Data Clustering: K-means and Hierarchical Clustering

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CS5350/6350: Machine Learning

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- Given: N unlabeled examples $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$; the number of partitions K
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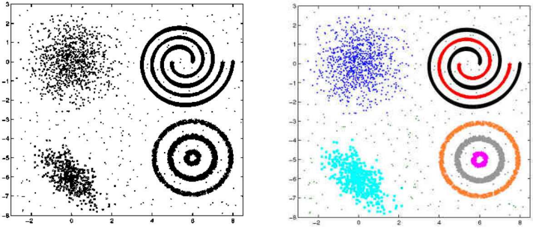

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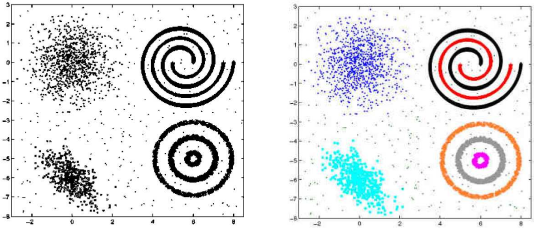

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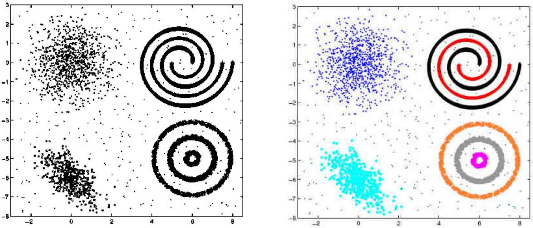

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- The only information clustering uses is the similarity between examples
- Clustering groups examples based of their mutual similarities
- A good clustering is one that achieves:
  - High within-cluster similarity
  - Low inter-cluster similarity

Picture courtesy: "Data Clustering: 50 Years Beyond K-Means", A.K. Jain (2008)

Notions of Similarity

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  - Manhattan distance: $d(\mathbf{x}, \mathbf{z}) = \sum_{d=1}^{D} |x_d - z_d|$
  - Kernelized (non-linear) distance: $d(\mathbf{x}, \mathbf{z}) = ||\phi(\mathbf{x}) - \phi(\mathbf{z})||$

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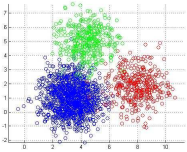
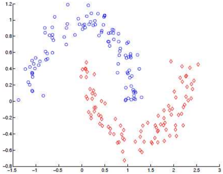



  - For the left figure above, Euclidean distance may be reasonable
  - For the right figure above, kernelized distance seems more reasonable

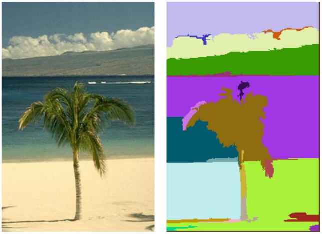
Similarity is Subjective

- Similarity is often hard to define

- Different similarity criteria can lead to different clusterings

Data Clustering: Some Real-World Examples

- Clustering images based on their perceptual similarities
- Image segmentation (clustering pixels)



- Clustering webpages based on their content
- Clustering web-search results
- Clustering people in social networks based on user properties/preferences
- .. and many more..

Picture courtesy: `http://people.cs.uchicago.edu/∼pff/segment/`

Types of Clustering

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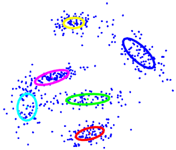
   - Partitions are independent of each other



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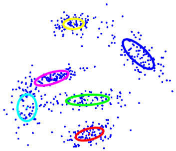
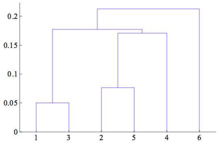
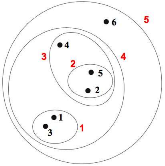


2. **Hierarchical clustering** (e.g., agglomerative clustering, divisive clustering)
   - Partitions can be visualized using a tree structure (a dendrogram)
   - Does not need the number of clusters as input
   - Possible to view partitions at different levels of granularities (i.e., can refine/coarsen clusters) using different K





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  - Assign each of example $\mathbf{x}_n$ to its closest cluster center

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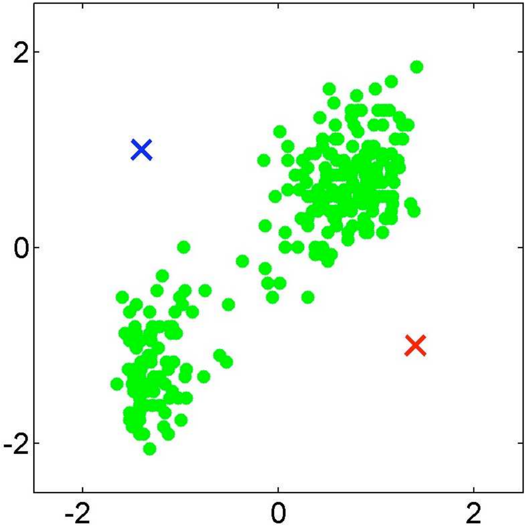
- A possible convergence criteria: cluster centers do not change anymore

