Support Vector Machines (Contd.), Classification Loss Functions and Regularizers

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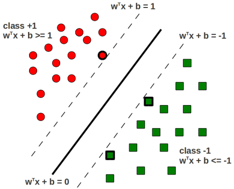
CS5350/6350: Machine Learning

September 13, 2011

SVM (Recap)

- SVM finds the maximum margin hyperplane that separates the classes



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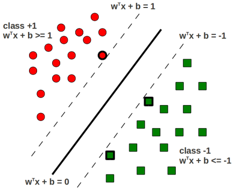

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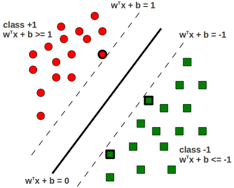

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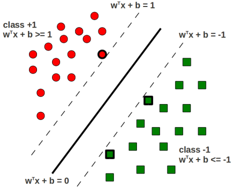

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- The optimization problem for the separable case (no misclassified training example)

$$\text{Minimize} \quad f(\mathbf{w}, b) = \frac{||\mathbf{w}||^2}{2}$$

$$\text{subject to} \quad y_n(\mathbf{w}^T\mathbf{x}_n + b) \geq 1, \qquad n = 1, \ldots, N$$

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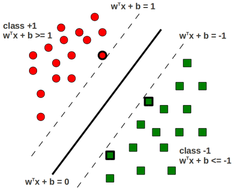

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- This is a Quadratic Program (QP) with N linear inequality constraints

SVM: The Optimization Problem

- Our optimization problem is:

$$\boxed{\begin{array}{ll} \text{Minimize} \quad f(\mathbf{w}, b) = \dfrac{||\mathbf{w}||^2}{2} & \\ \text{subject to} \quad 1 \leq y_n(\mathbf{w}^T\mathbf{x}_n + b), & \qquad n = 1, \ldots, N \end{array}}$$

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- Introducing Lagrange Multipliers α_n ($n = \{1, \ldots, N\}$), one for each constraint, leads to the Primal Lagrangian:

$$\text{Minimize} \quad L_P(\mathbf{w}, b, \alpha) = \frac{||\mathbf{w}||^2}{2} + \sum_{n=1}^{N} \alpha_n \{1 - y_n(\mathbf{w}^T\mathbf{x}_n + b)\}$$
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- We can now solve this Lagrangian
  - i.e., optimize $L(\mathbf{w}, b, \alpha)$ w.r.t. $\mathbf{w}$, b, and α
  - .. making use of the Lagrangian Duality theory..

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- Take (partial) derivatives of L_P w.r.t. $\mathbf{w}$, b and set them to zero

$$\frac{\partial L_P}{\partial \mathbf{w}} = 0 \Rightarrow \mathbf{w} = \sum_{n=1}^{N} \alpha_n y_n \mathbf{x}_n,$$

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  - Several off-the-shelf solvers exist to solve such QPs
  - Some examples: `quadprog` (MATLAB), CVXOPT, CPLEX, IPOPT, etc.

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- Once we have the α_n's, **w** and b can be computed as:

$$\mathbf{w} = \sum_{n=1}^{N} \alpha_n y_n \mathbf{x}_n$$

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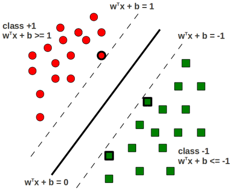
  - α_n is non-zero only if $\mathbf{x}_n$ lies on one of the two margin boundaries, i.e., for which $y_n(\mathbf{w}^T\mathbf{x}_n + b) = 1$
  - These examples are called support vectors
  - Support vectors "support" the margin boundaries



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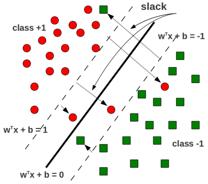

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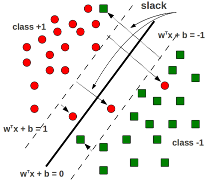

