Section 7.4
Connectivity

We extent the notion of a *path* to undirected graphs.

An informal definition (see the text for a formal definition):

There is a *path v0, v1, v2, . . . , vn* from vertex *v0* to vertex *vn* if there is a sequence of edges (joining the vertices in sequence) which can be followed from *v0* to *vn*. The path has length n.

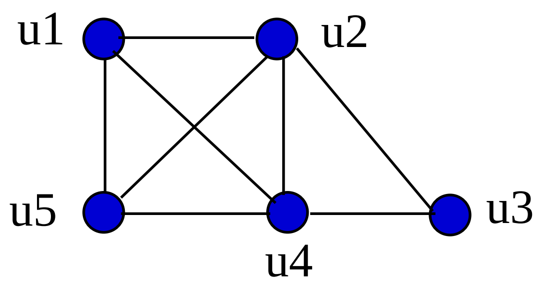
The path is a *circuit* if the path begins and ends with the same vertex.

A path is *simple* if it does not contain the same <u>edge</u> more than once.

Note: There is nothing to prevent traversing an edge back and forth to produce arbitrarily long paths. This is usually not interesting which is why we define a *simple* path.

Examples:

Let G1 be the following graph:



There are many paths from u1 to u3 in G1:

1) u1, u4, u2, u3; length = 3, the path is simple

2) u1, u5, u4, u1, u2, u3; length = 5, the path is simple and it contains a circuit u1, u5, u4, u1.

3) u1, u2, u5, u4, u3; length = 4, the path is simple

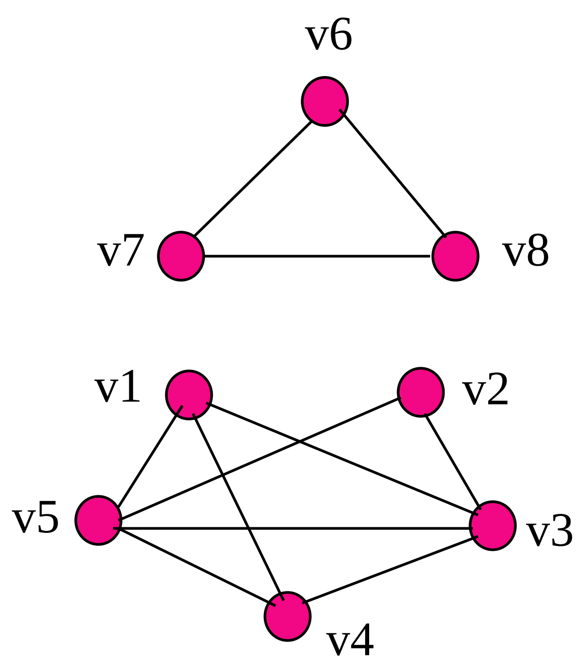
How many simple paths are there?

Connectedness

Definition: A simple graph is *connected* if there is a path between every pair of distinct vertices.

Example:

Let G be the following graph:



The graph G2 is not connected since there is no path from v7 to v1.

Theorem: There is a simple path between every pair of distinct vertices in a connected graph.

Proof:

Because the graph is connected there is a path between u and v. Throw out all redundant circuits to make the path simple.

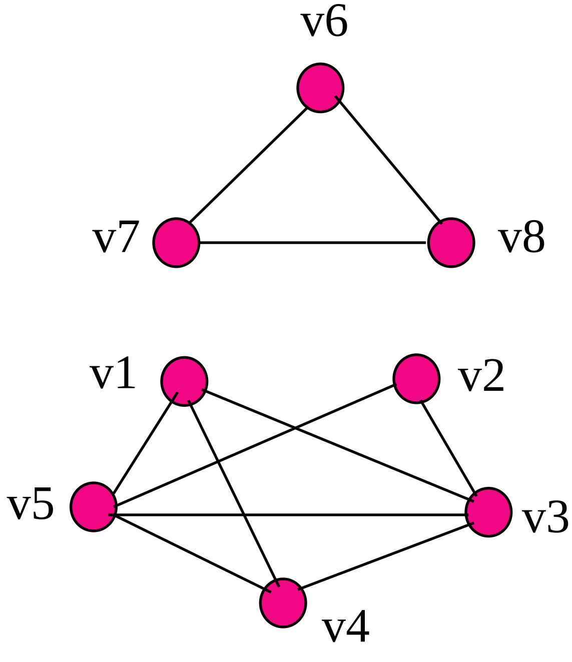
Example:

In example 2) in the graph G1 above there is a circuit containing the vertex u1. Eliminate all edges in the path before the second occurrence of u1.

Definition: The maximally connected subgraphs of G are called the *connected components* or just the *components*.

Example:

Let G be the following graph:



The components of the graph G are

$$G1 = (V1 = \{v6, v7, v8\}, E1)$$

and

$$G2 = (V2 = \{v1, v2, v3, v4, v5\}, E2)$$

where E1 and E2 contain all the edges which join the vertices in V1 and V2 respectively.

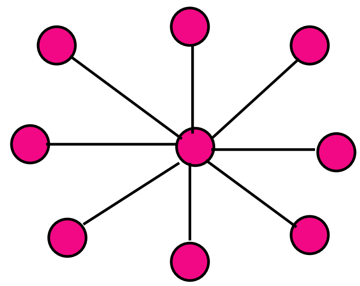
If one can remove a vertex (and all incident edges) and produce a graph with more components, the vertex is called a *cut vertex* or *articulation point*.

Similarly if removal of an edge creates more components the edge is called a *cut edge* or *bridge*.