K-means: Initialization (assume $K = 2$)



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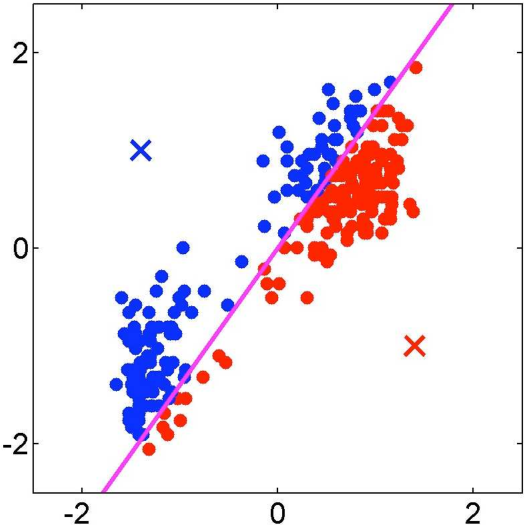
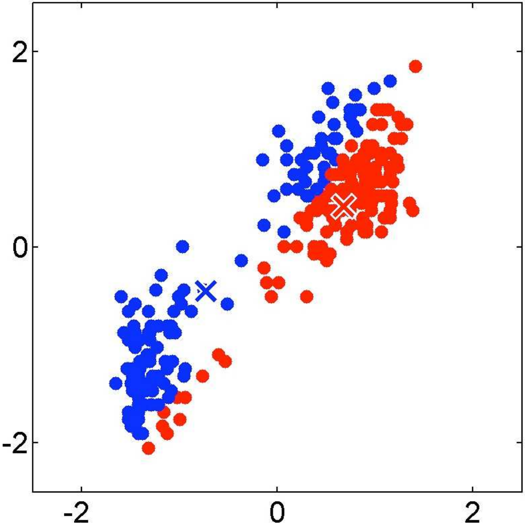
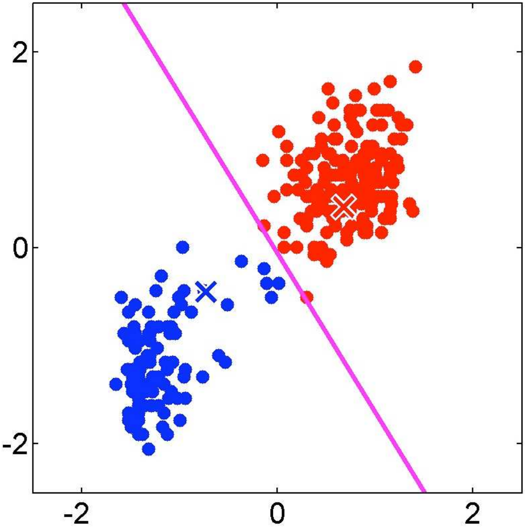


K-means iteration 1: Recomputing the cluster centers



K-means iteration 2: Assigning points



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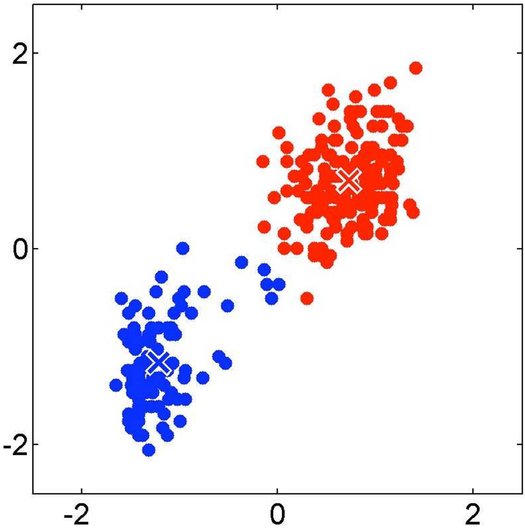
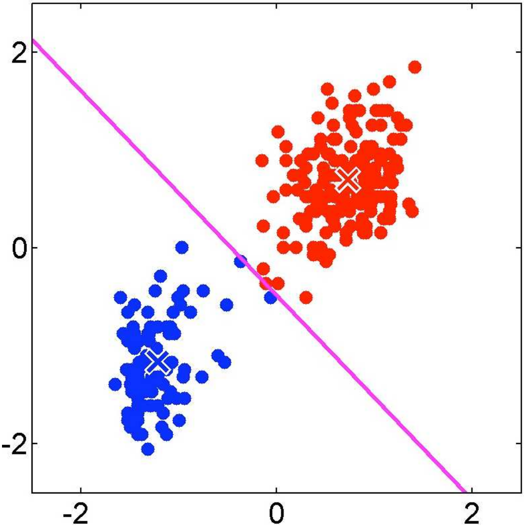
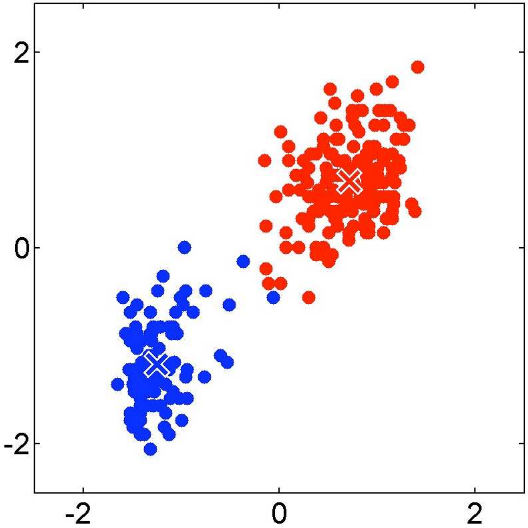


