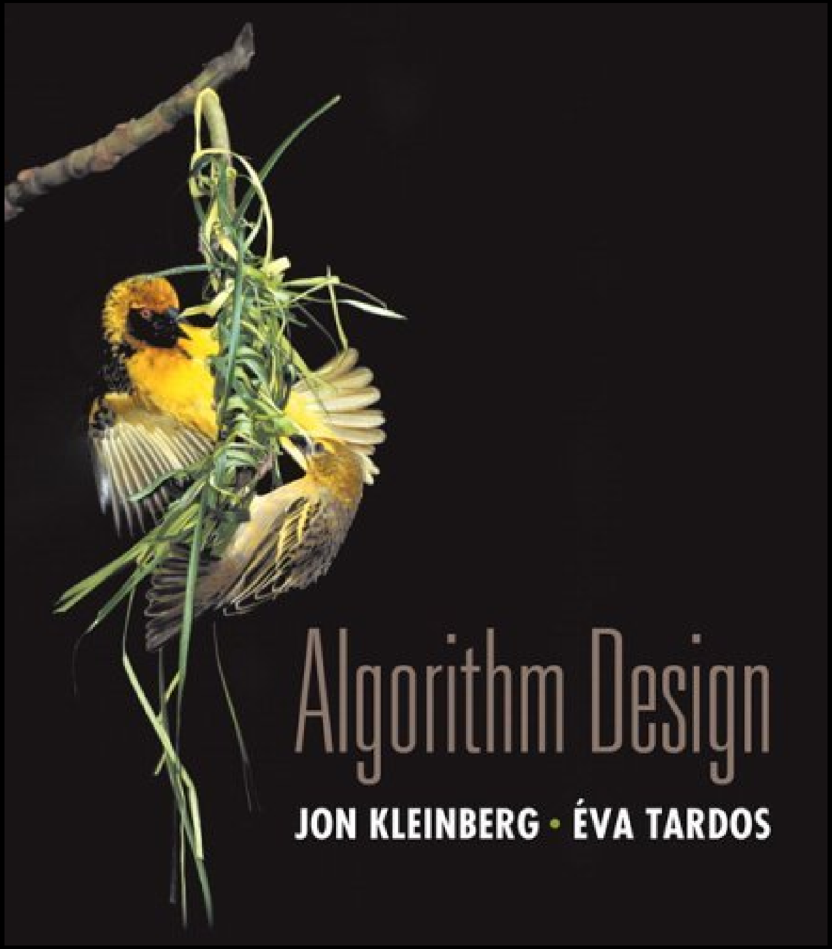


Chapter 4

Greedy Algorithms

Greedy algorithms

Greedy approaches

Seek to maximize the overall utility of some process by making the immediately optimal choice at each sub-stage of the process.

Greedy solutions

May solve some problems optimally, but not for many others.

Greedy Analysis Strategies

Greedy algorithm stays ahead (e.g. Interval Scheduling).

Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

Structural (e.g. Interval Partition).

Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

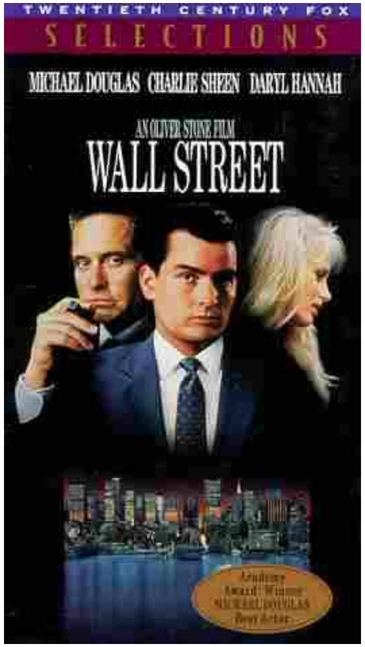
Exchange argument (e.g. Scheduling to Minimize Lateness).

Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

Other greedy algorithms. Kruskal, Prim, Dijkstra*, Huffman, …

4.1 Interval Scheduling

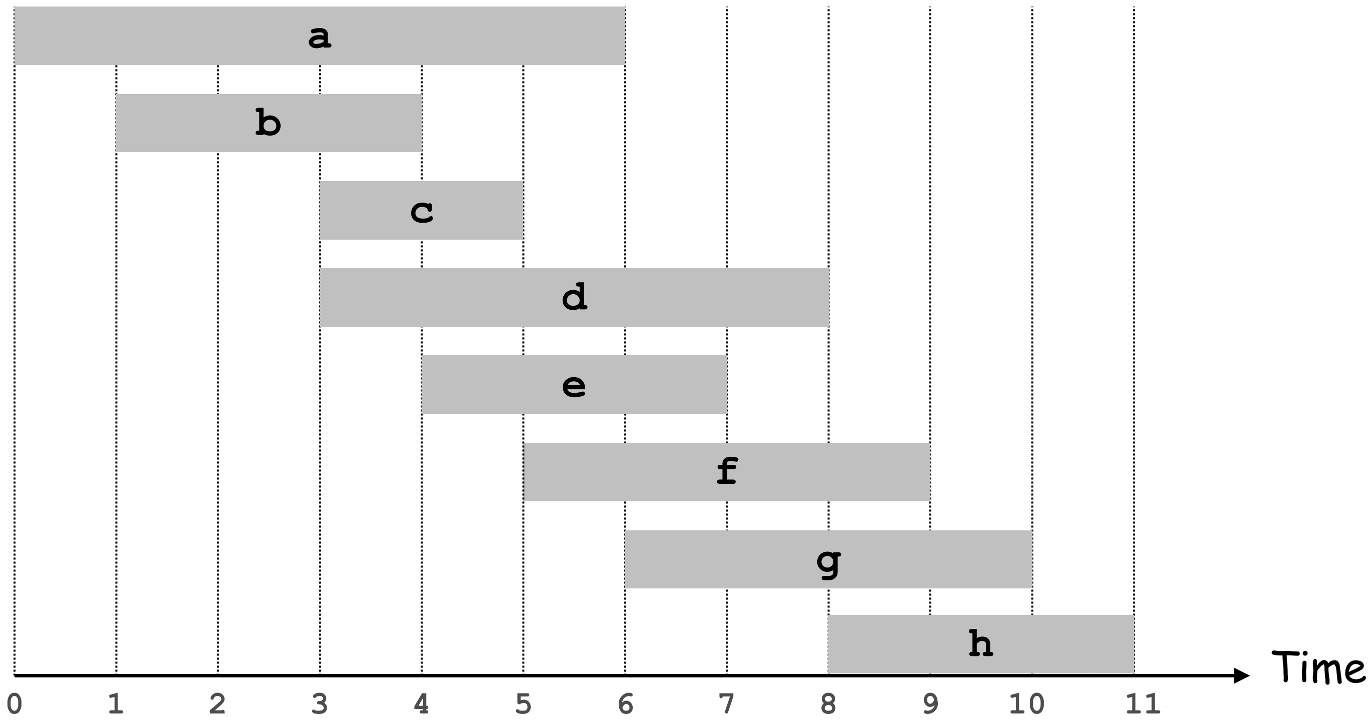
> Greed is good. Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit.
>
> *- Gordon Gecko (Michael Douglas)*


TWENTIETH CENTURY FOX
SELECTIONS
MICHAEL DOUGLAS CHARLIE SHEEN DARYL HANNAH
AN OLIVER STONE FILM
WALL STREET
Academy Award Winner MICHAEL DOUGLAS Best Actor

Interval Scheduling

Interval scheduling.

- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some natural order.
Take each job provided it's compatible with the ones already taken.

[Earliest start time] Consider jobs in ascending order of s_j.

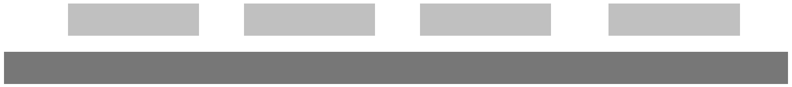
[Earliest finish time] Consider jobs in ascending order of f_j.

[Shortest interval] Consider jobs in ascending order of $f_j - s_j$.

