Section 7.4
Closures of Relations

**Definition:** The *closure* of a relation $R$ with respect to property P is the relation obtained by adding the minimum number of ordered pairs to $R$ to obtain property P.

In terms of the digraph representation of $R$

- To find the reflexive closure - add loops.
- To find the symmetric closure - add arcs in the opposite direction.
- To find the transitive closure - if there is a path from $a$ to $b$, add an arc from $a$ to $b$.

Note: Reflexive and symmetric closures are easy. Transitive closures can be very complicated.

**Definition:** Let $A$ be a set and let $\Delta = \{<x, x> | x \text{ in } A\}$. $\Delta$ is called the *diagonal relation* on $A$ (sometimes called the *equality* relation $E$).
Note that $D$ is the smallest (has the fewest number of ordered pairs) relation which is reflexive on $A$.

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**Reflexive Closure**

**Theorem:** Let $R$ be a relation on $A$. The reflexive closure of $R$, denoted $r(R)$, is $R \cup \Delta$.

- Add loops to all vertices on the digraph representation of $R$.
- Put 1’s on the diagonal of the connection matrix of $R$.

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**Symmetric Closure**

**Definition:** Let $R$ be a relation on $A$. Then $R^{-1}$ or the inverse of $R$ is the relation $R^{-1} = \{< y, x > | < x, y > \in R \}$

Note: to get $R^{-1}$

- reverse all the arcs in the digraph representation of $R$
- take the transpose $M^T$ of the connection matrix $M$ of $R$. 

Note: This relation is sometimes denoted as $R^T$ or $R^c$ and called the *converse* of $R$.

The composition of the relation with its inverse does not necessarily produce the diagonal relation (recall that the composition of a bijective function with its inverse is the identity).

**Theorem:** Let $R$ be a relation on $A$. The *symmetric closure* of $R$, denoted $s(R)$, is the relation $R \cup R^{-1}$.

**Examples:**

![Diagram of a relation $R$]
Examples:

- If $A = \mathbb{Z}$, then $r(\neq) = \mathbb{Z} \times \mathbb{Z}$
- If $A = \mathbb{Z}^+$, then $s(<) = \neq$.

What is the (infinite) connection matrix of $s(<)$?

- If $A = \mathbb{Z}$, then $s(\leq) = \square$
Theorem: Let $R_1$ and $R_2$ be relations from $A$ to $B$. Then

- $(R^{-1})^{-1} = R$
- $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$
- $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$
- $(A \times B)^{-1} = B \times A$
- $\emptyset^{-1} = \emptyset$
- $R^{-1} = (R_1 - R_2)^{-1}$
- $(R_1 - R_2)^{-1} = R_1^{-1} - R_2^{-1}$
- If $A = B$, then $(R_1R_2)^{-1} = R_2^{-1}R_1^{-1}$
- If $R_1 \subseteq R_2$ then $R_1^{-1} \subseteq R_2^{-1}$

Theorem: $R$ is symmetric iff $R = R^{-1}$

Paths

Definition: A path of length $n$ in a digraph $G$ is a sequence of edges $<x_0, x_1><x_1, x_2> \ldots <x_{n-1}, x_n>$. The terminal vertex of the previous arc matches with the initial vertex of the following arc.
If \( x_0 = x_n \) the path is called a *cycle* or *circuit*. Similarly for relations.

Theorem: Let \( R \) be a relation on \( A \). There is a path of length \( n \) from \( a \) to \( b \) iff \( <a, b> \in R^n \).

Proof: (by induction)

- **Basis:** An arc from \( a \) to \( b \) is a path of length 1 which is in \( R^1 = R \). Hence the assertion is true for \( n = 1 \).

- **Induction Hypothesis:** Assume the assertion is true for \( n \).

Show it must be true for \( n+1 \).

There is a path of length \( n+1 \) from \( a \) to \( b \) iff there is an \( x \) in \( A \) such that there is a path of length 1 from \( a \) to \( x \) and a path of length \( n \) from \( x \) to \( b \).

From the Induction Hypothesis,

\[ <a, x> \in R \]

and since \( <x, b> \) is a path of length \( n \),

\[ <x, b> \in R^n. \]

If \[ <a, x> \in R \]
and

\[ <x, b> \in R^n, \]

then

\[ <a, b> \in R^n \circ R = R^{n+1} \]

by the inductive definition of the powers of \( R \).

Q. E. D.

Useful Results for Transitive Closure

Theorem:

If \( A \subset B \) and \( C \subset B \), then \( A \cup C \subset B \).

