Agents don’t have complete knowledge about the world.

Agents need to make decisions based on their uncertainty.

It isn’t enough to assume what the world is like.

Example: wearing a seat belt.

An agent needs to reason about its uncertainty.

When an agent makes an action under uncertainty, it is gambling \( \Rightarrow \) probability.
Probability

- Probability is an agent’s measure of belief in some proposition — subjective probability.
- Example: Your probability of a bird flying is your measure of belief in the flying ability of an individual based only on the knowledge that the individual is a bird.
  - Other agents may have different probabilities, as they may have had different experiences with birds or different knowledge about this particular bird.
  - An agent’s belief in a bird’s flying ability is affected by what the agent knows about that bird.
Belief in proposition, \( f \), can be measured in terms of a number between 0 and 1 — this is the probability of \( f \).

- The probability \( f \) is 0 means that \( f \) is believed to be definitely false.
- The probability \( f \) is 1 means that \( f \) is believed to be definitely true.

Using 0 and 1 is purely a convention.

\( f \) has a probability between 0 and 1, doesn’t mean \( f \) is true to some degree, but means you are ignorant of its truth value. Probability is a measure of your ignorance.
A random variable is a term in a language that can take one of a number of different values.

The domain of a variable $X$, written $\text{dom}(X)$, is the set of values $X$ can take.

A tuple of random variables $\langle X_1, \ldots, X_n \rangle$ is a complex random variable with domain $\text{dom}(X_1) \times \cdots \times \text{dom}(X_n)$. Often the tuple is written as $X_1, \ldots, X_n$.

Assignment $X = x$ means variable $X$ has value $x$.

A proposition is a Boolean formula made from assignments of values to variables.
A **possible world** specifies an assignment of one value to each random variable.

\( \omega \models X = x \)
means variable \( X \) is assigned value \( x \) in world \( \omega \).

Logical connectives have their standard meaning:

\[
\begin{align*}
\omega \models \alpha \land \beta & \text{ if } \omega \models \alpha \text{ and } \omega \models \beta \\
\omega \models \alpha \lor \beta & \text{ if } \omega \models \alpha \text{ or } \omega \models \beta \\
\omega \models \neg \alpha & \text{ if } \omega \not\models \alpha
\end{align*}
\]

Let \( \Omega \) be the set of all possible worlds.
Semantics of Probability: finite case

For a finite number of possible worlds:

- Define a nonnegative measure $\mu(\omega)$ to each world $\omega$ so that the measures of the possible worlds sum to 1. The measure specifies how much you think the world $\omega$ is like the real world.

- The probability of proposition $f$ is defined by:

$$P(f) = \sum_{\omega \models f} \mu(\omega).$$
Axioms of Probability: finite case

Three axioms define what follows from a set of probabilities:

**Axiom 1** \(0 \leq P(f)\) for any formula \(f\).

**Axiom 2** \(P(\tau) = 1\) if \(\tau\) is a tautology.

**Axiom 3** \(P(f \lor g) = P(f) + P(g)\) if \(\neg(f \land g)\) is a tautology.

- These axioms are sound and complete with respect to the semantics.
In the general case, probability defines a measure on sets of possible worlds. We define $\mu(S)$ for some sets $S \subseteq \Omega$ satisfying:

- $\mu(S) \geq 0$
- $\mu(\Omega) = 1$
- $\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$ if $S_1 \cap S_2 = \emptyset$.

Or sometimes $\sigma$-additivity:

$$
\mu\left(\bigcup_{i} S_i\right) = \sum_{i} \mu(S_i) \text{ if } S_i \cap S_j = \emptyset \text{ for } i \neq j
$$

Then $P(\alpha) = \mu(\{\omega|\omega \models \alpha\})$. 
A probability distribution on a random variable $X$ is a function $\text{dom}(X) \rightarrow [0, 1]$ such that

$$x \mapsto P(X = x).$$

This is written as $P(X)$.

This also includes the case where we have tuples of variables. E.g., $P(X, Y, Z)$ means $P(\langle X, Y, Z \rangle)$.

When $\text{dom}(X)$ is infinite sometimes we need a probability density function...
Conditioning

- Probabilistic conditioning specifies how to revise beliefs based on new information.
- You build a probabilistic model taking all background information into account. This gives the **prior probability**.
- All other information must be conditioned on.
- If **evidence** $e$ is the all of the information obtained subsequently, the **conditional probability** $P(h|e)$ of $h$ given $e$ is the **posterior probability** of $h$. 
Evidence $e$ rules out possible worlds incompatible with $e$. Evidence $e$ induces a new measure, $\mu_e$, over possible worlds

$$\mu_e(S) = \begin{cases} 
  c \times \mu(S) & \text{if } \omega \models e \text{ for all } \omega \in S \\
  0 & \text{if } \omega \not\models e \text{ for all } \omega \in S
\end{cases}$$

We can show $c = \frac{1}{P(e)}$.

The conditional probability of formula $h$ given evidence $e$ is

$$P(h|e) = \mu_e(\{\omega : \omega \models h\}) = \frac{P(h \land e)}{P(e)}$$
Chain Rule

\[ P(f_1 \land f_2 \land \ldots \land f_n) \]
\[ = \quad P(f_n|f_1 \land \cdots \land f_{n-1}) \times \]
\[ P(f_1 \land \cdots \land f_{n-1}) \]
\[ = \quad P(f_n|f_1 \land \cdots \land f_{n-1}) \times \]
\[ P(f_{n-1}|f_1 \land \cdots \land f_{n-2}) \times \]
\[ P(f_1 \land \cdots \land f_{n-2}) \]
\[ = \quad P(f_n|f_1 \land \cdots \land f_{n-1}) \times \]
\[ P(f_{n-1}|f_1 \land \cdots \land f_{n-2}) \]
\[ \times \cdots \times P(f_3|f_1 \land f_2) \times P(f_2|f_1) \times P(f_1) \]
\[ = \quad \prod_{i=1}^{n} P(f_i|f_1 \land \cdots \land f_{i-1}) \]
The chain rule and commutativity of conjunction ($h \land e$ is equivalent to $e \land h$) gives us:

$$P(h \land e) = P(h|e) \times P(e) = P(e|h) \times P(h).$$

If $P(e) \neq 0$, you can divide the right hand sides by $P(e)$:

$$P(h|e) = \frac{P(e|h) \times P(h)}{P(e)}.$$

This is **Bayes’ theorem.**
Why is Bayes’ theorem interesting?

- Often you have causal knowledge:
  \[ P(\text{symptom} \mid \text{disease}) \]
  \[ P(\text{light is off} \mid \text{status of switches and switch positions}) \]
  \[ P(\text{alarm} \mid \text{fire}) \]
  \[ P(\text{image looks like } \mathcal{T} \mid \text{a tree is in front of a car}) \]

- and want to do evidential reasoning:
  \[ P(\text{disease} \mid \text{symptom}) \]
  \[ P(\text{status of switches} \mid \text{light is off and switch positions}) \]
  \[ P(\text{fire} \mid \text{alarm}) \]
  \[ P(\text{a tree is in front of a car} \mid \text{image looks like } \mathcal{T}) \]