# Section 7.1
# Relations and Their Properties

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**Definition:** A *binary relation* R from a set A to a set B is a subset $R \quad A \times B$.

Note: there are no constraints on relations as there are on functions.

We have a common graphical representation of relations:

**Definition:** A *Directed graph* or a *Digraph* D from A to B is a collection of *vertices* $V \quad A \quad B$ and a collection of *edges* $R \quad A \times B$. If there is an ordered pair e = <x, y> in *R* then there is an *arc* or *edge* from x to y in D. The elements x and y are called the *initial* and *terminal* vertices of the edge e.

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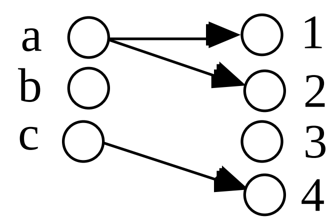
Examples:

- Let A = { a, b, c}
- B = {1, 2, 3, 4}
- R is defined by the ordered pairs or edges

{<a, 1>, <a, 2>, <c, 4>}

can be represented by the digraph D:

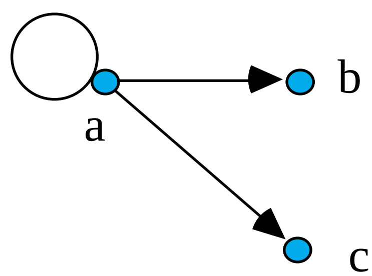



**Definition:** A binary relation R *on a set A* is a subset of $A \times A$ or a relation from $A$ to $A$.

Example:

- A = {a, b, c}
- R = {<a, a>, <a, b>, <a, c>}.

Then a digraph representation of R is:




Note: An arc of the form <x, x> on a digraph is called a *loop*.

Question: How many binary relations are there on a set A?

## Special Properties of Binary Relations

Given:

- A Universe U
- A binary relation R on a subset A of U

**Definition:** R is *reflexive* iff

$$x[x \quad U \rightarrow < x, x > \quad R]$$

Note: if U =      then the implication is true vacuously

The void relation on a void Universe is reflexive!

Note: If U is not void then <u>all</u> vertices in a reflexive relation must have loops!

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**Definition:** R is *symmetric* iff

$$x \quad y[< x, y > \quad R \rightarrow < y, x > \quad R]$$

Note: <u>If</u> there is an arc <x, y> there must be an arc <y, x>.

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**Definition:** R is *antisymmetric* iff

$$x \quad y[< x,y > \quad R \quad < y,x > \quad R \rightarrow \ x = y]$$

Note: If there is an arc from x to y there cannot be one from y to x if x≠y.

You should be able to show that logically: if <x, y> is in R and x ≠ y then <y, x> is not in R.

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**Definition:** R is *transitive* iff

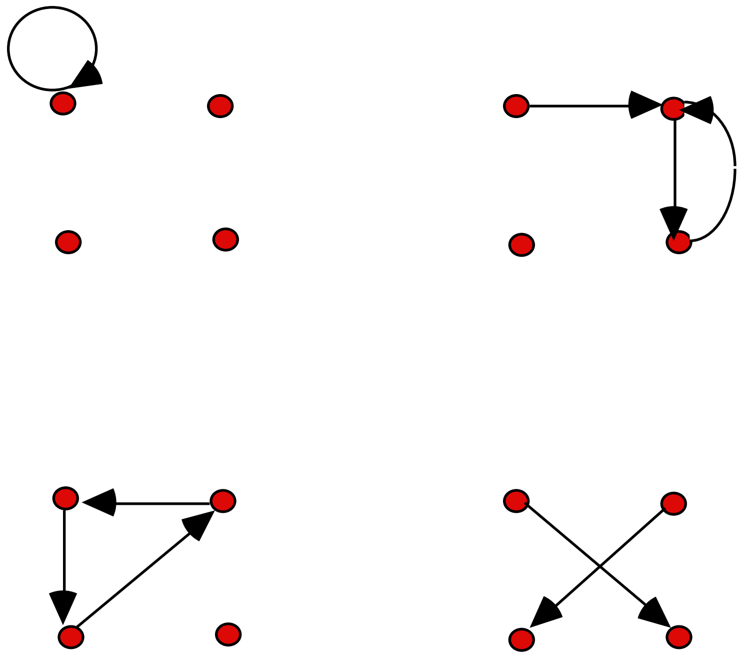
$$x \quad y \quad z[< x,y > \quad R \quad < y,z > \quad R \rightarrow < x,z > \quad R]$$

Note: if there is an arc from x to y and one from y to z then there must be one from x to z.

This is the most difficult one to check. We will develop algorithms to check this later.