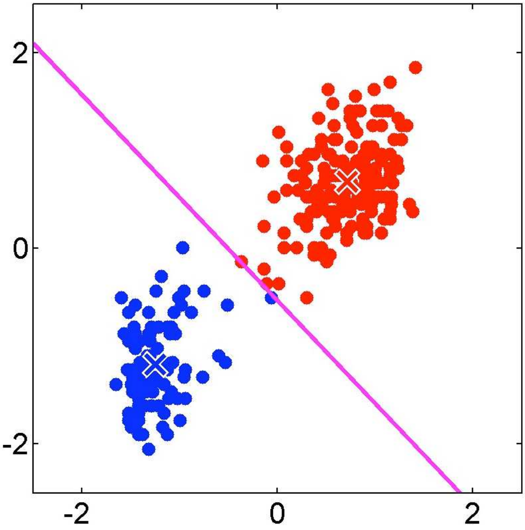
K-means iteration 3: Assigning points



K-means iteration 3: Recomputing the cluster centers



K-means iteration 4: Assigning points



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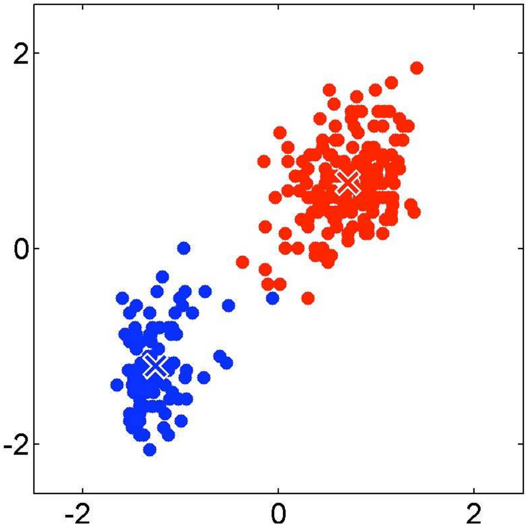

K-means: The Objective Function

The K-means objective function

- Let $\mu_1, \ldots, \mu_K$ be the K cluster centroids (means)
- Let $r_{nk} \in \{0, 1\}$ be indicator denoting whether point $\mathbf{x}_n$ belongs to cluster k
- K-means objective minimizes the total distortion (sum of distances of points from their cluster centers)

$$J(\mu, r) = \sum_{n=1}^{N} \sum_{k=1}^{K} r_{nk} ||\mathbf{x}_n - \mu_k||^2$$

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- Note: Exact optimization of the K-means objective is NP-hard
- The K-means algorithm is a heuristic that converges to a local optimum

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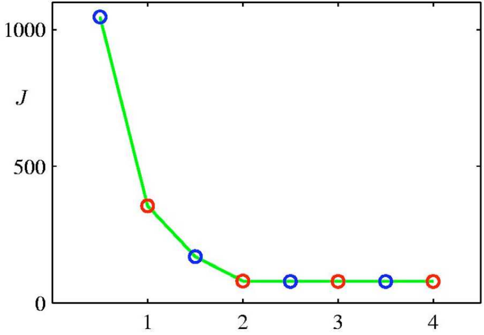
- One way to select K for the K-means algorithm is to try different values of K, plot the K-means objective versus K, and look at the "elbow-point" in the plot



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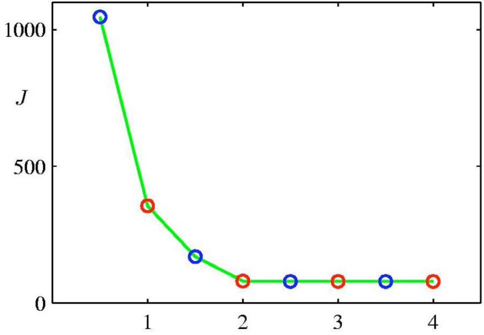

