Hyperplane based Classification: Perceptron and (Intro to) Support Vector Machines

Piyush Rai

CS5350/6350: Machine Learning

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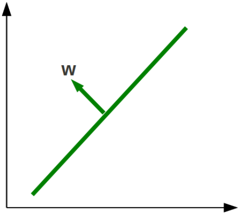
Hyperplane

- Separates a D-dimensional space into two half-spaces



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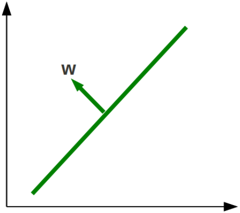

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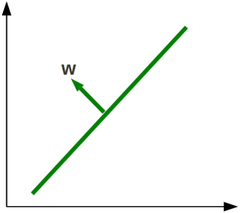

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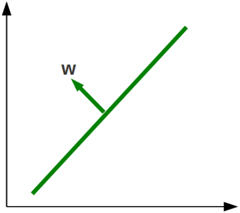

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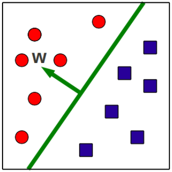
- $\mathbf{w}$ is orthogonal to any vector lying on the hyperplane
- Assumption: The hyperplane passes through origin. If not,
  - have a *bias* term b; we will then need both $\mathbf{w}$ and b to define it
  - $b > 0$ means moving it parallely along $\mathbf{w}$ ($b < 0$ means in opposite direction)

Linear Classification via Hyperplanes

- Linear Classifiers: Represent the decision boundary by a hyperplane **w**



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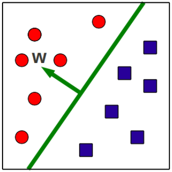

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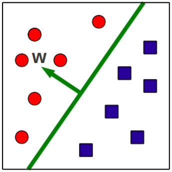

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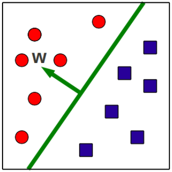

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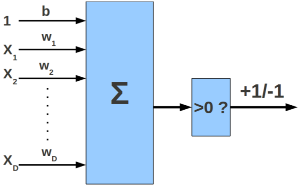

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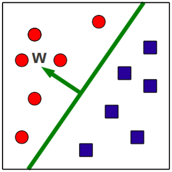

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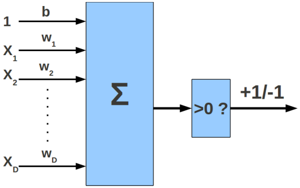

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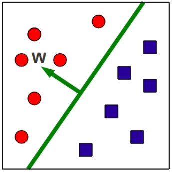

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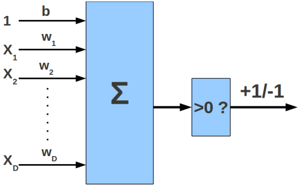
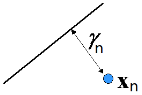


- **Question:** What about the points **x** for which $\mathbf{w}^T\mathbf{x} + b = 0$?
- **Goal:** To learn the hyperplane $(\mathbf{w}, b)$ using the training data

Concept of Margins

- Geometric margin γ_n of an example $\mathbf{x}_n$ is its distance from the hyperplane

$$\gamma_n = \frac{\mathbf{w}^T\mathbf{x}_n + b}{||\mathbf{w}||}$$



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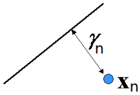

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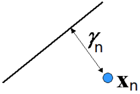

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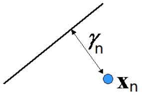

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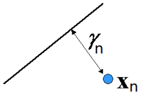

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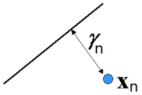

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  - large margin ⇒ high confidence

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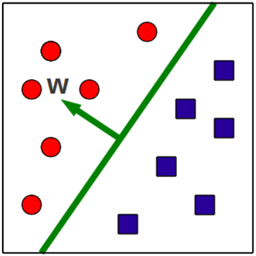

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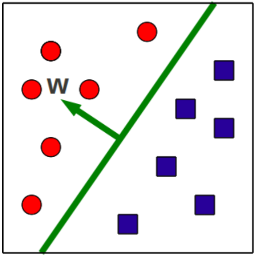

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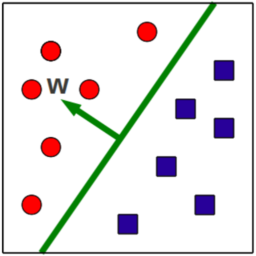

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  - .. or use a *combination* of multiple perceptrons (Neural Networks)

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    - Often batch problems can be solved using online learning!