Examples:

A. B.



C. D.

A:
not reflexive
symmetric
antisymmetric
transitive

B:
not reflexive
not symmetric
not antisymmetric
not transitive

C:
not reflexive
not symmetric
antisymmetric
not transitive

D:
not reflexive
not symmetric
antisymmetric
transitive

# Combining Relations

## Set operations

A very large set of potential questions -

Let R1 and R2 be binary relations on a set A:

If R1 has property 1

and

R2 has property 2,

does

R1 * R2 have property 3

where * represents an arbitrary binary set operation?

Example:

If

- R1 is symmetric,

and

- R2 is antisymmetric,

does it follow that

- R1   R2 is transitive?

If so, prove it. Otherwise find a counterexample.

Example:

Let R1 and R2 be transitive on A. Does it follow that

R1    R2

is transitive?

Consider

- A = {1, 2}
- R1 = {<1,2>}
- R2 = {<2, 1>}

Then R1    R2 = {<1, 2>, <2,1>} which is <u>not</u> transitive! (Why?)

## Composition

**Definition:** Suppose

- R1 is a relation from A to B
- R2 is a relation from B to C.

Then the composition of R2 with R1, denoted R2∘R1 is the relation from A to C:

If <x. y> is a member of R1 and <y, z> is a member of R2 then <x, z> is a member of R2∘R1.

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Note: For <x, z> to be in the composite relation R2∘R1 there must exist a y in B . . . .

Note: We read them right to left as in functions.

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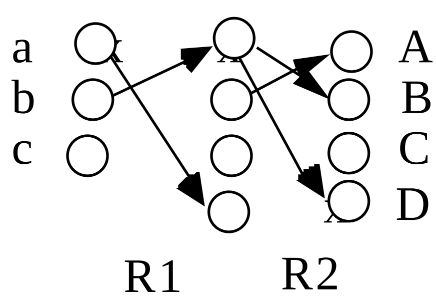
Example:




R2∘R1 = {<b, D>,  <b, B>}

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**Definition:** Let R be a binary relation on A. Then

*Basis*: $R^1 = R$

*Induction*: $R^{n+1} = R^n \circ R$

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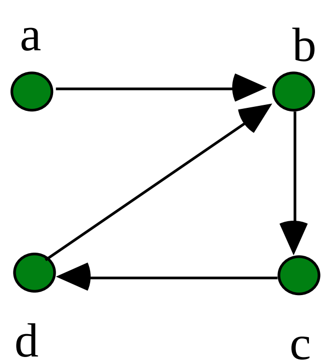
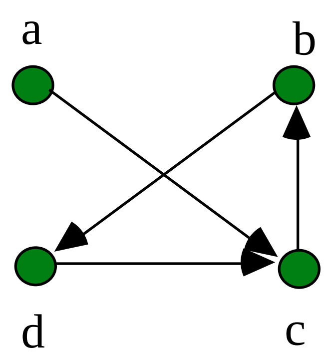
Note: an ordered pair <x, y> is in $R^n$ iff there is a *path* of length n from x to y following the arcs (in the direction of the arrows) of R.

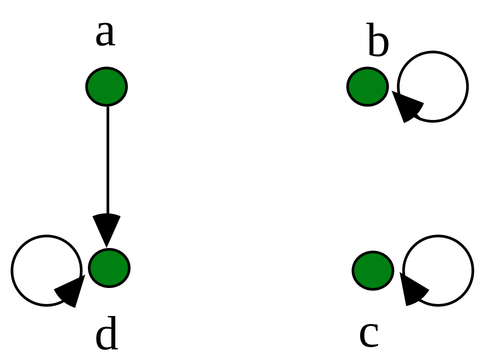
Example:



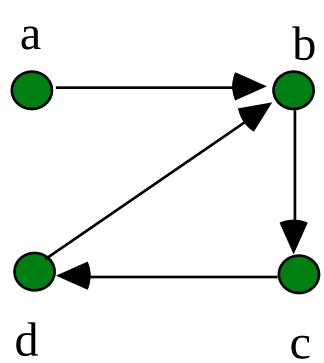

R




$R^2$




$R^3$




$R^4$

Very Important
**Theorem:**

R is transitive iff $R^n$     R for n > 0.

Proof:

1. $R$ transitive $\rightarrow$ $R^n$     $R$

   Use a direct proof and a proof by induction:

   • Assume $R$ is transitive.

   • Now show $R^n$     $R$ by induction.

*Basis*: Obviously true for n = 1.

*Induction:*

• The induction hypothesis:

'assume true for n'.

• Show it must be true for n + 1.

$R^{n+1} = R^n \circ R$ so if <x, y> is in $R^{n+1}$ then there is a z such that <x, z> is in $R^n$ and <z, y> is in $R$.

But since $R^n$     $R$, <x, z> is in $R$.

$R$ is transitive so <x, y> is in $R$.

Since <x, y> was an arbitrary edge the result follows.

2. $R^n \quad R \to R$ transitive

Use the fact that $R^2 \quad R$ and the definition of transitivity. Proof left to the ......

Q. E. D.