

Section 7.2 Graph Terminology

Undirected Graphs

Definition: Two vertices u, v in V are *adjacent* or *neighbors* if there is an edge e between u and v .

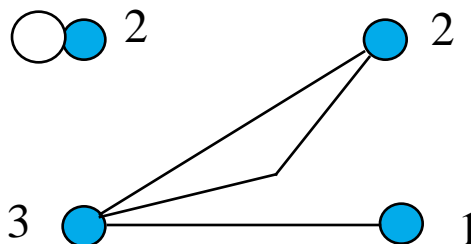
The edge e *connects* u and v .

The vertices u and v are endpoints of e .

Definition: The *degree* of a vertex v , denoted $\deg(v)$, is the number of edges for which it is an endpoint.

A loop contributes twice in an undirected graph.

Example:



- If $\deg(v) = 0$, v is called *isolated*.

- If $\deg(v) = 1$, v is called *pendant*.
-

The Handshaking Theorem:

Let $G = (V, E)$. Then

$$2|E| = \sum_{v \in V} \deg(v)$$

Proof:

Each edge represents contributes twice to the degree count of all vertices.

Q. E. D.

Example:

If a graph has 5 vertices, can each vertex have degree 3?
4?

- The sum is $3 \cdot 5 = 15$ which is an odd number. Not possible.

- The sum is $20 = 2 | E |$ and $20/2 = 10$. May be possible.

Theorem: A graph has an even number of vertices of odd degree.

Proof:

Let V_1 = vertices of odd degree

V_2 = vertices of even degree

The sum must be even. But

- odd times odd = odd
- odd times even = even
- even times even = even
- even plus odd = odd

It doesn't matter whether V_2 has odd or even cardinality.

V_1 cannot have odd cardinality.

Q. E. D.

Example:

It is not possible to have a graph with 3 vertices each of which has degree 1.

Directed Graphs

Definition: Let $\langle u, v \rangle$ be an edge in G . Then u is an *initial vertex* and is *adjacent to* v and v is a *terminal vertex* and is *adjacent from* u .

Definition: The *in degree* of a vertex v , denoted $\deg^-(v)$ is the number of edges which terminate at v .

Similarly, the *out degree* of v , denoted $\deg^+(v)$, is the number of edges which initiate at v .

Theorem: $|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$

Special Simple Graphs

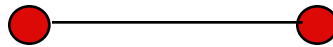
- Complete graphs - K_n : the simple graph with
 - n vertices
 - exactly one edge between every pair of distinct vertices.

Maximum redundancy in local area networks and processor connection in parallel machines.

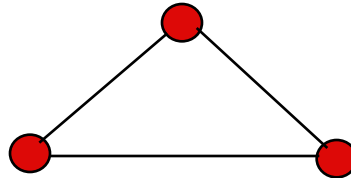
Examples:



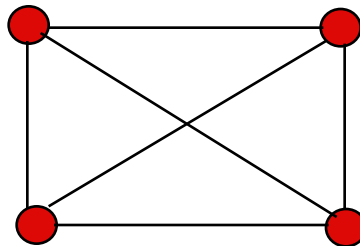
K_1



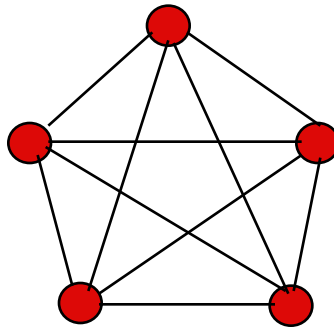
K_2



K_3



K_4

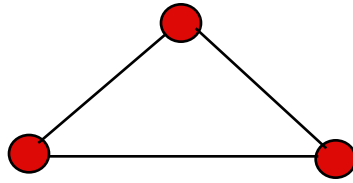


K_5

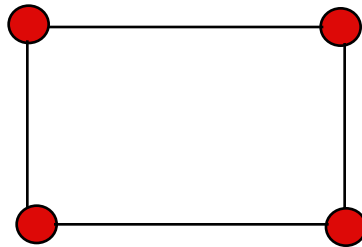
Note: K_5 is important because it is the simplest nonplanar graph: It cannot be drawn in a plane with nonintersecting edges.

- Cycles:

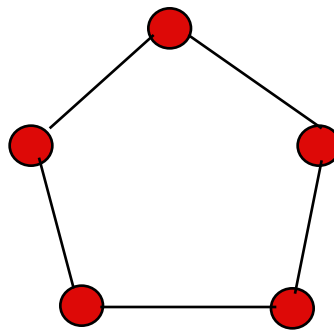
C_n is an n vertex graph which is a cycle. Local area networks are sometimes configured this way called *Ring* networks.