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- Given: Sequence of N training examples $\{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_N, y_N)\}$
- Initialize: $\mathbf{w} = [0, \ldots, 0], b = 0$
- Repeat until convergence:
  - For $n = 1, \ldots, N$
    - if $sign(\mathbf{w}^T\mathbf{x}_n + b) \neq y_n$ (i.e., mistake is made)

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  - Completed one pass over the data (each example seen once)

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The Perceptron Algorithm: Formally

- Given: Sequence of N training examples $\{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_N, y_N)\}$
- Initialize: $\mathbf{w} = [0, \ldots, 0], b = 0$
- Repeat until convergence:
  - For $n = 1, \ldots, N$
    - if $sign(\mathbf{w}^T\mathbf{x}_n + b) \neq y_n$ (i.e., mistake is made)

$$\begin{aligned} \mathbf{w} &= \mathbf{w} + y_n\mathbf{x}_n \\ b &= b + y_n \end{aligned}$$

- Stopping condition: stop when either
  - All training examples are classified correctly
    - May overfit, so less common in practice
  - A fixed number of iterations completed, or some convergence criteria met
  - Completed one pass over the data (each example seen once)
    - E.g., examples arriving in a streaming fashion and can't be stored in memory (more passes just not possible)
- Note: $sign(\mathbf{w}^T\mathbf{x}_n + b) \neq y_n$ is equivalent to $y_n(\mathbf{w}^T\mathbf{x}_n + b) < 0$

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- Let's look at a misclassified positive example ($y_n = +1$)
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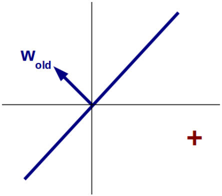
$$\mathbf{w}_{new} = \mathbf{w}_{old} + \mathbf{x}$$



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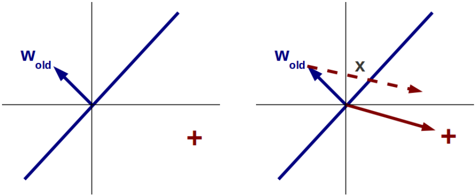

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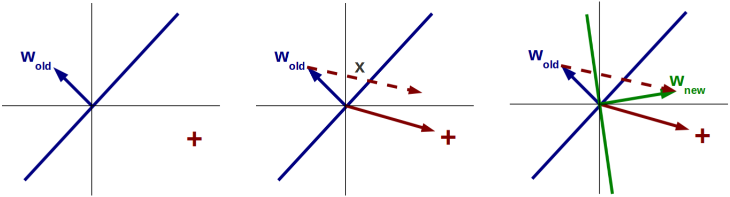

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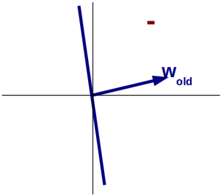

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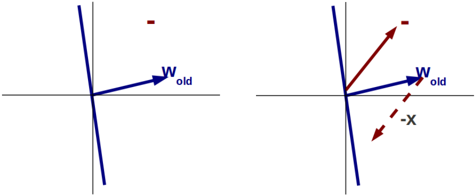

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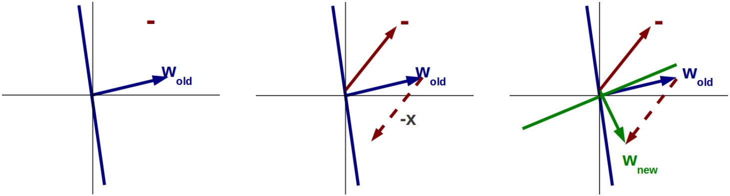

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- Using (1), (2), and $||\mathbf{w}_*|| = 1$, we get $k\gamma < \mathbf{w}_{k+1}^T \mathbf{w}_* \leq ||\mathbf{w}_{k+1}|| \leq R\sqrt{k}$

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- $\mathbf{w}_{k+1}^T \mathbf{w}_* = \mathbf{w}_k^T \mathbf{w}_* + y_n \mathbf{w}_*^T \mathbf{x}_n \geq \mathbf{w}_k^T \mathbf{w}_* + \gamma$ (why is this nice?)
- Repeating iteratively k times, we get $\mathbf{w}_{k+1}^T \mathbf{w}_* > k\gamma$ (1)
- $||\mathbf{w}_{k+1}||^2 = ||\mathbf{w}_k||^2 + 2y_n \mathbf{w}_k^T \mathbf{x}_n + ||\mathbf{x}||^2 \leq ||\mathbf{w}_k||^2 + R^2$ (since $y_n \mathbf{w}_k^T \mathbf{x}_n \leq 0$)
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$$\boxed{k \leq R^2/\gamma^2}$$

Convergence of Perceptron

Theorem (Block & Novikoff): If the training data is linearly separable with margin γ by a unit norm hyperplane $\mathbf{w}_*$ ($||\mathbf{w}_*|| = 1$) with $b = 0$, then perceptron converges after R^2/γ^2 mistakes during training (assuming $||\mathbf{x}|| < R$ for all $\mathbf{x}$).