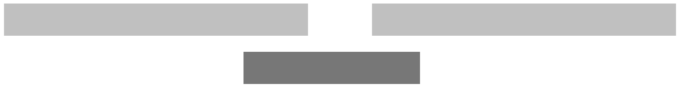
[Fewest conflicts] For each job j, count the number of conflicting jobs c_j. Schedule in ascending order of c_j.

Interval Scheduling: Greedy Algorithms

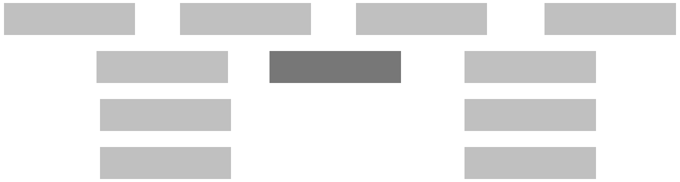
Greedy template. Consider jobs in some natural order.
Take each job provided it's compatible with the ones already taken.



counterexample for earliest start time



counterexample for shortest interval



counterexample for fewest conflicts

Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that $f_1 \le f_2 \le ... \le f_n$.

  set of jobs selected
$A \leftarrow \phi$
for j = 1 to n {
   if (job j compatible with A)
      $A \leftarrow A \cup \{j\}$
}
return A
```

Implementation. $O(n \log n)$.

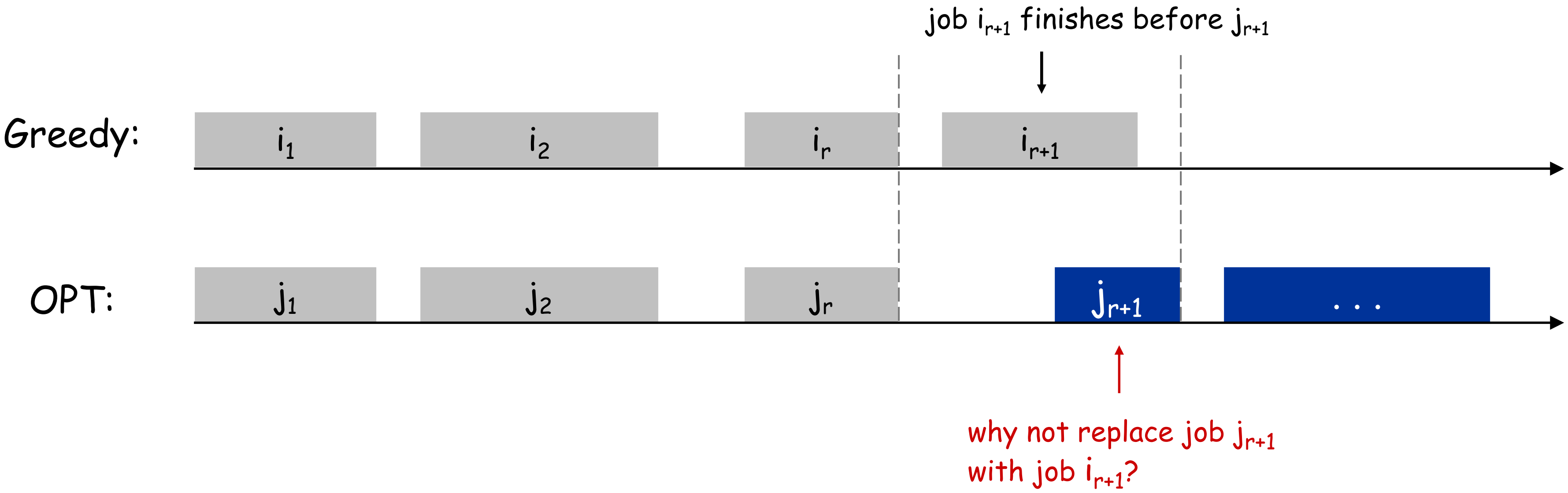
- Remember job j^* that was added last to A.
- Job j is compatible with A if $s_j \ge f_{j^*}$.

Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

Assume greedy is not optimal, and let's see what happens.
Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
Let $j_1, j_2, \ldots j_m$ denote set of jobs in the optimal solution with
$i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



why not replace job j_{r+1}
with job i_{r+1}?

Interval Scheduling: Analysis

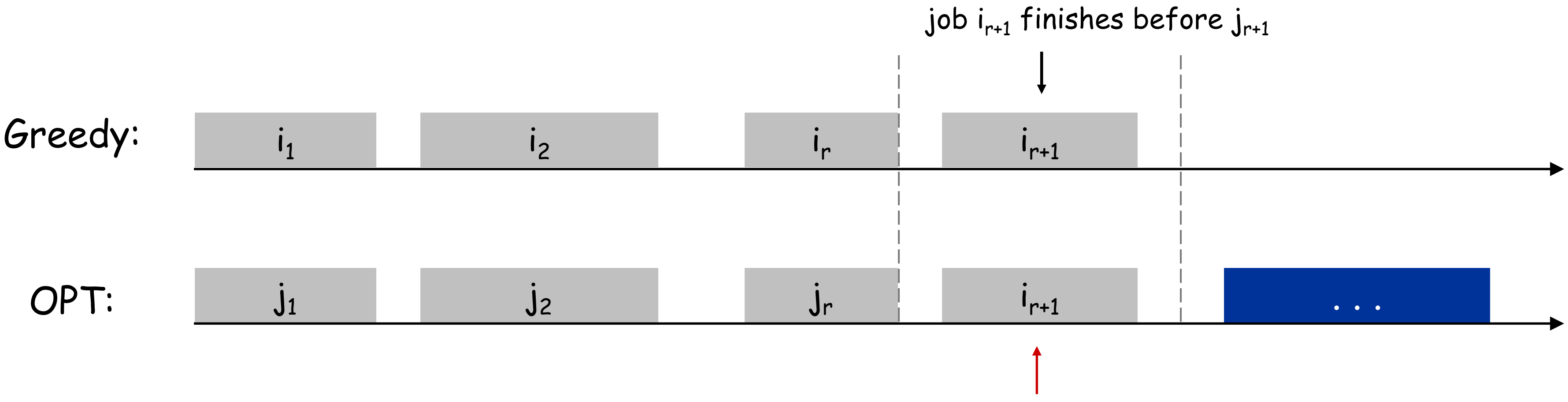
Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

Assume greedy is not optimal, and let's see what happens.

Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.

Let $j_1, j_2, \ldots j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



solution still feasible and optimal, but contradicts maximality of r.

Interval Scheduling: Extensions

Online: must make decisions as time proceeds, without knowledge of future inputs.

Weighted Interval Scheduling Problems: Each request has a different value. Dynamic programming solution.

4.1 Interval Partitioning

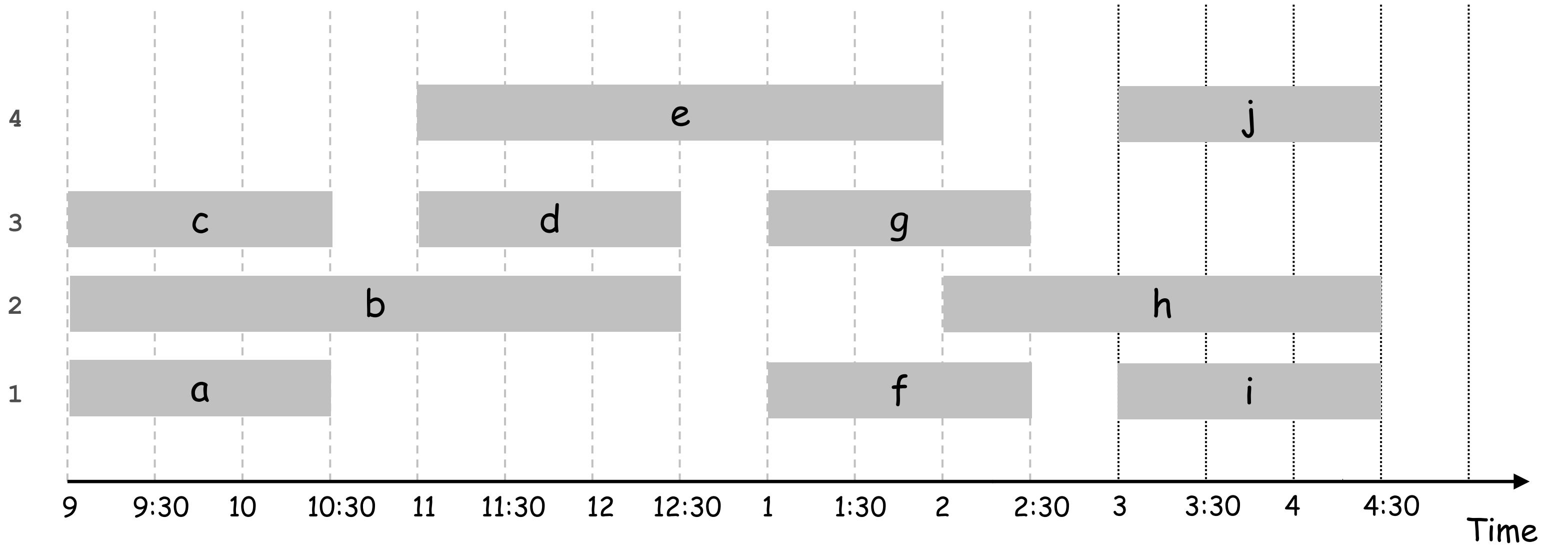
Interval Partitioning

Interval partitioning.

Lecture j starts at s_j and finishes at f_j.

Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.



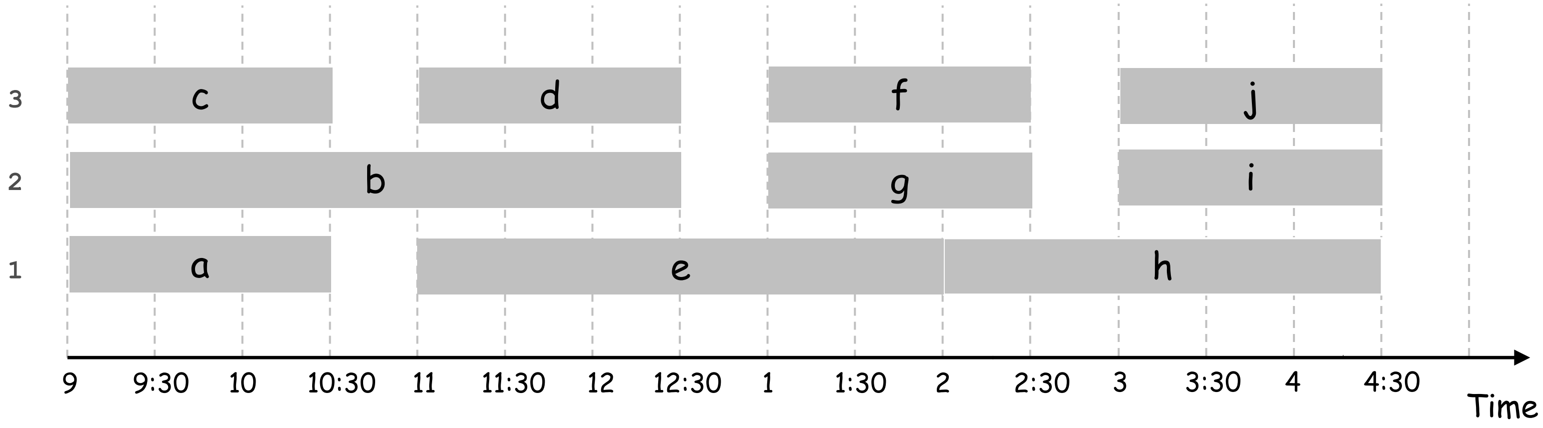
Interval Partitioning

Interval partitioning.

Lecture j starts at s_j and finishes at f_j.

Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.



Interval Partitioning: Lower Bound on Optimal Solution

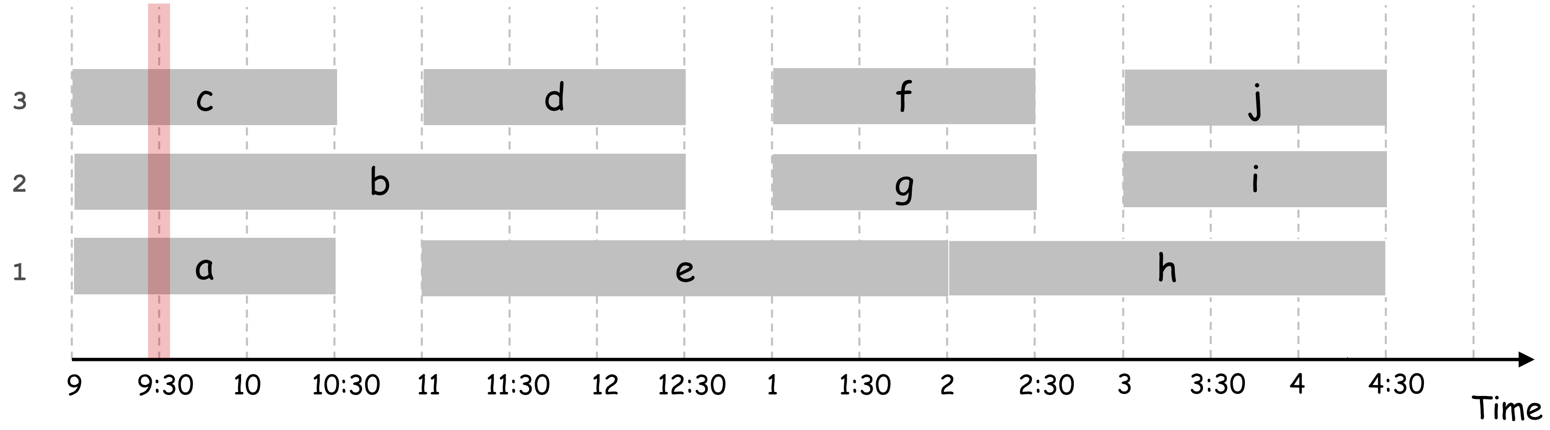
Def. The depth of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed ≥ depth.

Ex: Depth of schedule below = 3 ⇒ schedule below is optimal.

a, b, c all contain 9:30

Q. Does there always exist a schedule equal to depth of intervals?



Interval Partitioning: Greedy Algorithm

Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that $s_1 \le s_2 \le \ldots \le s_n$.
d ← 0 ←  number of allocated classrooms

for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation. O(n log n).

- For each classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

Interval Partitioning: Greedy Analysis

Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.

Pf.

- Let d = number of classrooms that the greedy algorithm allocates.
- Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms.
- These d jobs each end after s_j.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- Key observation ⇒ all schedules use ≥ d classrooms. ▪

4.2 Scheduling to Minimize Lateness

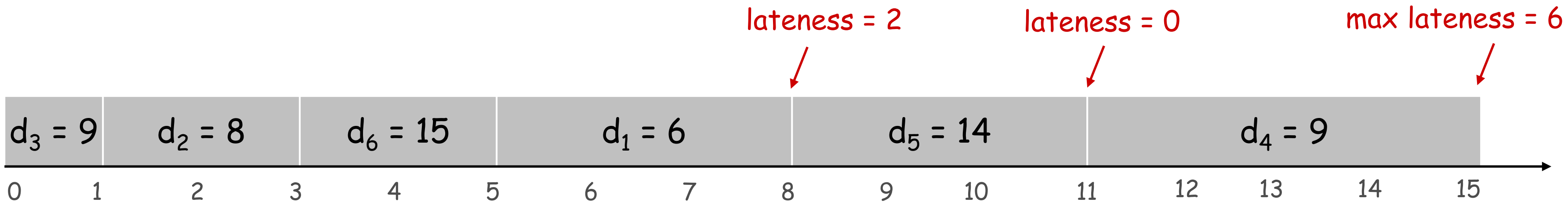
Scheduling to Minimizing Lateness

Minimizing lateness problem.

- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j.
- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max \{ 0, f_j - d_j \}$.
- Goal: schedule all jobs to minimize maximum lateness $L = \max \ell_j$.

Ex:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

[Shortest processing time first] Consider jobs in ascending order of processing time t_j.

[Earliest deadline first] Consider jobs in ascending order of deadline d_j.

[Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

[Shortest processing time first] Consider jobs in ascending order of processing time t_j.

	1	2
t_j	1	10
d_j	100	10

counterexample

[Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

	1	2
t_j	1	10
d_j	2	10

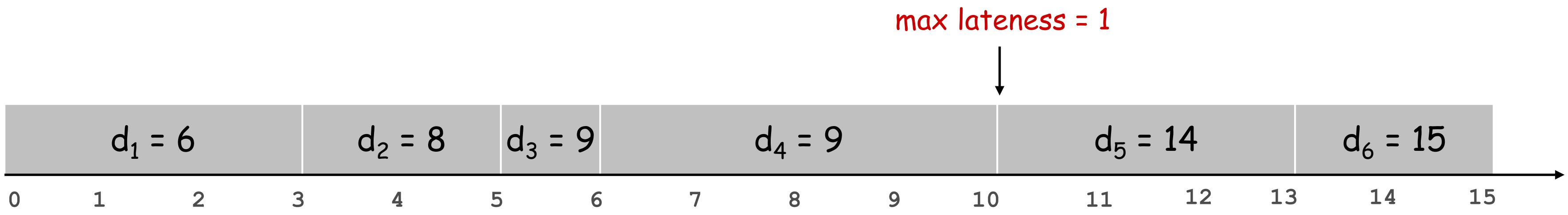
counterexample

Minimizing Lateness: Greedy Algorithm

Greedy algorithm. Earliest deadline first.

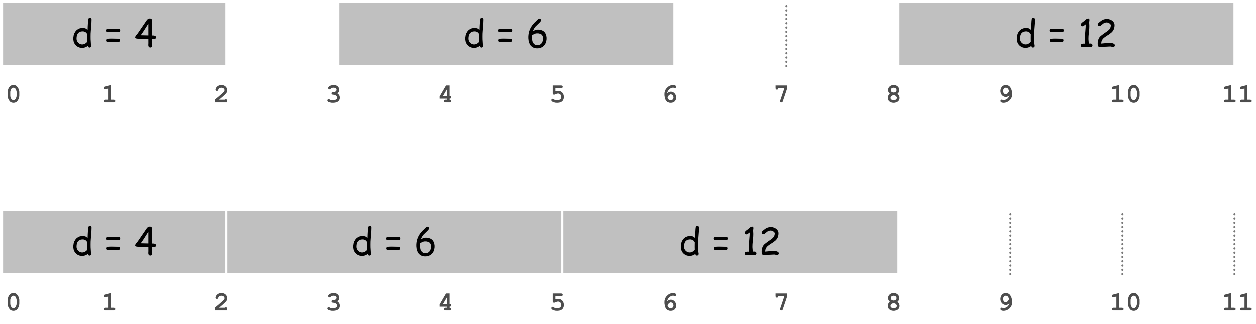
```
Sort n jobs by deadline so that $d_1 \le d_2 \le … \le d_n$