C_3



C_4

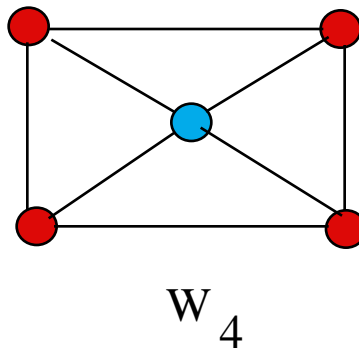
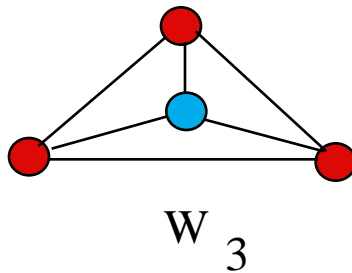


C_5

- Wheels:

Add one additional vertex to the cycle C_n and add an edge from each vertex to the new vertex to produce W_n .

Provides redundancy in local area networks.



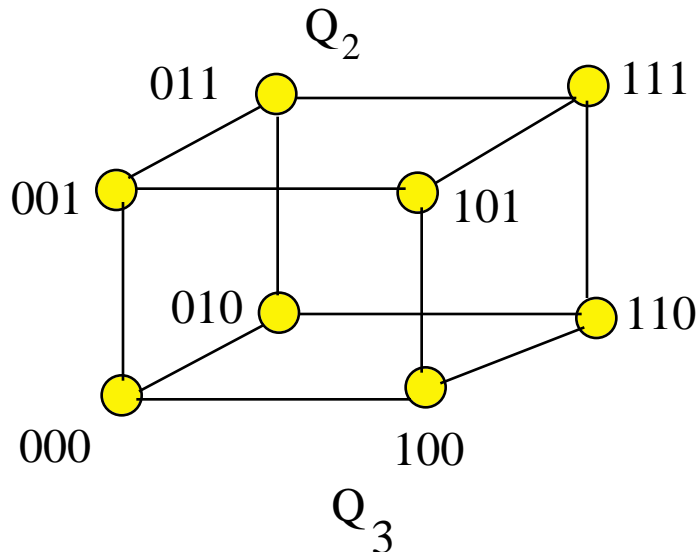
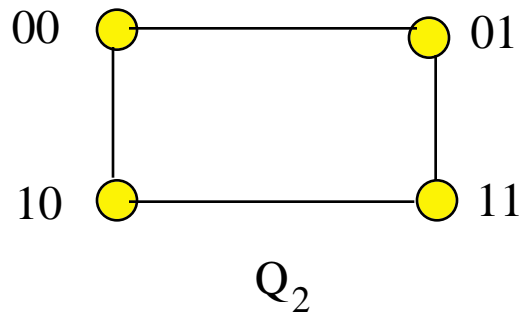
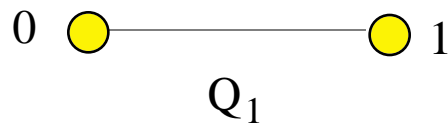
- n-Cubes:

Q_n is the graph with 2^n vertices representing bit strings of length n .

An edge exists between two vertices that differ by one bit position.

A common way to connect processors in parallel machines.

Intel Hypercube.



Bipartite Graphs

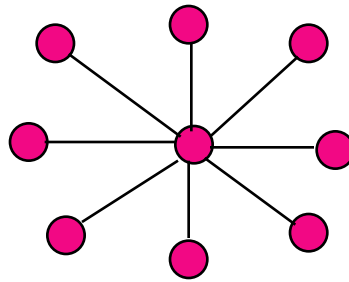
Definition: A simple graph G is *bipartite* if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2 .

Note: There are no edges which connect vertices in V_1 or in V_2 .

A bipartite graph is *complete* if there is an edge from every vertex in V_1 to every vertex in V_2 , denoted $K_{m,n}$ where $m = |V_1|$ and $n = |V_2|$.