Proof:

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Nice Thing: Convergence rate does not depend on the number of training examples N or the data dimensionality D. Depends only on the margin!!!

Perceptron: some additional notes

- The Perceptron loss function (without any regularization on **w**):

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Homework

1. Consider a perceptron with 2 inputs, 1 output, and threshold activation function. If the two weights are w_1=1 and w_2=1, and the threshed bias is b=-1.5, then what is the output for input (0, 0)? What about the inputs (1,0), (0, 1), (1, 1)?
Draw the decision function for this perceptron, and write down its equation.
Does it correspond to any particular logic gate?

2. Work out perceptrons that construct logical operators NOT, NAND, and NOR.

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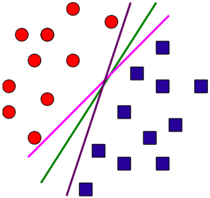

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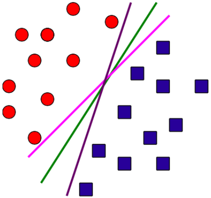

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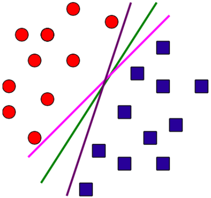

- Intuitively, we want the hyperplane having the maximum margin
- Large margin leads to good generalization on the test data
  - We will see this formally when we cover Learning Theory

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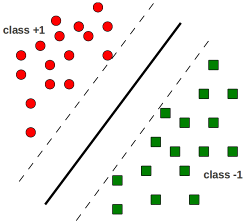
- *Additionally* uses the Maximum Margin Principle
  - Finds the hyperplane with *maximum separation margin* on the training data



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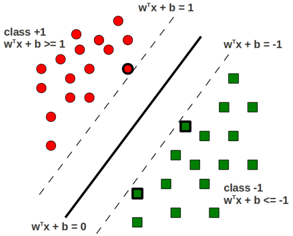

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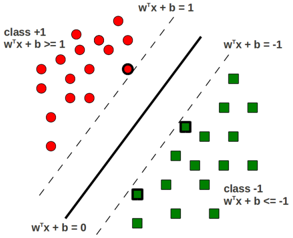

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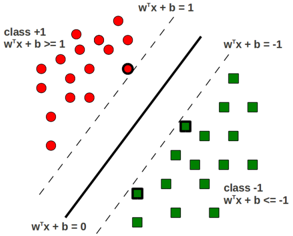

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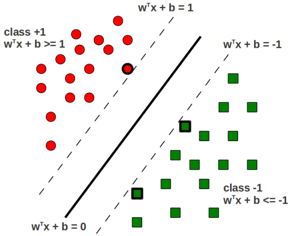

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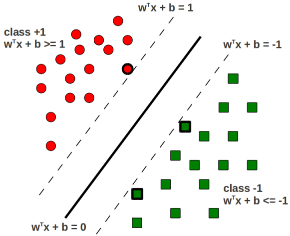

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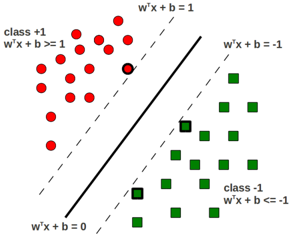

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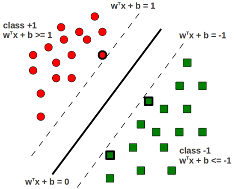

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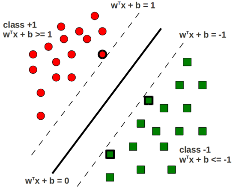

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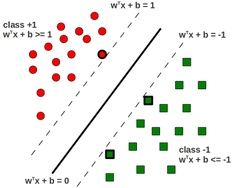

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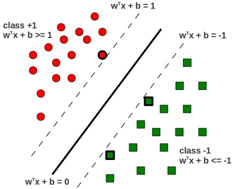

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- This is a Quadratic Program (QP) with N linear inequality constraints

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Solving the SVM Optimization Problem

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- We can now solve this Lagrangian
  - i.e., optimize $L(\mathbf{w}, b, \alpha)$ w.r.t. $\mathbf{w}$, b, and α
  - .. making use of the Lagrangian Duality theory..

Next class..

- Solving the SVM optimization problem
- Allowing misclassified training examples (**non-zero loss**)
- Introduction to kernel methods (nonlinear SVMs)

Homework

Given is the following dataset:

Class 1: $(1, 1)^T$, $(1, 2)^T$, $(2, 1)^T$,
Class 2: $(0, 0)^T$, $(1, 0)^T$, $(0, 1)^T$.

Plot the data points and find the optimal separating line.
What are the support vectors, and what is the margin?