

Section 8.3

Representing Graphs and Graph Isomorphism

We wish to be able to determine when two graphs are identical except perhaps for the labeling of the vertices.

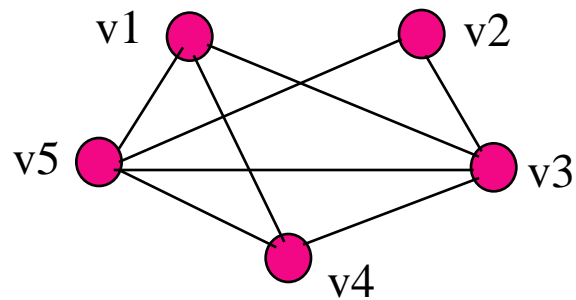
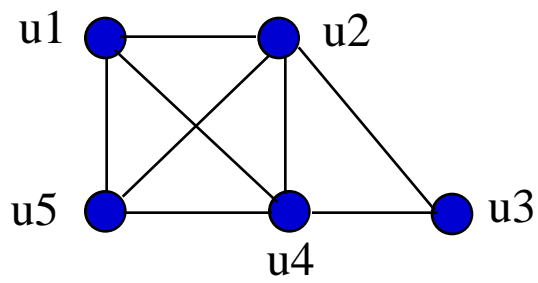
We derive some alternate representations which are extensions of connection matrices we have seen before.

Adjacency Matrices

$A_{ij} = 1$ if there is an edge from vertex i to vertex j
 $= 0$ else

Examples:

G1 and G2:



$$G1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$G2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

In pseudographs, $A_{ij} = \underline{\text{number}}$ of edges from vertex i to vertex j .

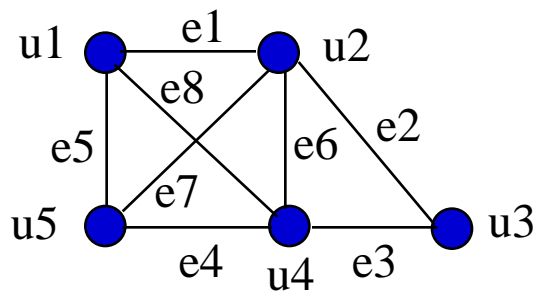
Incidence Matrices

$A_{ij} = 1$ if edge j is incident with vertex i
 $= 0$ else

Note: this method requires labeling of edges. Only 2 1's per column.

Examples:

G1:

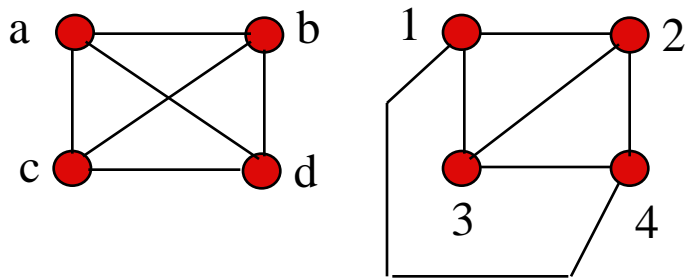


$$A = \begin{matrix} & \begin{matrix} e1 & e2 & e3 & e4 & e5 & e6 & e7 & e8 \end{matrix} \\ \begin{matrix} u1 \\ u2 \\ u3 \\ u4 \\ u5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

Isomorphism

The two graphs below are really the same graph.

One is drawn so that no edges intersect (planar).



We say these graphs are *isomorphic*.

Definition: Let $G1 = (V1, E1)$ and $G2 = (V2, E2)$ be simple graphs. The graphs $G1$ and $G2$ are *isomorphic* iff

- There exists a bijection $f: V1 \rightarrow V2$
- if for all $v1$ and $v2$ in $V1$,

$v1$ and $v2$ are adjacent in $G1$

iff

$f(v1)$ and $f(v2)$ are adjacent in $G2$,.

