Section 8.2
Graph Terminology

Undirected Graphs

Definition: Two vertices $u, v$ in $V$ are adjacent or neighbors if there is an edge $e$ between $u$ and $v$.

The edge $e$ connects $u$ and $v$.

The vertices $u$ and $v$ are endpoints of $e$.

Definition: The degree of a vertex $v$, denoted $\text{deg}(v)$, is the number of edges for which it is an endpoint.

A loop contributes twice in an undirected graph.

Example:

- If $\text{deg}(v) = 0$, $v$ is called isolated.
• If \( \deg(v) = 1 \), \( v \) is called **pendant**.

The Handshaking Theorem:

Let \( G = (V, E) \). Then

\[
2|E| = \sum_{v \in V} \deg(v)
\]

Proof:

Each edge contributes twice to the degree count of all vertices.

Q. E. D.

Example:

If a graph has 5 vertices, can each vertex have degree 3? 4?

• The sum is \( 3 \cdot 5 = 15 \) which is an odd number. Not possible.

• The sum is \( 20 = 2 \cdot |E| \) and \( 20/2 = 10 \). May be possible.
**Theorem:** A graph has an even number of vertices of odd degree.

Proof:

Let \( V_1 = \) vertices of odd degree

\( V_2 = \) vertices of even degree

The sum must be even. But

- odd times odd = odd
- odd times even = even
- even times even = even
- even plus odd = odd

It *doesn't matter* whether \( V_2 \) has odd or even cardinality.

\( V_1 \) cannot have odd cardinality.

Q. E. D.

Example:

It is not possible to have a graph with 3 vertices each of which has degree 1.
Directed Graphs

**Definition:** Let \( u, v \) be an edge in \( G \). Then \( u \) is an *initial vertex* and is *adjacent to* \( v \) and \( v \) is a *terminal vertex* and is *adjacent from* \( u \).

Definition: The *in degree* of a vertex \( v \), denoted \( \deg^-(v) \) is the number of edges which terminate at \( v \).

Similarly, the *out degree* of \( v \), denoted \( \deg^+(v) \), is the number of edges which initiate at \( v \).

Theorem: \( |E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) \)

**Special Simple Graphs**

- Complete graphs - \( K_n \): the simple graph with
  - \( n \) vertices
  - exactly one edge between every pair of distinct vertices.

Maximum redundancy in local area networks and processor connection in parallel machines.
Examples:

- $K_1$
- $K_2$
- $K_3$
- $K_4$
Note: $K_5$ is important because it is the simplest nonplanar graph: It cannot be drawn in a plane with nonintersecting edges.

- Cycles:

$C_n$ is an $n$ vertex graph which is a cycle. Local area networks are sometimes configured this way called *Ring* networks.
$C_3$

$C_4$

$C_5$
• Wheels:

Add one additional vertex to the cycle $C_n$ and add an edge from each vertex to the new vertex to produce $W_n$.

Provides redundancy in local area networks.
• n-Cubes:

$Q_n$ is the graph with $2^n$ vertices representing bit strings of length $n$.

An edge exists between two vertices that differ by one bit position.

A common way to connect processors in parallel machines.

Intel Hypercube.
Bipartite Graphs

**Definition:** A simple graph $G$ is bipartite if $V$ can be partitioned into two disjoint subsets $V_1$ and $V_2$ such that every edge connects a vertex in $V_1$ and a vertex in $V_2$.

Note: There are no edges which connect vertices in $V_1$ or in $V_2$.

A bipartite graph is *complete* if there is an edge from every vertex in $V_1$ to every vertex in $V_2$, denoted $K_{m,n}$ where $m = |V_1|$ and $n = |V_2|$.

**Examples:**

- Suppose bigamy is permitted but not same sex marriages and males are in $V_1$ and females in $V_2$ and an edge represents a marriage. If every male is married to every female then the graph is complete.

- Supplier, warehouse transportation models are bipartite and an edge indicates that a given supplier sends inventory to a given warehouse.

- A Star network is a $K_{1,n}$ bipartite graph.
\[ K_{1,8} \]

- \( C_k \) for \( k \) even is a bipartite graph: even numbered vertices in \( V_1 \), odd numbered in \( V_2 \).
• Is the following graph bipartite?

If $a$ is in $V_1$ then $e$, $d$ and $b$ must be in $V_2$ (why?). Then $c$ is in $V_1$ and there is no inconsistency.

We rearrange the graph as follows:

---

**New Graphs from Old**

**Definition:** $(W, F)$ is a subgraph of $G = (V, E)$ if

$$W \subseteq V \text{ and } F \subseteq E.$$
**Definition:** If $G_1$ and $G_2$ are simple then

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$$

and the graph is simple.

________________

**Examples:**

- Find the subgraphs of $Q_1$:

  0 \[ \text{---} \] 1

  0 \[ \text{---} \] 1

  0

  1

- Count the number of subgraphs of a given graph.

- Find the union of the two graphs $G_1$ and $G_2$: 
Note: The important properties of a graph do not depend on how we draw it. We want to be able to identify two graphs that are the same (up to labeling of the vertices).