t ← 0
for j = 1 to n
   Assign job j to interval [t, t + $t_j$]
   $s_j$ ← t, $f_j$ ← t + $t_j$
   t ← t + $t_j$
output intervals [$s_j$, $f_j$]
```



Minimizing Lateness: No Idle Time

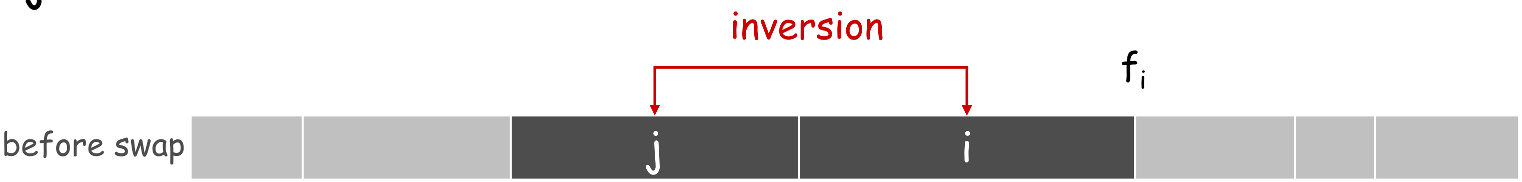
Observation. There exists an optimal schedule with no idle time.



Observation. The greedy schedule has no idle time.

Minimizing Lateness: Inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that: i < j but j scheduled before i.



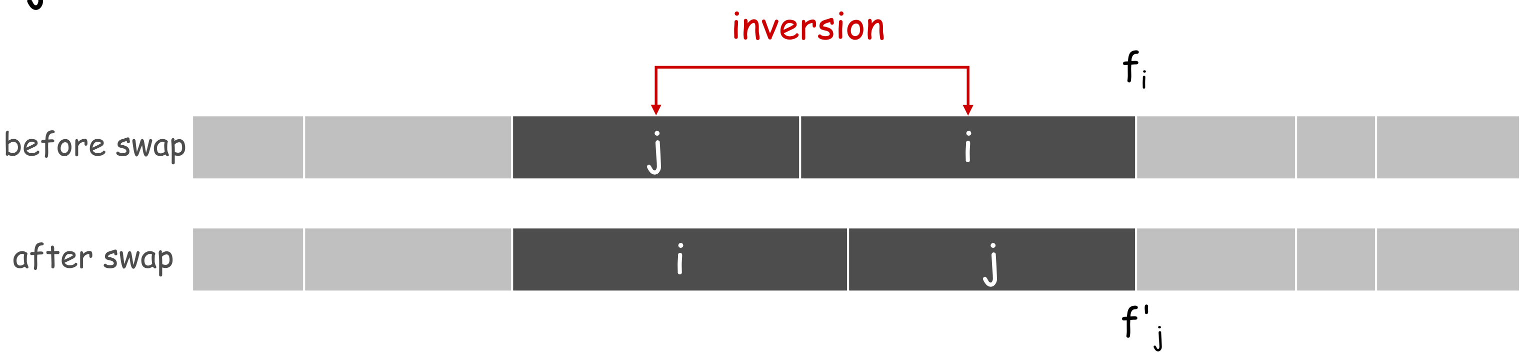
[as before, we assume jobs are numbered so that $d_1 \leq d_2 \leq \ldots \leq d_n$]

Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

Minimizing Lateness: Inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that: i < j but j scheduled before i.



Claim. Swapping two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards.

$\ell'_k = \ell_k$ for all $k \neq i, j$

$\ell'_i \leq \ell_i$

If job j is late:

$$\begin{aligned} \ell'_j &= f'_j - d_j && \text{(definition)} \\ &= f_i - d_j && (j \text{ finishes at time } f_i) \\ &\leq f_i - d_i && (i < j) \\ &\leq \ell_i && \text{(definition)} \end{aligned}$$

Minimizing Lateness: Analysis of Greedy Algorithm

Theorem. Greedy schedule S is optimal.

Pf. Define S* to be an optimal schedule that has the fewest number of inversions, and let's see what happens.

- Can assume S* has no idle time.
- If S* has no inversions, then S = S*.
- If S* has an inversion, let i-j be an adjacent inversion.
  - swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
  - this contradicts definition of S* ▪

Minimizing Lateness : Extension

Each job has a different starting time, instead of one common starting time.

The earliest starting time is called release time.

Interval partition with different release time is hard (NP-hard).

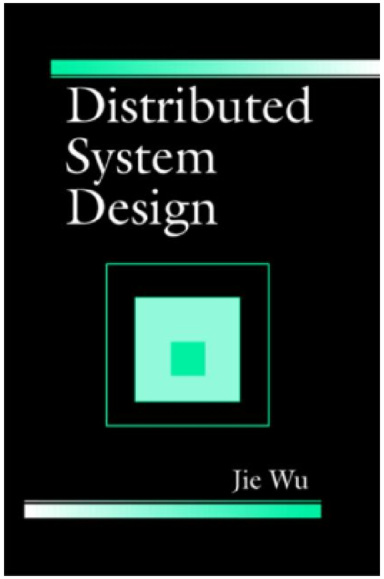
General Job Scheduling

Job with precedence order: task precedence graphs

Jobs with communication: task communication graphs

Pipeline models: There are m jobs and n machines, with each job runs on these machines following a certain order

Flow shop: in a common order by all jobs
Job shop: in a particular order by its own
Open shop: in an arbitrary order



Manufactory assemble lines

Offloading ML code in edge computing

General Design and Proof Strategies

Proof by Contradiction

Proof by Induction

Design and Proof by Mapping to a New Problem (e.g., Ant Lifetime)

Proof by Accounting Method (e.g., Group Size)

Design and Proof by a More General Problem (e.g., Shortest Paths)

Easy Solution but Difficult Proof (e.g., Optimal Caching)

Difficult Problem but Easy Solution and Proof (e.g., Shortest Paths)

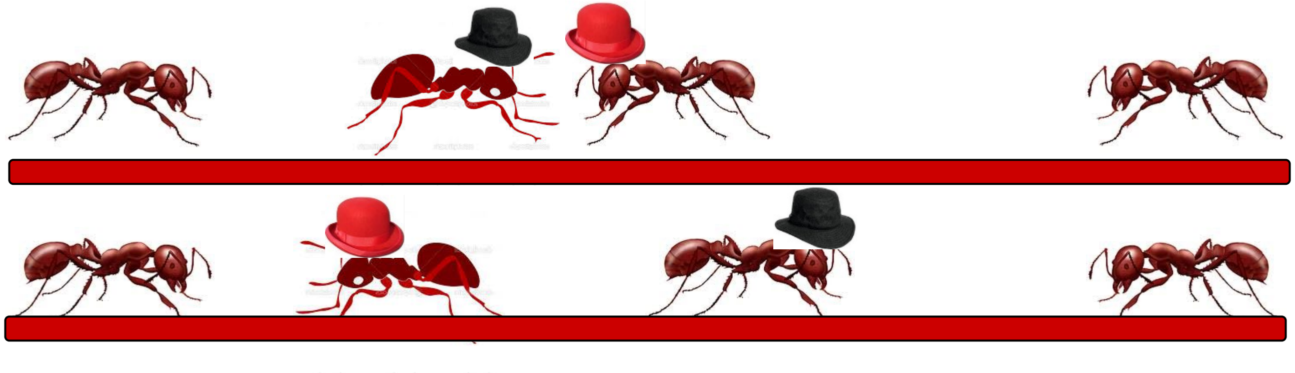
Ant Lifetime

Comm. of ACM, March 2013

Ants always march at 1 cm/sec in whichever direction they are facing, and reverse directions when they collide

Ant X stays in the middle of 25 ants on a 1 meter-long stick

How long must we wait before X has fallen off the stick?



Solution: Introduce a new variable: a hat on each ant