Theorem:

If \( R \subset S \) and \( T \subset U \) then \( R \circ T \subset S \circ U \).

Corollary:

If \( R \subset S \) then \( R^n \subset S^n \).

Theorem:

If \( R \) is transitive then so is \( R^n \).
Trick proof: Show \((R^n)^2 = (R^2)^n \subset R^n\)

**Theorem:** If \(R^k = R^j\) for some \(j > k\), then \(R^{j+m} = R^n\) for some \(n \leq j\).

We don’t get any new relations beyond \(R^j\).

As soon as you get a power of \(R\) that is the same as one you had before, STOP.

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**Transitive Closure**

Recall that the transitive closure of a relation \(R\), \(t(R)\), is the smallest transitive relation containing \(R\).

Also recall

\[
R \text{ is transitive iff } R^n \text{ is contained in } R \text{ for all } n
\]

Hence, if there is a path from \(x\) to \(y\) then there must be an arc from \(x\) to \(y\), or \(<x, y>\) is in \(R\).

Example:

- If \(A = \mathbb{Z}\) and \(R = \{<i, i+1>\}\) then \(t(R) = <\)

- Suppose \(R\): is the following:
What is $t(R)$?

Definition: The connectivity relation or the star closure of the relation $R$, denoted $R^*$, is the set of ordered pairs $<a, b>$ such that there is a path (in $R$) from $a$ to $b$:

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

Examples:

- Let $A = \mathbb{Z}$ and $R = \{<i, i+1>\}$. $R^* = <$.

- Let $A$ = the set of people, $R = \{<x, y> | \text{person x is a parent of person y}\}$. $R^* =$ ?
**Theorem:** $t(R) = R^*$. 

Proof:

Note: this is not the same proof as in the text.

We must show that $R^*$

1) is a transitive relation

2) contains $R$

3) is the smallest transitive relation which contains $R$

Proof:

Part 2):

Easy from the definition of $R^*$.

Part 1):

Suppose $<x, y>$ and $<y, z>$ are in $R^*$.

Show $<x, z>$ is in $R^*$.

By definition of $R^*$, $<x, y>$ is in $R^m$ for some $m$ and $<y, z>$ is in $R^n$ for some $n$. 

Then $<x, z>$ is in $R^m R^n = R^{m+n}$ which is contained in $R^*$. Hence, $R^*$ must be transitive.
Part 3):

Now suppose \( S \) is any transitive relation that contains \( R \).

We must show \( S \) contains \( R^* \) to show \( R^* \) is the smallest such relation.

\[ R \subseteq S \text{ so } R^2 \subseteq S^2 \subseteq S \text{ since } S \text{ is transitive} \]

Therefore \( R^n \subseteq S^n \subseteq S \) for all \( n \). (why?)

Hence \( S \) must contain \( R^* \) since it must also contain the union of all the powers of \( R \).

Q. E. D.

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In fact, we need only consider paths of length \( n \) or less.

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**Theorem:** If \( |A| = n \), then any path of length \( > n \) must contain a cycle.

Proof:

If we write down a list of more than \( n \) vertices representing a path in \( R \), some vertex must appear at least twice in the list (by the Pigeon Hole Principle).
Thus $R^k$ for $k > n$ doesn’t contain any arcs that don’t already appear in the first $n$ powers of $R$.

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**Corollary:** If $|A| = n$, then $t(R) = R^* = R \cup R^2 \cup \ldots \cup R^n$

**Corollary:** We can find the connection matrix of $t(R)$ by computing the join of the first $n$ powers of the connection matrix of $R$.

Powerful Algorithm!

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Example:

![Graph](image)

Do the following in class:

\[
R^2:
\]

\[
R^3:
\]
\[ R^4: \]
\[ R^5: \]
\[ \cdot \]
\[ \cdot \]
\[ \cdot \]
\[ t(R) = R^*: \]

So that you don’t get bored, here are some problems to discuss on your next blind date:

1) Do the closure operations commute?
   
   - Does \( st(R) = ts(R) \)?
   
   - Does \( rt(R) = tr(R) \)?
   
   - Does \( rs(R) = sr(R) \)?

2) Do the closure operations distribute
   
   - Over the set operations?
   
   - Over inverse?
   
   - Over complement?
   
   - Over set inclusion?
Examples:

• Does $t(R_1 - R_2) = t(R_1) - t(R_2)$?

• Does $r(R^{-1}) = [r(R)]^{-1}$?