Examples:

• There are no cut edges or vertices in the graph G above. Removal of any vertex or edge does not create additional components.

• In the star network the center vertex is a cut vertex. All edges are cut edges.

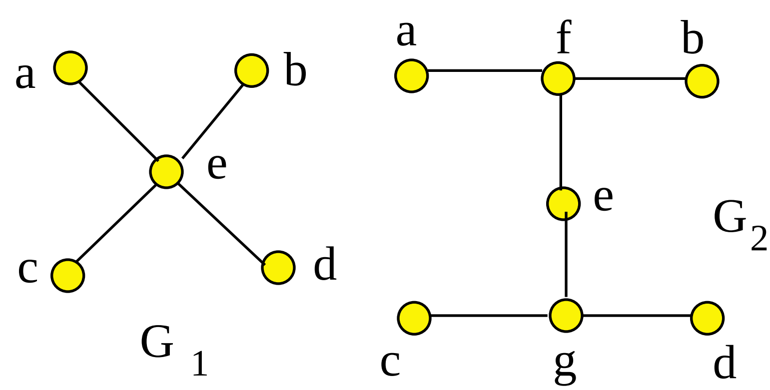


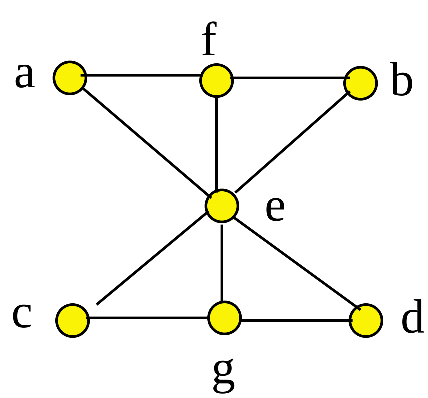
$K_{1,8}$

• In the following graphs G1 and G2 every edge is a cut edge.

In the union, no edge is a cut edge.

The vertex e is a cut vertex in all graphs.





G_1 G_2

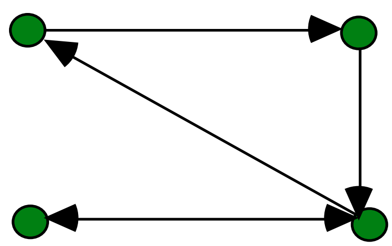
Connectedness in Directed Graphs

Definition: A directed graph is *strongly connected* if there is (directed) path between every pair of vertices.

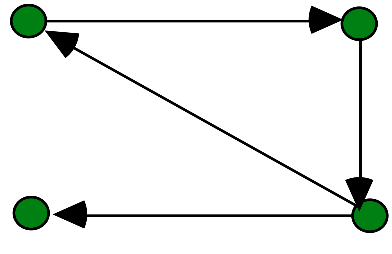
If you can eliminate the arrows (turn the graph into an undirected one) and the graph is connected then the directed graph is *weakly connected*.

Examples:

- strongly connected (hence weakly connected)



- not strongly connected but weakly connected.



Paths and Isomorphism

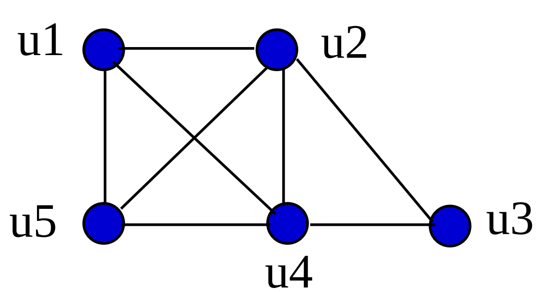
Isomorphic graphs must have 'isomorphic' paths. If one has a simple circuit if length k then so must the other.

Theorem: Let M be the adjacency matrix for the graph G. Then the (i, j) entry of M^r is the number of paths of length r from vertex i to vertex j.

Note: This is the standard power of M, not the boolean product.

Proof below. First an example.

Example:



$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 3 & 2 & 2 & 2 & 2 \\ 2 & 4 & 1 & 3 & 2 \\ 2 & 1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 4 & 2 \\ 2 & 2 & 2 & 2 & 3 \end{bmatrix}$$

$$M^3 = \begin{bmatrix} 6 & 9 & 4 & 9 & 7 \\ 9 & 8 & 7 & 9 & 9 \\ 4 & 7 & 2 & 7 & 4 \\ 9 & 9 & 7 & 8 & 9 \\ 7 & 9 & 4 & 9 & 6 \end{bmatrix}$$

Proof: (by induction, of course!)

• *Basis*: true for paths of length 1 since the adjacency matrix (by definition) represents those paths.

• *Induction Hypothesis*: Assume $M^n(i, j)$ is the number of paths of length n from vertex i to vertex j (recall that M and all its powers are symmetric).

• To show: $M^{(n+1)}(i, j)$ is the number of paths of length (n+1) from vertex i to vertex j:

We know $M^{(n+1)} = M^n \bullet M$ and by definition of matrix product

$$M^n M(i,j) = \sum_{k=1}^{p} M^n(i,k) M(k,j)$$

• $M^n(i,k)$ is the number of paths from vertex i to vertex k of length n by the induction hypothesis.

• $M(k,j)$ is 1 if there is a path of length 1 from vertex k to vertex j.

Hence

• $M^n(i,k)M(k,j)$ is the number of paths of length (n+1) from vertex i to vertex j through the intermediate vertex k.

• To get the total number of paths of length (n+1) from vertex i to vertex j, by the rule of sum we add the number of paths through all intermediate vertices 1 to p.

Q. E. D.
