: R is *symmetric* iff

$$\forall x, y [\langle x, y \rangle \in R \rightarrow \langle y, x \rangle \in R]$$

Note: If there is an arc $\langle x, y \rangle$ there must be an arc $\langle y, x \rangle$.

Definition: R is *antisymmetric* iff

$$x \neq y [\langle x, y \rangle \in R \wedge \langle y, x \rangle \in R \implies x = y]$$

Note: If there is an arc from x to y there cannot be one from y to x if $x \neq y$.

You should be able to show that logically: if $\langle x, y \rangle$ is in R and $x \neq y$ then $\langle y, x \rangle$ is not in R .

Definition: R is *transitive* iff

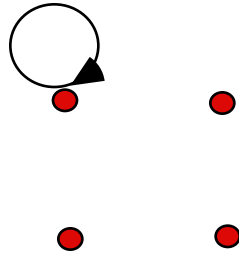
$$x \neq y \neq z [\langle x, y \rangle \in R \wedge \langle y, z \rangle \in R \implies \langle x, z \rangle \in R]$$

Note: if there is an arc from x to y and one from y to z then there must be one from x to z .

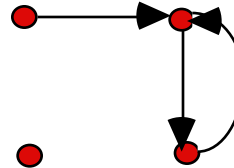
This is the most difficult one to check. We will develop algorithms to check this later.

Examples:

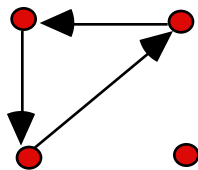
A.



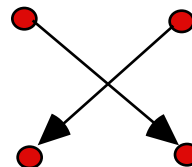
B.



C.



D.



A: not reflexive
symmetric
antisymmetric
transitive

B: not reflexive
not symmetric
not antisymmetric
not transitive

C: not reflexive
not symmetric
antisymmetric
not transitive

D: not reflexive
not symmetric
antisymmetric
transitive

Combining Relations

Set operations

A very large set of potential questions -

Let R_1 and R_2 be binary relations on a set A :

If R_1 has property 1

and

R_2 has property 2,

does

$R_1 * R_2$ have property 3

where $*$ represents an arbitrary binary set operation?

Example:

If

- R_1 is symmetric,

and

- R_2 is antisymmetric,

does it follow that

- $R_1 \cap R_2$ is transitive?

If so, prove it. Otherwise find a counterexample.

Example:

Let R_1 and R_2 be transitive on A . Does it follow that

$$R_1 \cap R_2$$

is transitive?

Consider

- $A = \{1, 2\}$
- $R_1 = \{\langle 1, 2 \rangle\}$
- $R_2 = \{\langle 2, 1 \rangle\}$

Then $R_1 \cap R_2 = \{\langle 1, 2 \rangle, \langle 2, 1 \rangle\}$ which is not transitive!
(Why?)

Composition

Definition: Suppose

- $R1$ is a relation from A to B
- $R2$ is a relation from B to C .

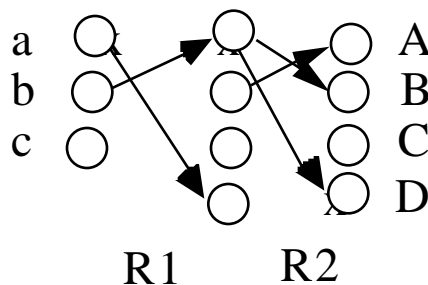
Then the composition of $R2$ with $R1$, denoted $R2 \circ R1$ is the relation from A to C :

If $\langle x, y \rangle$ is a member of $R1$ and $\langle y, z \rangle$ is a member of $R2$ then $\langle x, z \rangle$ is a member of $R2 \circ R1$.

Note: For $\langle x, z \rangle$ to be in the composite relation $R2 \circ R1$ there must exist a y in B

Note: We read them right to left as in functions.

Example:



$$R2 \circ R1 = \{ \langle b, D \rangle, \langle b, B \rangle \}$$

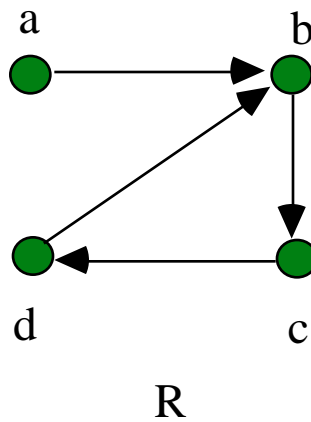
Definition: Let R be a binary relation on A . Then

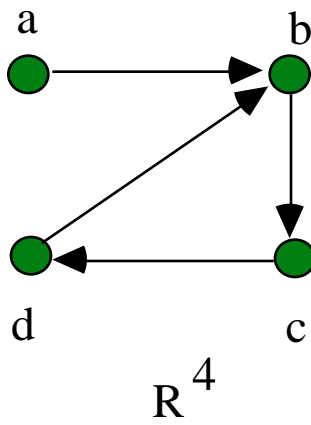
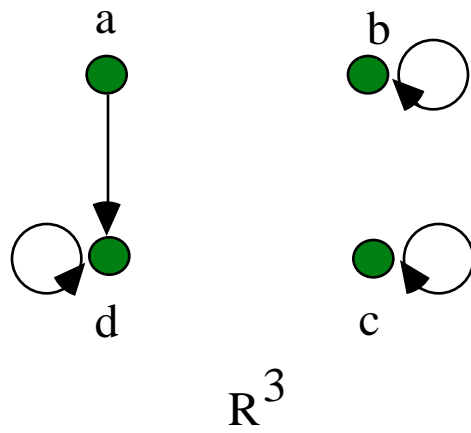
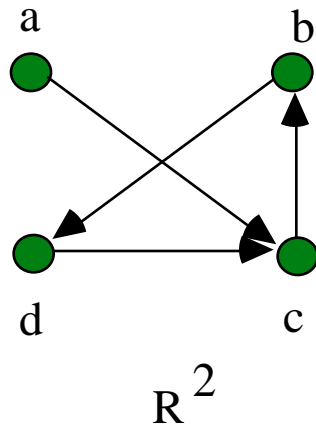
$$\text{Basis: } R^1 = R$$

$$\text{Induction: } R^{n+1} = R^n \circ R$$

Note: an ordered pair $\langle x, y \rangle$ is in R^n iff there is a *path* of length n from x to y following the arcs (in the direction of the arrows) of R .

Example:





Very Important
Theorem:

R is transitive iff $R^n \subseteq R$ for $n > 0$.

Proof:

1. R transitive $\iff R^n \subseteq R$

Use a direct proof and a proof by induction:

- Assume R is transitive.
- Now show $R^n \subseteq R$ by induction.

Basis: Obviously true for $n = 1$.

Induction:

- The induction hypothesis:

'assume true for n '.

- Show it must be true for $n + 1$.

$R^{n+1} = R^n \circ R$ so if $\langle x, y \rangle$ is in R^{n+1} then there is a z such that $\langle x, z \rangle$ is in R^n and $\langle z, y \rangle$ is in R .

But since $R^n \subseteq R$, $\langle x, z \rangle$ is in R .

R is transitive so $\langle x, y \rangle$ is in R .

Since $\langle x, y \rangle$ was an arbitrary edge the result follows.

2. $R^n \subseteq R$ if R is transitive

Use the fact that $R^2 \subseteq R$ and the definition of transitivity. Proof left to the student.

Q. E. D.