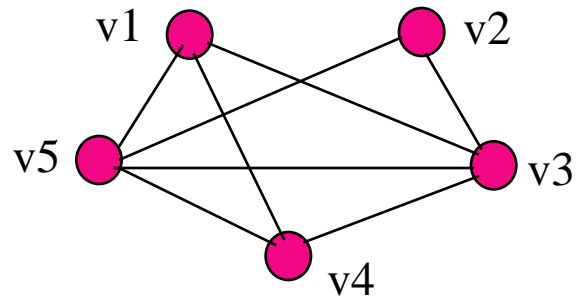
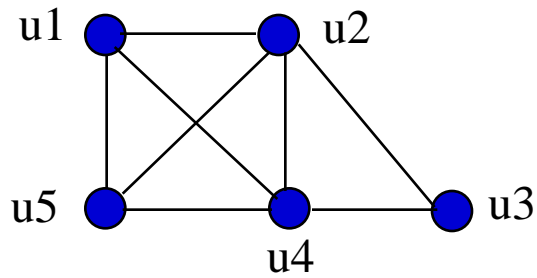
Note: This is a hard problem to solve. Normally to prove existence we must construct the isomorphism.

Invariants - things that $G1$ and $G2$ must have in common to be isomorphic:

- the same number of vertices
 - the same number of edges
 - degrees of corresponding vertices are the same.
 - if one is bipartite, the other must be
 - if one is complete, the other must be
 - if one is a wheel, the other must be
- etc.
-

Example:

Determine if the following two graphs $G1$ and $G2$ are isomorphic.



Solution:

Check . . .

- They have the same number of vertices = 5
- They have the same number of edges = 8
- They have the same number of vertices with the same degrees: 2, 3, 3, 4, 4.
- Now we try to construct the isomorphism f using the degrees of vertices to help us.

- $\deg(u_3) = \deg(v_2) = 2$ so

$$f(u_3) = v_2$$

is our only choice.

• $\deg(u_1) = \deg(u_5) = \deg(v_1) = \deg(v_4) = 3$ so we must have either

i) $f(u_1) = v_1$ and $f(u_5) = v_4$

or

ii) $f(u_1) = v_4$ and $f(u_5) = v_1$

Perhaps either choice will work.

• Finally since $\deg(u_2) = \deg(u_4) = \deg(v_3) = \deg(v_5) = 4$ we must have either

i) $f(u_2) = v_3$ and $f(u_4) = v_5$

or

ii) $f(u_2) = v_5$ and $f(u_4) = v_3$.

We first try the relabeling using i) in each case to get the function

$$3 \quad 2,1 \quad 1,5 \quad 4,2 \quad 3,4 \quad 5$$

• permute the rows and columns of the adjacency matrix of G_1 using the above map to see if we get the adjacency matrix of G_2 .

or

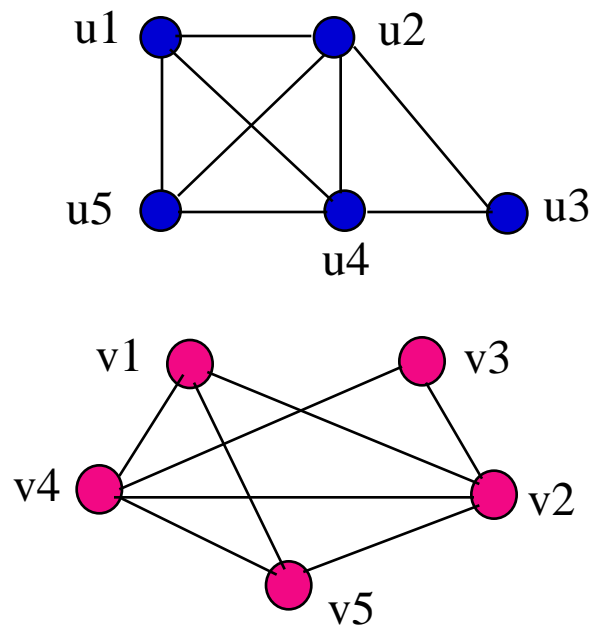
• change the labels of the graph G_2 to produce the graph G_2^* according to the above permutation and recalculate the adjacency matrix.

Recall:

$$G1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$G2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The new labeling of $G2$, $G2^*$, becomes



The new adjacency matrix becomes:

$$G2^* = \begin{matrix} & & 0 & 1 & 0 & 1 & 1 \\ & & 1 & 0 & 1 & 1 & 1 \\ & 0 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 0 \end{matrix}$$

which is the same adjacency matrix as for $G1$. Hence we have found an isomorphism!

Observation: Doing these by hand is a bummer!

Questions:

- suppose we had tried the relabelings implied by cases ii) instead?
- what is the worst case complexity?