Exchange hats when two ants collide

New problem: Lifetime of each hat (1-to-1 bijection between hat and ant)

Group Size

A group of students (n>1) is partitioned into k groups in 2020 and then is re-partitioned into k+1 groups in 2021. (Each group has at least one student.)

Proof that there exist at least 2 students. Their 2021 group size is smaller than 2020 group size.

E.g. 2020: (1, 3), (4, 5, 6, 10), (2, 7, 8, 9). 2021: (1, 2), (7, 8), (4, 6, 9), (3, 5, 10)

Proof by the accounting method and by contradiction

- Assign $1 to each group, which is then equally divided among group members.
- Check each student's payment difference between 2021 and 2020.
- The total payment difference between 2021 and 2020 should be $1

4.3 Optimal Caching

Optimal Offline Caching

Caching.

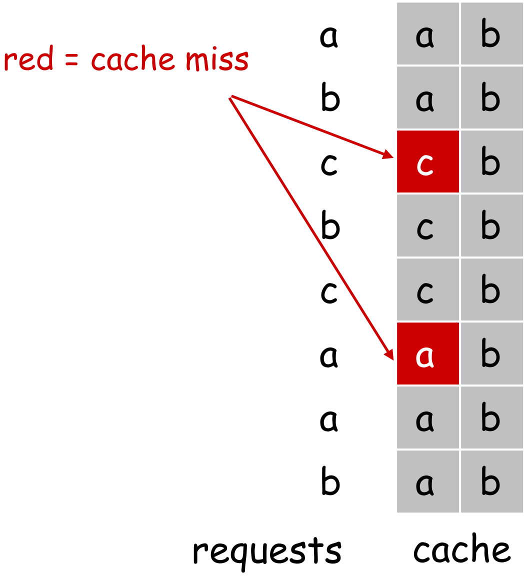
- Cache with capacity to store k items.
- Sequence of m item requests $d_1, d_2, \ldots, d_m$.
- Cache hit: item already in cache when requested.
- Cache miss: item not already in cache when requested: must bring requested item into cache, and evict some existing item, if full.

Goal. Eviction schedule that minimizes number of cache misses.

Ex: k = 2, initial cache = ab,

requests: a, b, c, b, c, a, a, b.

Optimal eviction schedule: 2 cache misses.

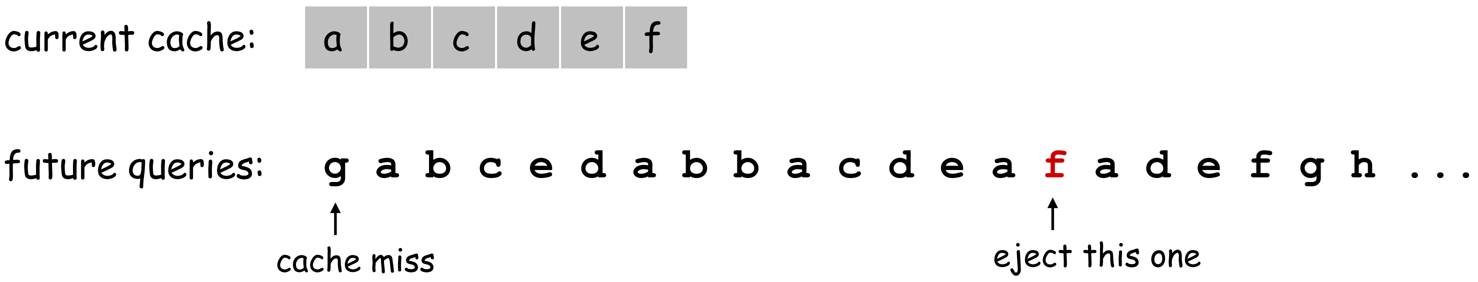


red = cache miss

requests	cache	
a	a	b
b	a	b
c	c	b
b	c	b
c	c	b
a	a	b
a	a	b
b	a	b

Optimal Offline Caching: Farthest-In-Future

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.



current cache: a b c d e f

future queries: g a b c e d a b b a c d e a f a d e f g h . . .

cache miss

eject this one

Theorem. [Bellady, 1960s] FF is optimal eviction schedule.

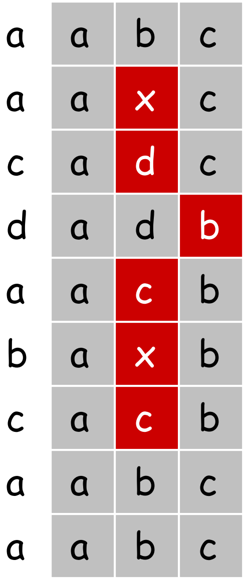
Pf. Algorithm and theorem are intuitive

Proof is subtle: exchange argument by swapping decisions

Reduced Eviction Schedules

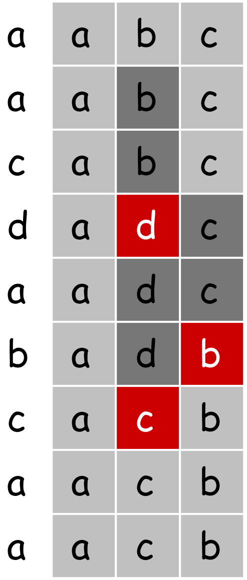
Def. A reduced schedule is a schedule that only inserts an item into the cache in a step in which that item is requested.

Intuition. Can transform an unreduced schedule into a reduced one with no more cache misses.



a	a	b	c
a	a	x	c
c	a	d	c
d	a	d	b
a	a	c	b
b	a	x	b
c	a	c	b
a	a	b	c
a	a	b	c

an unreduced schedule



a	a	b	c
a	a	b	c
c	a	b	c
d	a	d	c
a	a	d	c
b	a	d	b
c	a	c	b
a	a	c	b
a	a	c	b

a reduced schedule

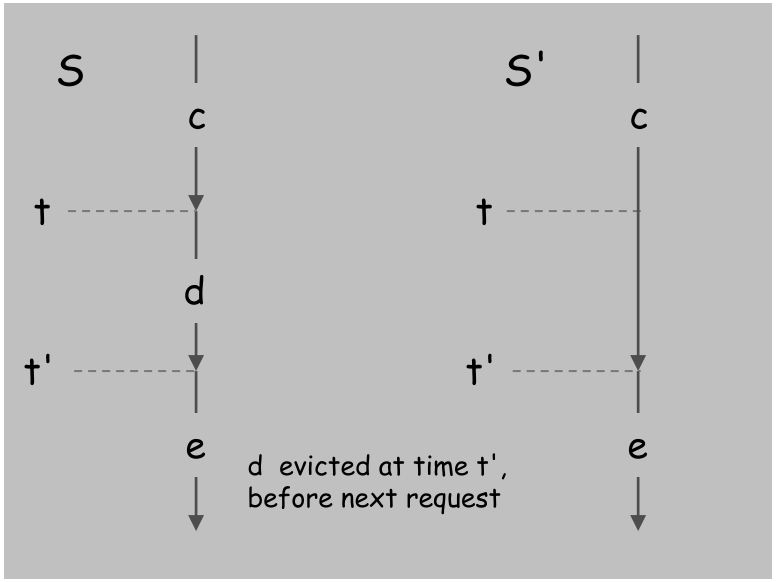
Reduced Eviction Schedules

Claim. Given any unreduced schedule S, can transform it into a reduced schedule S' with no more cache misses.

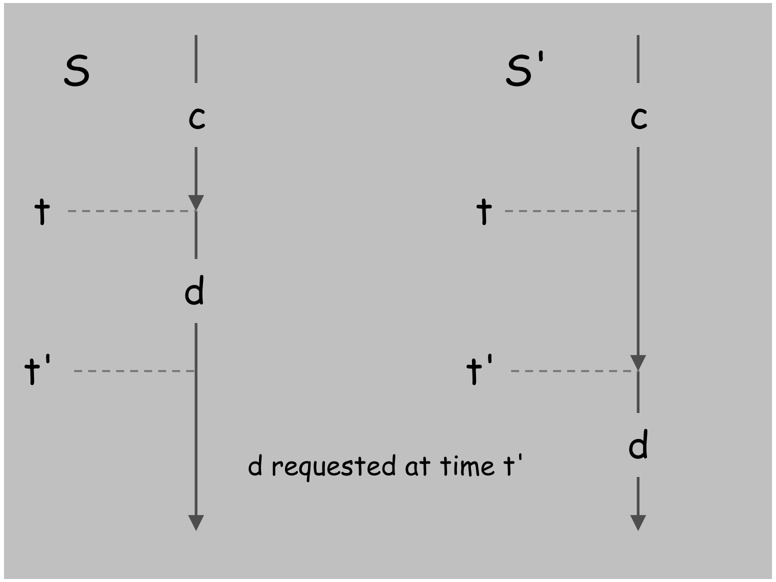
Pf. (by induction on number of unreduced items)

doesn't enter cache at requested time

- Suppose S brings d into the cache at time t, without a request.
- Let c be the item S evicts when it brings d into the cache.
- Case 1: d evicted at time t', before next request for d.
- Case 2: d requested at time t' before d is evicted. ▪



Case 1



Case 2

Farthest-In-Future: Analysis

Theorem. FF is optimal eviction algorithm.

Pf. (by induction on number or requests j)