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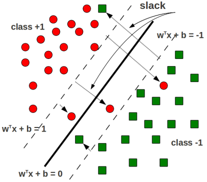

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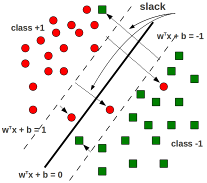

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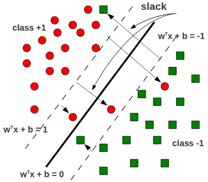

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- $\xi_n \geq 0, \forall n$, misclassification when $\xi_n > 1$

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  - Large $C \Rightarrow C\sum_{n=1}^{N} \xi_n$ dominates $\Rightarrow$ prefer small # of misclassified examples
    - .. at the expense of having a small margin

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- Comparison note: Terms in red font were not there in the separable case

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- Again a Quadratic Programming problem in α
- Given α, the solution for $\mathbf{w}, b$ has the same form as the separable case
- **Note:** α is again sparse. Nonzero α_n's correspond to the support vectors

Support Vectors in the non-separable case

- The separable case has only one type of support vectors
  - .. ones that lie on the margin boundaries $\mathbf{w}^T\mathbf{x} + b = -1$ and $\mathbf{w}^T\mathbf{x} + b = +1$

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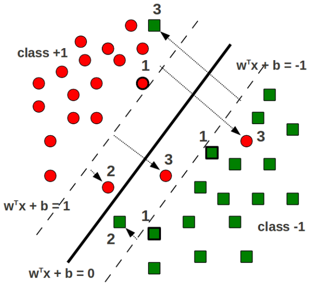

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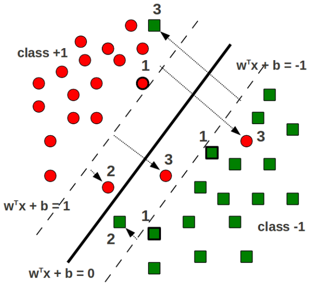

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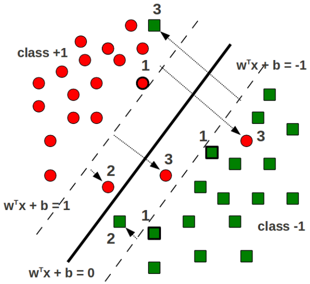

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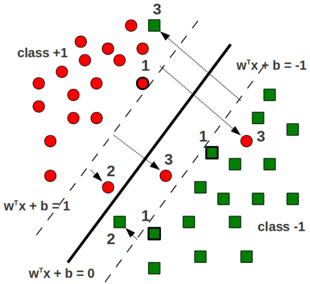

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- Popular SVM implementations: libSVM, SVMLight, SVM-struct, etc.
  - Also `http://www.kernel-machines.org/software`

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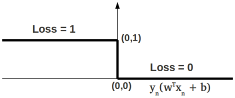

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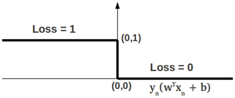

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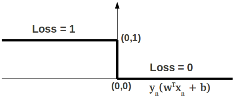

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- **Different loss function approximations and regularizers lead to specific algorithms** (e.g., Perceptron, SVM, Logistic Regression, etc.).

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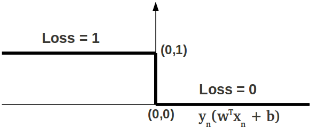

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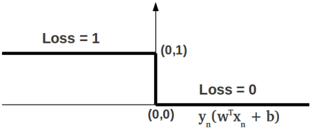

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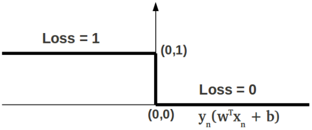

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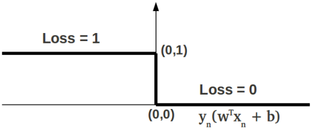

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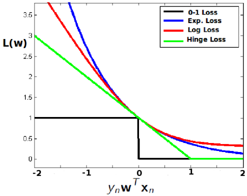

