Chapter 7

7.1 Let T be the outcome of a roll with a fair die.

a)  
Outcomes | 1 | 2 | 3 | 4 | 5 | 6 |
---------|---|---|---|---|---|---|
Probability Mass | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 |
Cumulative Dist. | 1/6 | 2/6 | 3/6 | 4/6 | 5/6 | 1 |

b)  
E[T] = 1 × 1/6 + 2 × 1/6 + 3 × 1/6 + 4 × 1/6 + 5 × 1/6 + 6 × 1/6 = 3.5
Var(T) = 1/6 × (1 – 3.5)^2 + 1/6 × (2 – 3.5)^2 + 1/6 × (3 – 3.5)^2 + 1/6 × (4 – 3.5)^2 + 1/6 × (5 – 3.5)^2 + 1/6 × (6 – 3.5)^2
= 1/6(6.25 + 2.25 + .25 + .25 + 2.25 + 6.25) = 2.917

7.2 X is a random variable with the following distribution

P(X=-1) = 1/5
P(X=0) = 2/5
P(X=1 = 2/5

a)  
E[X] = -1 × 1/5 + 0 × 2/5 + 1 × 2/5 = 1/5

b) Y = X^2

P(Y = 0) = 2/5
P(Y = 1) = 3/5
E[Y] = 0 × 2/5 + 1 × 3/5 = 3/5

c)  
The change of variable formula for expectation is
(-1)^2 × 1/5 + 0^2 × 2/5 + 1^2 × 2/5 = 1/5 + 0 + 2/5 = 3/5 which agrees with b)

d)  
Var(X) = 1/5 × (-1 – 1/5)^2 + 2/5 × (0 – 1/5)^2 + 2/5 × (1 – 1/5)^2
= 1/5 × 1.44 + 2/5 × 0.04 + 2/5 × .64 = 0.552

7.9 U is a random variable with distribution U(a, b)

a) The probability density function F(u) = 1/(b – a) if a ≤ u ≤ b, 0 otherwise.
The anti-derivative of u × F(u) is
u^2 / (2 × (b-a))
So the definite integral is (b^2 – a^2) / (2 × (b-a)) = (b – a)(b + a)/2(b – a)
= (a + b)/2.
b) To find the $\text{Var}(U)$ we integrate $\frac{1}{(b - 1)} \times (u - \frac{a + b}{2})^2$ from $a$ to $b$. This gives $\frac{(a + b)^2}{12}$.

7.13
For any random variable $X$,
\[ 0 \leq \text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \]
Re-arranging terms gives
\[ \mathbb{E}[X]^2 \geq (\mathbb{E}[X])^2. \]

7.14
We choose an arbitrary point from a square with vertices at $(2,1)$, $(3,1)$, $(2,2)$, and $(3,2)$. Let $A$ be the random variable which is the area of the triangle formed by the chosen point and the points $(2,1)$ and $(3,1)$.

This area will be given by the half the product of the $(y-1)$ and the distance from $(2,1)$ to $(3,1)$ (which is of course 1), where $y$ is the $y$-coordinate of the chosen point. Thus $A = \frac{(Y-1)}{2}$. $Y$ is uniformly distributed between 1 and 2. Its expected value is 1.5

Thus $\mathbb{E}[A] = \mathbb{E}[Y] / 2 - \frac{1}{2} = 1.5 / 2 - .5 = 0.25$.