> Invariant: There exists an optimal reduced schedule S that makes the same eviction schedule as S_{FF} through the first j+1 requests.

Let S be reduced schedule that satisfies invariant through j requests.

We produce S' that satisfies invariant after j+1 requests.

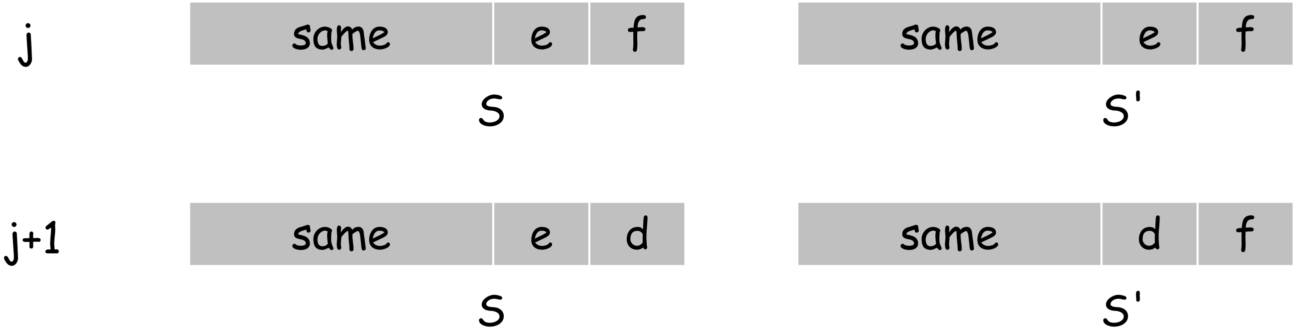
- Consider (j+1)st request $d = d_{j+1}$.
- Since S and S_{FF} have agreed up until now, they have the same cache contents before request j+1.
- Case 1: (d is already in the cache). S' = S satisfies invariant.
- Case 2: (d is not in the cache and S and S_{FF} evict the same element). S' = S satisfies invariant.

Farthest-In-Future: Analysis

Pf. (continued)

Case 3: (d is not in the cache; S_{FF} evicts e; S evicts f $\neq$ e).

- begin construction of S' from S by evicting e instead of f



j

same	e	f

S

same	e	f

S'

j+1

same	e	d

S

same	d	f

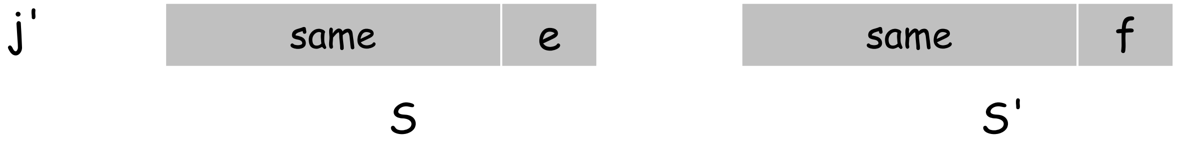
S'

- now S' agrees with S_{FF} on first j+1 requests; we show that having element f in cache is no worse than having element e

Farthest-In-Future: Analysis

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

must involve e or f (or both)



Case 3a: g = e. Can't happen with Farthest-In-Future since there must be a request for f before e.

Case 3b: g = f. Element f can't be in cache of S, so let e' be the element that S evicts.

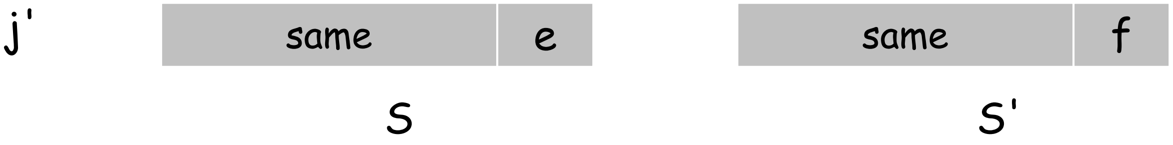
- if e' = e, S' accesses f from cache; now S and S' have same cache
- if e' ≠ e, S' evicts e' and brings e into the cache; now S and S' have the same cache

Note: S' is no longer reduced, but can be transformed into a reduced schedule that agrees with S_{FF} through step j+1

Farthest-In-Future: Analysis

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

must involve e or f (or both)



j'

same | e

S

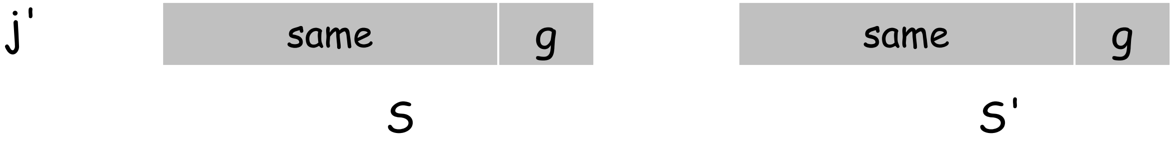
same | f

S'

otherwise S' would take the same action

Case 3c: g ≠ e, f. S must evict e.

Make S' evict f; now S and S' have the same cache. ▪



j'

same | g

S

same | g

S'

Caching Perspective

Online vs. offline algorithms.

Offline: full sequence of requests is known a priori.

Online (reality): requests are not known in advance.

Caching is among most fundamental online problems in CS.

LIFO. Evict page brought in most recently.

LRU. Evict page whose most recent access was earliest.

↑ FF with direction of time reversed!

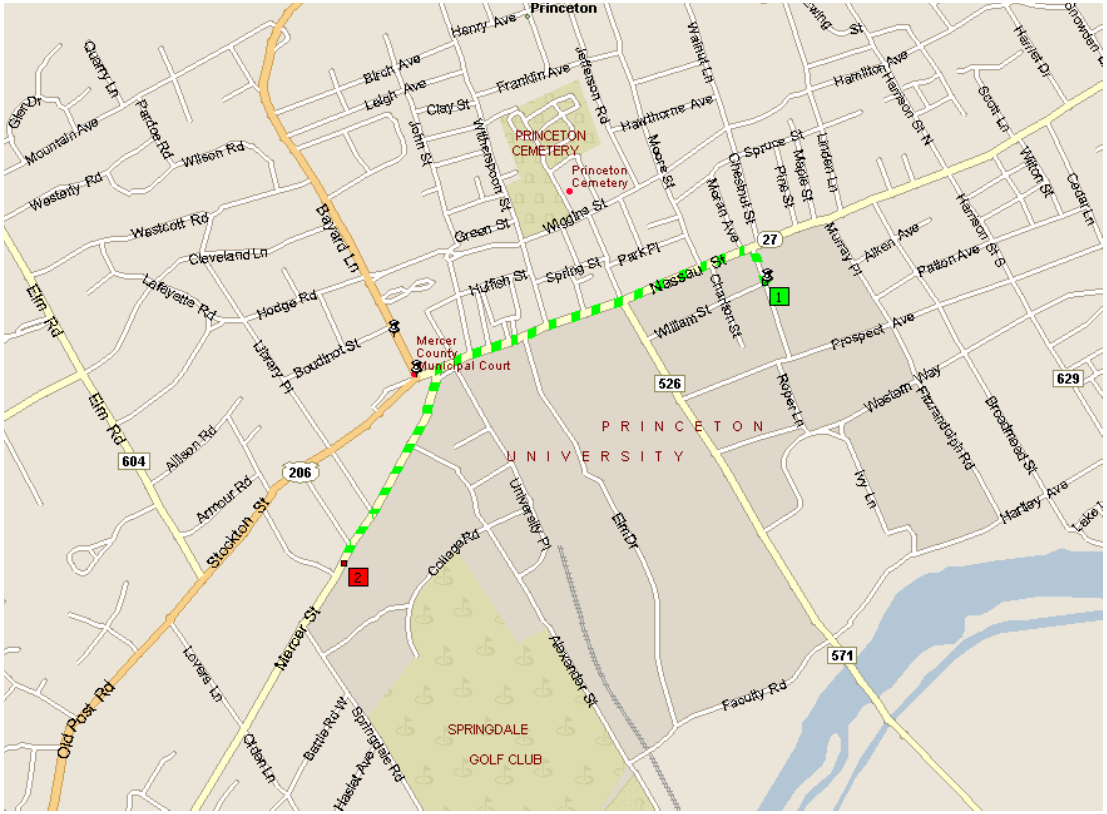
Theorem. FF is optimal offline eviction algorithm.

Provides basis for understanding and analyzing online algorithms.

LRU is k-competitive, with a better version through random caching

LIFO is arbitrarily bad.

4.4 Shortest Paths in a Graph



shortest path from Princeton CS department to Einstein's house

Shortest Path Problem

Shortest path network.

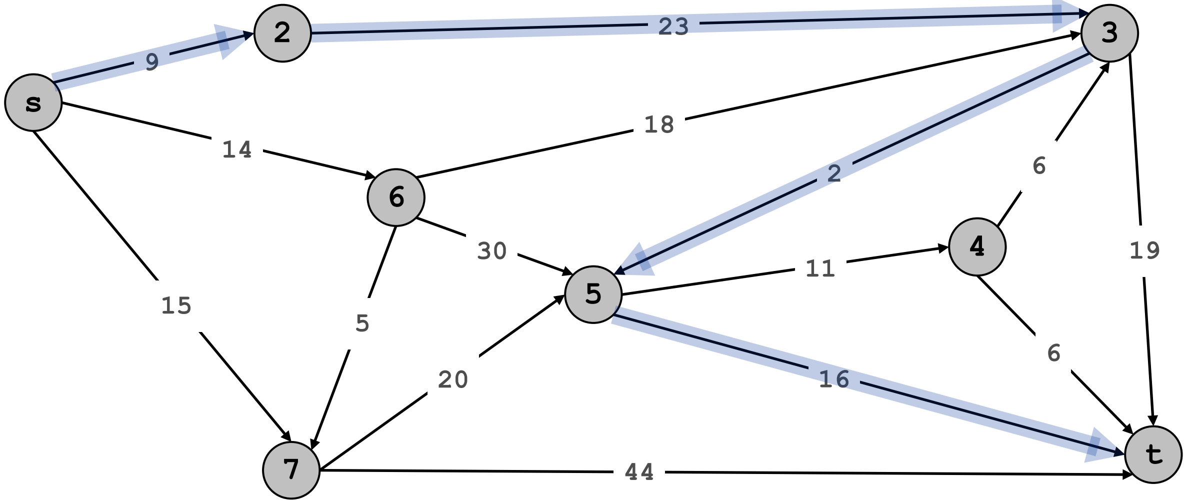
Directed graph G = (V, E).

Source s, destination t.

Length ℓ_e = length of edge e.

Shortest path problem: find shortest directed path from s to t.

cost of path = sum of edge costs in path



Cost of path s-2-3-5-t