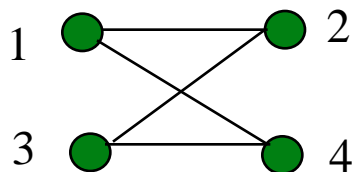
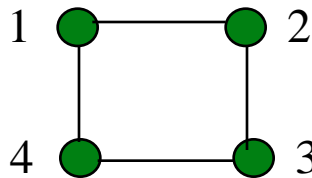
Examples:

- Suppose bigamy is permitted but not same sex marriages and males are in V_1 and females in V_2 and an edge represents a marriage. If every male is married to every female then the graph is complete.
- Supplier, warehouse transportation models are bipartite and an edge indicates that a given supplier sends inventory to a given warehouse.
- A Star network is a $K_{1,n}$ bipartite graph.

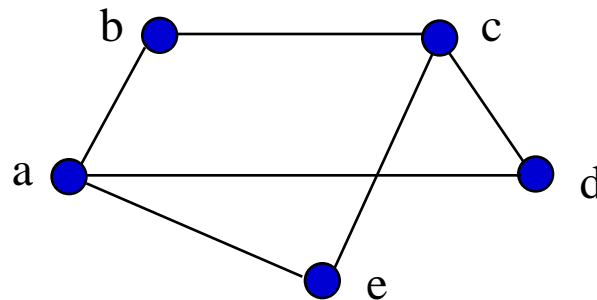


$K_{1,8}$

• C_k for k even is a bipartite graph: even numbered vertices in $V1$, odd numbered in $V2$.



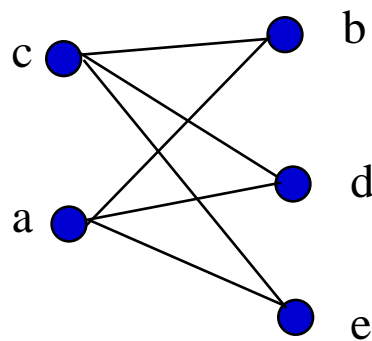
- Is the following graph bipartite?



If a is in $V1$ then e , c and b must be in $V2$ (why?).

Then c is in $V1$ and there is no inconsistency.

We rearrange the graph as follows:



New Graphs from Old

Definition: (W, F) is a *subgraph* of $G = (V, E)$ if

$$W \subseteq V \text{ and } F \subseteq E.$$

Definition: If G_1 and G_2 are simple then

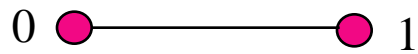
$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$$

and the graph is simple.

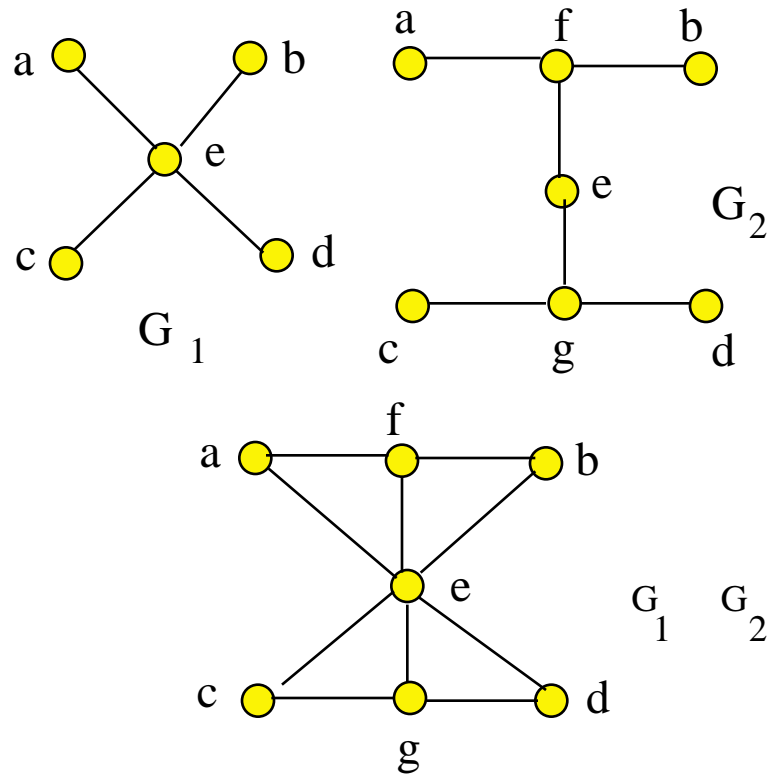


Examples:

- Find the subgraphs of Q_1 :



- Find the union of the two graphs G_1 and G_2 :



Note: The important properties of a graph do not depend on how we draw it. We want to be able to identify two graphs that are the same (up to labeling of the vertices).