- For the above plot, $K = 2$ is the elbow point

Picture courtesy: "Pattern Recognition and Machine Learning, Chris Bishop (2006)

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  - Try multiple initializations and choose the best result
  - Other smarter initialization schemes (e.g., look at the K-means++ algorithm by Arthur and Vassilvitskii)

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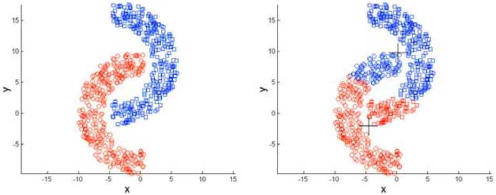
  - Reason: Median is more robust than mean in presence of outliers
- Works well only for round shaped, and of roughly equal sizes/density clusters
- Does badly if the clusters have non-convex shapes
  - Spectral clustering or kernelized K-means can be an alternative

K-means Limitations Illustrated

Non-convex/non-round-shaped clusters: Standard K-means fails!



K-means Limitations Illustrated

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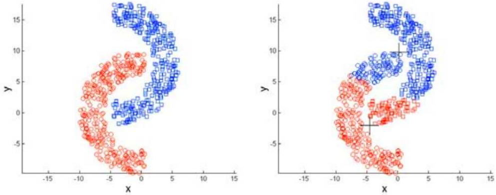
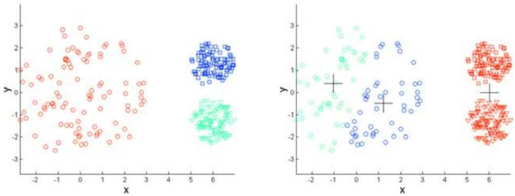
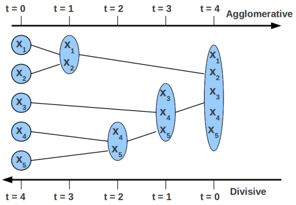


Clusters with different densities



Picture courtesy: Christof Monz (Queen Mary, Univ. of London)

Hierarchical Clustering



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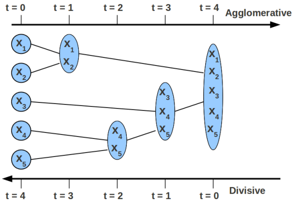

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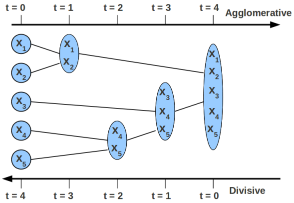

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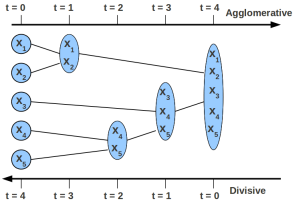

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- Agglomerative is more popular and simpler than divisive (but less accurarate)

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- **Average-link:** compromise between single and complexte linkage

$$d(R, S) = \frac{1}{|R||S|} \sum_{\mathbf{x}_R \in R, \mathbf{x}_S \in S} d(\mathbf{x}_R, \mathbf{x}_S)$$

Hierarchical Clustering: (Dis)similarity between clusters

- We know how to compute the dissimilarity $d(\mathbf{x}_i, \mathbf{x}_j)$ between two examples
- How to compute the dissimilarity between two clusters R and S?
- **Min-link or single-link:** results in chaining (clusters can get very large)

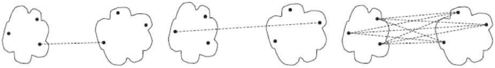
$$d(R, S) = \min_{\mathbf{x}_R \in R, \mathbf{x}_S \in S} d(\mathbf{x}_R, \mathbf{x}_S)$$

- **Max-link or complete-link:** results in small, round shaped clusters

$$d(R, S) = \max_{\mathbf{x}_R \in R, \mathbf{x}_S \in S} d(\mathbf{x}_R, \mathbf{x}_S)$$

- **Average-link:** compromise between single and complexte linkage

$$d(R, S) = \frac{1}{|R||S|} \sum_{\mathbf{x}_R \in R, \mathbf{x}_S \in S} d(\mathbf{x}_R, \mathbf{x}_S)$$



(a) MIN (single link.) (b) MAX (complete link.) (c) Group average.

Flat vs Hierarchical Clustering

- Flat clustering produces a single partitioning
- Hierarchical Clustering can give different partitionings depending on the level-of-resolution we are looking at
- Flat clustering needs the number of clusters to be specified
- Hierarchical clustering doesn't need the number of clusters to be specified
- Flat clustering is usually more efficient run-time wise
- Hierarchical clustering can be slow (has to make several merge/split decisions)
- No clear consensus on which of the two produces better clustering