= 9 + 23 + 2 + 16
= 50.

Dijkstra's Algorithm

Dijkstra's algorithm.

Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.

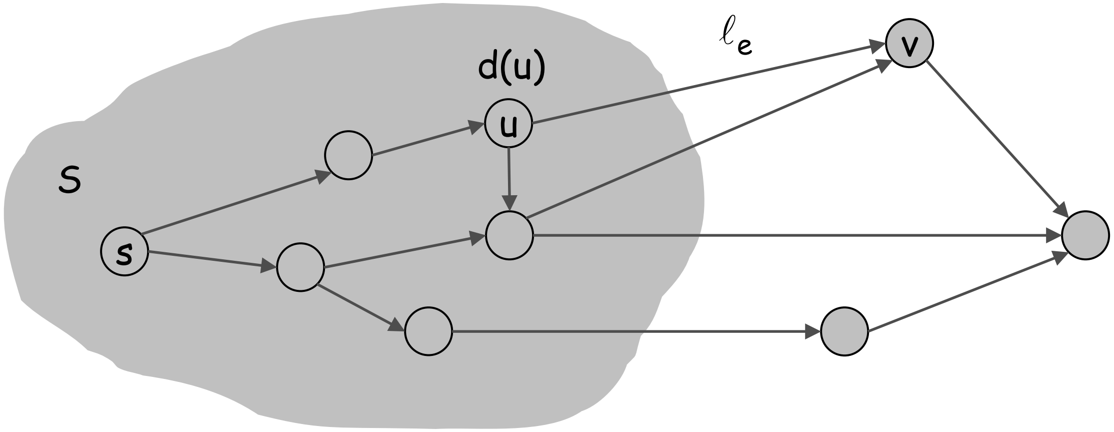
Initialize S = { s }, d(s) = 0.

Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e ,$$

add v to S, and set d(v) = π(v).

shortest path to some u in explored part, followed by a single edge (u, v)



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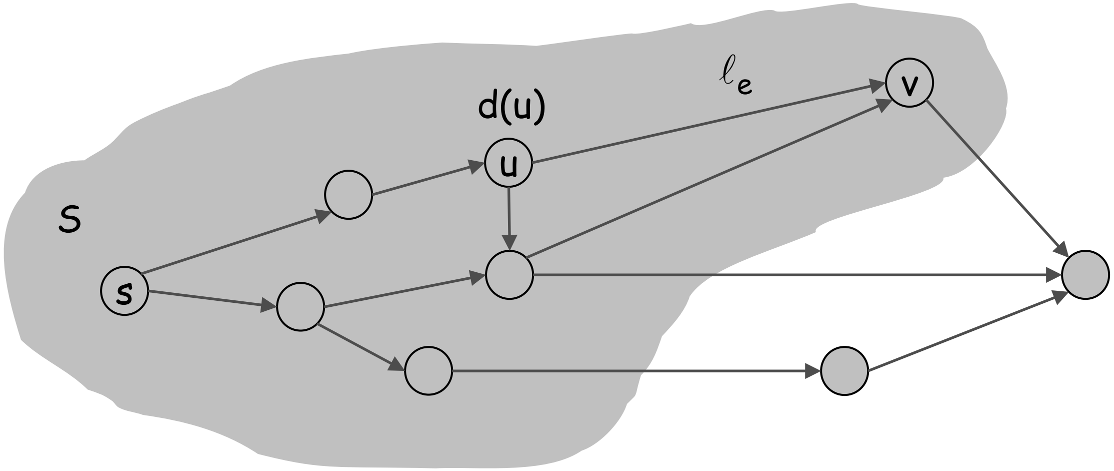


Dijkstra's Algorithm: Proof of Correctness

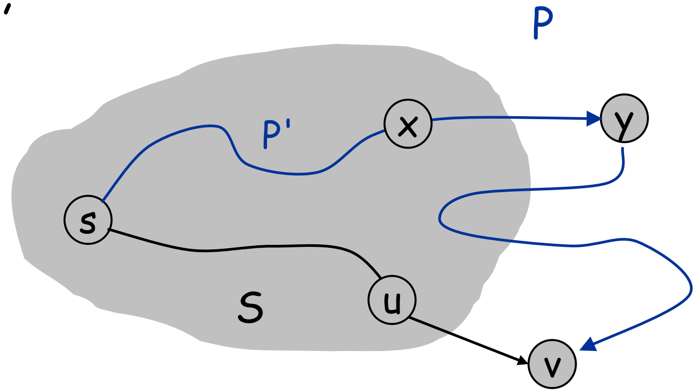
Invariant. For each node u ∈ S, d(u) is the length of the shortest s-u path.

Pf. (by induction on |S|)

Base case: |S| = 1 is trivial.

Inductive hypothesis: Assume true for |S| = k ≥ 1.

- Let v be next node added to S, and let u-v be the chosen edge.
- The shortest s-u path plus (u, v) is an s-v path of length $\pi(v)$.
- Consider any s-v path P. We'll see that it's no shorter than $\pi(v)$.
- Let x-y be the first edge in P that leaves S, and let P' be the subpath to x.
- P is already too long as soon as it leaves S.



$$\ell(P) \underset{\substack{\uparrow \\ \text{nonnegative} \\ \text{weights}}}{\geq} \ell(P') + \ell(x,y) \underset{\substack{\uparrow \\ \text{inductive} \\ \text{hypothesis}}}{\geq} d(x) + \ell(x, y) \underset{\substack{\uparrow \\ \text{defn of } \pi(y)}}{\geq} \pi(y) \underset{\substack{\uparrow \\ \text{Dijkstra chose v} \\ \text{instead of y}}}{\geq} \pi(v)$$

Dijkstra's Algorithm: Implementation

For each unexplored node, explicitly maintain $\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e$.

Next node to explore = node with minimum $\pi(v)$.

When exploring v, for each incident edge e = (v, w), update

$$\pi(w) = \min \{ \; \pi(w), \; \pi(v) + \ell_e \}.$$

Efficient implementation. Maintain a priority queue of unexplored nodes, prioritized by $\pi(v)$.

PQ Operation	Dijkstra	Array	Binary heap	d-way Heap	Fib heap †
Insert	n	n	log n	$d \log_d n$	1
ExtractMin	n	n	log n	$d \log_d n$	log n
ChangeKey	m	1	log n	$\log_d n$	1
IsEmpty	n	1	1	1	1
Total		n^2	m log n	$m \log_{m/n} n$	m + n log n

† Individual ops are amortized bounds

Shortest Path Problem: More Discussion

Bellman-Ford algorithm

- Iterative algorithm that converges to the shortest distance for each node.

Dijkstra's algorithm for improvement

- Start from both ends (source and destination).
- Execute both runs alternatively.
- Stop when a common exploded node is found.

Practicality

- Dijkstra vs. Bellman-Ford (Internet OSPF vs. Internet ISIS)

Similar algorithm (that gradually "grows" a set from the source)

. Prim's solution for minimum spanning tree (MST)

. Prim vs. Kruskal for MST

Practical Application: Coin Changing

Coin Changing

Goal. Given currency denominations: 1, 5, 10, 25, 100, devise a method to pay amount to customer using fewest number of coins.

Ex: 34¢.

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Ex: $2.89.

Coin Changing: Greedy Algorithm

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