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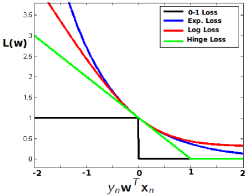

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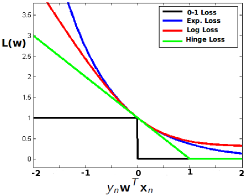

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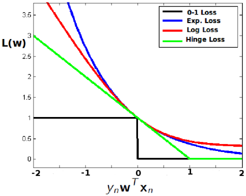

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  - Note: hinge loss is not smooth at (1,0) but subgradient descent can be used

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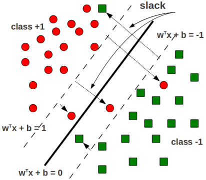

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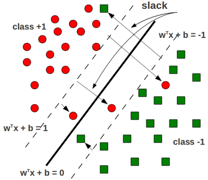

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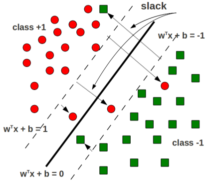

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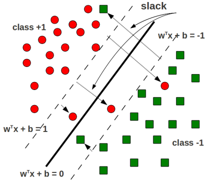

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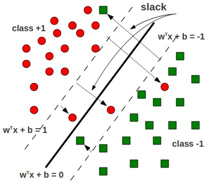

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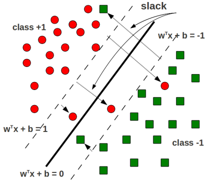

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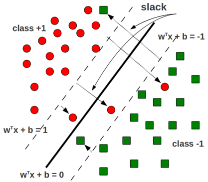

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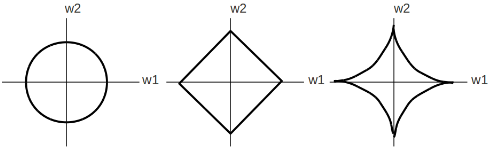

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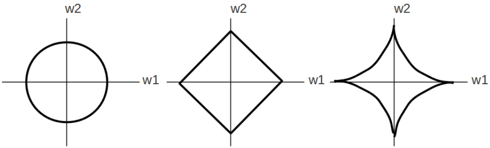

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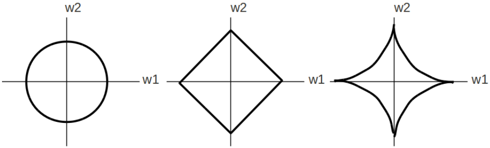

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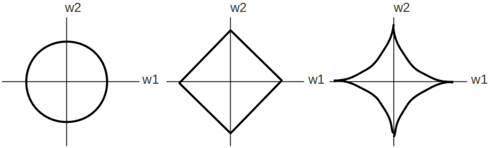

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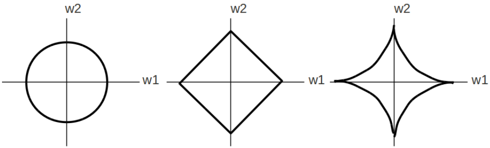

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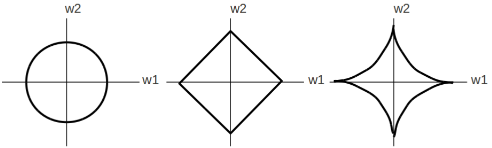

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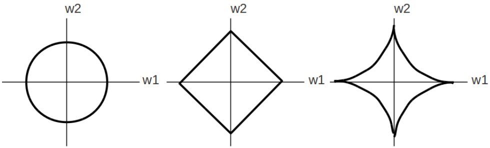

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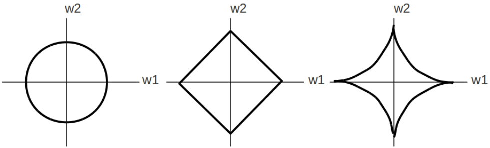

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