Parameter Estimation in Probabilistic Models, Linear Regression and Logistic Regression

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CS5350/6350: Machine Learning

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- **Goal:** Estimate parameter θ that best models/describes the data
- Several ways to define the "best"

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$$\boxed{\hat{\theta}_{MLE} = \arg\max_{\theta} \log \mathcal{L}(\theta) = \arg\max_{\theta} \sum_{n=1}^{N} \log P(\mathbf{d}_n \mid \theta)}$$

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- While doing MAP, we usually maximize the log of the posterior probability

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- MAP allows incorporating our prior knowledge about θ in its estimation

Linear Regression: The Probabilistic Formulation

- Each response generated by a linear model plus some Gaussian noise

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- Given data $\mathcal{D} = \{(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \ldots, (\mathbf{x}_N, y_N)\}$, we want to estimate the weight vector $\mathbf{w}$

Linear Regression: Maximum Likelihood Solution

- Log-likelihood:

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- For $\sigma = 1$ (or some constant) for each input, it's equivalent to the least-squares objective for linear regression

Linear Regression: Maximum-a-Posteriori Solution

- Let's assume a Gaussian prior distribution over the weight vector **w**

$$P(\mathbf{w}) = \mathcal{N}or(\mathbf{w} \mid 0, \lambda^{-1}\mathbf{I}) = \frac{1}{(2\pi)^{D/2}} \exp\left(-\frac{\lambda}{2}\mathbf{w}^\top \mathbf{w}\right)$$

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- For $\sigma = 1$ (or some constant) for each input, it's equivalent to the regularized least-squares objective

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Probabilistic Classification: Logistic Regression

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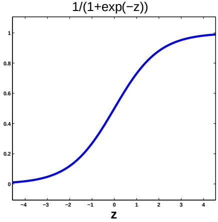
$$P(y \mid \mathbf{x}, \mathbf{w}) = \sigma(y\mathbf{w}^\top\mathbf{x}) = \frac{1}{1 + \exp(-y\mathbf{w}^\top\mathbf{x})}$$

- σ is the logistic function which maps all real number into (0,1)



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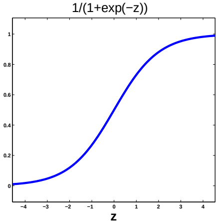

- σ is the logistic function which maps all real number into (0,1)
- This is the Logistic Regression model
  - Misnomer: Logistic Regression is a classification model :-)

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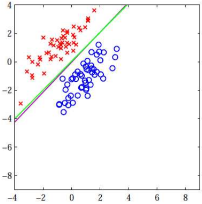
- The decision boundary is therefore linear ⇒ Logistic Regression is a linear classifier (note: it's possible to kernelize and make it nonlinear)



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$$P(\mathbf{w}) = \mathcal{N}or(\mathbf{w} \mid 0, \lambda^{-1}\mathbf{I}) = \frac{1}{(2\pi)^{D/2}} \exp\left(-\frac{\lambda}{2}\mathbf{w}^\top \mathbf{w}\right)$$

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- See "A comparison of numerical optimizers for logistic regression" by Tom Minka on optimization techniques (gradient descent and others) for logistic regression (both MLE and MAP)

Logistic Regression: MLE vs MAP (summary)

- MLE solution:

$$\boxed{\hat{\mathbf{w}}_{MLE} = \arg\min_{\mathbf{w}} \sum_{n=1}^{N} \log[1 + \exp(-y_n \mathbf{w}^\top \mathbf{x}_n)]}$$

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- Note: For MAP, different prior distributions lead to different regularizers
  - Gaussian prior on $\mathbf{w}$ regularizes the ℓ_2 norm of $\mathbf{w}$
  - Laplace prior $\exp(-C||\mathbf{w}||_1)$ on $\mathbf{w}$ regularizes the ℓ_1 norm of $\mathbf{w}$

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