```
Sort coins denominations by value: $c_1 < c_2 < … < c_n$.

  coins selected
$S \leftarrow \phi$
while ($x \neq 0$) {
   let k be largest integer such that $c_k \leq x$
   if ($k = 0$)
      return "no solution found"
   $x \leftarrow x - c_k$
   $S \leftarrow S \cup \{k\}$
}
return S
```

Q. Is cashier's algorithm optimal?

Coin-Changing: Analysis of Greedy Algorithm

Theorem. Greed is optimal for U.S. coinage: 1, 5, 10, 25, 100.

Pf. (by induction on x)

Consider optimal way to change $c_k \leq x < c_{k+1}$: greedy takes coin k.

We claim that any optimal solution must also take coin k.

- if not, it needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to x
- table below indicates no optimal solution can do this

Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by greedy algorithm. ▪

k	c_k	All optimal solutions must satisfy	Max value of coins 1, 2, …, k-1 in any OPT
1	1	$P \leq 4$	-
2	5	$N \leq 1$	4
3	10	$N + D \leq 2$	4 + 5 = 9
4	25	$Q \leq 3$	20 + 4 = 24
5	100	no limit	75 + 24 = 99

Coin-Changing: Analysis of Greedy Algorithm

Observation. Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.

Greedy: 100, 34, 1, 1, 1, 1, 1, 1.

Optimal: 70, 70.

Design Coin Denominations for Minimum Coin-Changing

Observation on changes 1: 1 (P), 2: 1+1, 3: 1+1+1, 4: 1+1+1+1, 5: 1 (N)
6: 1 (N)+1, 7: 1 (N) +1+1, 8: 1 (N)+1+1+1, 9: 1 (N)+1+1+1+1

A total of 25 coins is used for changes from 1 to 9, assuming each is equal.

Is the US system the best to cover from 1 to 9 for minimum changes?

- What is the best denominations using two coins to cover from 1 to 9?
- What is the best denominations using three coins to cover from 1 to 9?

Re-exam the whole US currency system (its denominations).

- Current system: 0.01, 0.05, 0.1, 0.25, 0.5, 1, 10, 20, 100
- It has flaws and why. How did it happen (?) (hint: use your imagination)
- Modular design: consistent changing rules for 0.1, 1, 10, and 100.

Some reflections

Greedy algorithm

- There is a local decision rule that one can use to construct optimal solutions.
- Usually, it follows a sequence, e.g., interval schedule and Dijkstra's shortest path

Local algorithm

- Local decision with no sequential propagation.
- Social network connectivity (Wu and Li, 1999):

A person can withdraw from a connected social network if all his friends are pair-wise (directed) connected without causing network disconnection.

Edsger W. Dijkstra

The question of whether computers can think is like the question of whether submarines can swim.

Do only what only you can do.

In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.

The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence.

APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums.