Searching for Sorted Sequences of Kings in Tournaments

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Abstract

A *tournament* T_n is an orientation of a complete graph on n vertices. A *king* in a tournament is a vertex from which every other vertex is reachable by a path of length at most 2. A *sorted sequence of kings* in a tournament T_n is an ordered list of its vertices $u_1, u_2, \ldots, u_n$ such that u_i dominates u_{i+1} ($u_i \rightarrow u_{i+1}$) and u_i is a king in the subtournament induced by $\{u_j : i \leq j \leq n\}$ for each $i = 1, 2, \ldots, n-1$. In particular, if T_n is transitive, searching for a sorted sequence of kings in T_n is equivalent to sorting a set of n numbers. In this paper, we try to find a sorted sequence of kings in a general tournament by asking the following type of binary questions: "What is the orientation of the edge between two specified vertices u, v?" The *cost* for finding a sorted sequence of kings is the minimum number of binary questions asked in order to guarantee the finding of a sorted sequence of kings. Using an adversary argument proposed in this paper, we show that the cost for finding a sorted sequence of kings in T_n is $\Theta(n^{3/2})$ in the worst case, thus settling the order of magnitude of this question. We also show that the cost for finding a king in T_n is $\Omega(n^{4/3})$ and $O(n^{3/2})$ in the worst case. Finally, we show a connection between a sorted sequence of kings and a median order in a tournament.

Key words: adversary argument, divide-and-conquer algorithm, king, recursive relation, sorted sequence of kings, tournament.

AMS subject classification: 05C20, 05C85, 68W40

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1 Introduction

A *digraph* $G = (V, E)$ contains a vertex set V and an edge set E, where each edge contains a tail u and a head v (and so the orientation $u \to v$). For a vertex u in G, the *outdegree* $d^+(G, u)$ of u is the number of edges with tail u, and the *indegree* $d^-(G, u)$ of u is the number of edges with head u. We will use $d^+(u)$ and $d^-(u)$ to denote $d^+(G, u)$ and $d^-(G, u)$, respectively, if G is specified from the context. A *tournament* T_n is an orientation of a complete graph on n vertices; that is, between any two distinct vertices u, v, there is exactly one edge: either $u \to v$ or $v \to u$ (but not both). This concept is used to model a tournament of n players where every two players compete a game and player u beats player v if and only if the vertex u dominates the vertex v ($u \to v$) in T_n. (Suppose no game has a draw.) A vertex u is called a *king* in T_n if every other vertex is reachable from u by a path of length at most 2; that is, for each $v \neq u$, either $u \to v$ or $u \to w \to v$ for some vertex w dependent of v. It is known [11] that a vertex with the maximum outdegree is always a king in T_n. A *sorted sequence of kings* in T_n is an ordered list of its vertices $u_1, u_2, \ldots, u_n$ such that $u_i \to u_{i+1}$ and u_i is a king in the subtournament induced by $\{u_j : i \leq j \leq n\}$ for each $i = 1, 2, \ldots, n-1$. In particular, if T_n is *transitive* (that is, $u \to v$ and $v \to w$ imply $u \to w$), searching for a sorted sequence of kings in T_n is equivalent to sorting a set of n numbers.

The existence of a sorted sequence of kings in any tournament was shown by Lou *et. al.* in [7] where an insertion sort algorithm with the worst case complexity of $O(n^3)$ was presented. A modified insertion sort algorithm with the worst case complexity of $O(n^2)$ was given by Wu and Sheng in [12]. It is easy to prove that a tournament has a unique sorted sequence of kings if and only if it is transitive. Thus in a general tournament the sorted sequence of kings is not unique. For example, Figure 1 shows the graph representation of a tournament. One sorted sequence of kings of the tournament is $u_2 \to u_4 \to u_1 \to u_5 \to u_3 \to u_6$, and another one is $u_2 \to u_6 \to u_4 \to u_1 \to u_5 \to u_3$.

In this paper, we try to find a sorted sequence of kings in T_n by asking the following type of binary questions: "What is the orientation of the edge between two specified vertices u, v?" The *cost* for finding a sorted sequence of kings is the minimum number of questions asked in order to guarantee the finding of a sorted sequence of kings. In particular, if we are told that T_n is transitive, the worst case cost for finding a sorted sequence of kings in T_n is $\Theta(n \log n)$, which is the number of comparisons needed to sort n numbers in the worst case. We set to determine the worst case cost for finding a sorted sequence of kings in T_n (which may not be transitive).

For a vertex u in a digraph, let $\Gamma^+(u) = \{v : u \to v\}$ be the *first out-neighborhood* of u, $\Gamma^{++}(u) = \{w : u \to v \to w \text{ for some } v\} \setminus \Gamma^+(u)$ be the *second out-neighborhood* of u, and $\Gamma^-(u) = \{v : v \to u\}$ be the *first in-neighborhood* of u. The following lemma follows

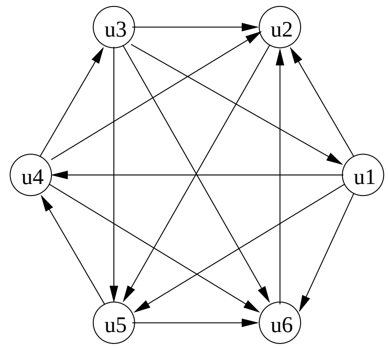

Figure 1: A sample tournament.

easily from the fact that each vertex in $\Gamma^+(u)$ is reachable from each vertex in $\Gamma^-(u)$ by a path of length at most 2.

Lemma 1 *For a vertex u in T_n, let $u_1, \ldots, u_t$ be a sorted sequence of kings in the subtournament of T_n induced by $\Gamma^-(u)$, and let $u_{t+2}, \ldots, u_n$ be a sorted sequence of kings in the subtournament of T_n induced by $\Gamma^+(u)$. Then $u_1, \ldots, u_t, u, u_{t+2}, \ldots, u_n$ form a sorted sequence of kings in T_n. In particular, u_1 is a king in T_n.*

One can apply the above lemma recursively to obtain a *divide-and-conquer algorithm* for the search of a sorted sequence of kings in T_n as follows:

1. Choose a pivot vertex u arbitrarily.
2. Use $n-1$ questions to find the edge orientation between u and every other vertex, and so obtain $\Gamma^-(u)$ and $\Gamma^+(u)$.
3. Apply the procedure recursively to $\Gamma^-(u)$ and $\Gamma^+(u)$.
4. Chain the outcomes of $\Gamma^-(u)$ and $\Gamma^+(u)$ with u in the way provided by Lemma 1.

This divide-and-conquer algorithm performs similarly to *quick sort*. It is well known that while the average performance of quick sort is better than that of *merge sort*, the worst case performance of quick sort ($\Theta(n^2)$) is poor compared with that of merge sort ($\Theta(n \log n)$). The average cost for finding a sorted sequence of kings by using the above divide-and-conquer algorithm is satisfactory ($\Theta(n \log n)$) since it is equivalent to applying quick sort to n numbers [2]. On the other hand, the worst case cost by using this algorithm is $n(n-1)/2$ if at each stage of divide-and-conquer either $\Gamma^+(u)$ or $\Gamma^-(u)$ is the empty set. Similarly, one can also have a divide-and-conquer algorithm to search for a king in T_n with satisfactory average cost $\Theta(n \log n)$ and poor worst case cost $n(n-1)/2$. The motivation of this paper is to provide alternative algorithms for the search of a king, and a sorted sequence of kings, respectively, in the case that avoiding the above mentioned worst case is crucial. We achieve this goal with some sacrifice in the average performance.

Let $f(n)$ and $g(n)$ be the cost in the worst case for finding a king, and a sorted sequence of kings, respectively, in T_n. Using an adversary argument proposed in this paper, we prove that

$$\frac{\sqrt{3}}{3}(n-1)^{3/2} - \frac{3}{2}n \leq g(n) \leq \frac{8\sqrt{2}}{3}n^{3/2} + 25n^{5/4}.$$

Therefore, the worst case asymptotic cost for finding a sorted sequence of kings in T_n is $\Theta(n^{3/2})$. We also prove that

$$\frac{3\sqrt[3]{2}}{4}(n-1)^{4/3} - \frac{3}{2}(n-1) \leq f(n) \leq \frac{4\sqrt{2}}{3}n^{3/2}.$$

That is, the worst case asymptotic cost for finding a king in T_n is $\Omega(n^{4/3})$ and $O(n^{3/2})$. Our proofs for both upper bounds provide algorithms for finding a king, and a sorted sequence of kings, respectively, in T_n both with a complexity of $O(n^{3/2})$. To prove the lower bounds for $f(n)$ and $g(n)$, we design a Pro-Small-Outdegree-Strategy for an *adversary argument* [5]. We prove that if the adversary uses this strategy, no algorithms can succeed with cost smaller than the above mentioned lower bounds in the worst case.

The paper is organized as follows: Section 2 introduces the idea of improving the worst cast performance with the sacrifice of the average performance, presents a Pro-Small-Outdegree-Strategy for the use of an adversary argument, and uses both ideas to prove an upper bound and a lower bound, respectively, for the worst case cost for finding a king in T_n. Section 3 is devoted to the proof that the worst case asymptotic cost for finding a sorted sequence of kings in T_n is $\Theta(n^{3/2})$. Section 4 shows a connection between a sorted sequence of kings and a median order in a tournament. Section 5 concludes the paper and raises two open problems.

2 Worst case cost for finding a king

If one uses the divide-and-conquer algorithm introduced in the previous section to search for a king in T_n, the worst case happens when $\Gamma^+(u) = \emptyset$ at each stage of divide-and-conquer. So in order to avoid the worst case, one should carefully choose a pivot vertex u with $\Gamma^+(u)$ large enough at each stage of divide-and-conquer. If we merge this idea into the divide-and-conquer algorithm, we call it the *revised-divide-and-conquer algorithm.* For this purpose, we need the following lemma for the choice of a pivot vertex.

Lemma 2 (Landau [6]) *Suppose $d_1^+ \le d_2^+ \le \ldots \le d_n^+$ is the outdegree sequence of T_n. Then, for each i,*

$$\frac{i-1}{2} \le d_i^+ \le \frac{n+i-2}{2}.$$

A simple idea to use a revised-divide-and-conquer algorithm to prove $f(n) = O(n^{3/2})$ is as follows. First, we can use n questions to obtain the orientation of all edges in a subtournament of roughly $\sqrt{2n}$ vertices. (Note that this part is the sacrifice of the average performance.) We choose a vertex with the maximum outdegree in this subtournament as the pivot vertex to apply Lemma 1. Then, at each stage of divide-and-conquer, at least $\sqrt{2n}/2$ vertices are eliminated with a total cost of $\binom{\sqrt{2n}}{2} + n - \sqrt{2n} < 2n$. Once the size of remaining vertices is sufficiently small, say $\sqrt{2n}/2$, using the direct method to find a king. Thus eliminating $n-1$ vertices with a king remaining costs at most $2n \cdot n/\left(\sqrt{2n}/2\right) = 2\sqrt{2}n^{3/2}$. We use recursive relation to prove an improved coefficient for $n^{3/2}$ in the next theorem.

Theorem 1 *One can find a king in T_n with cost at most $(4\sqrt{2})n^{3/2}/3$; that is,*

$$f(n) \le \frac{4\sqrt{2}}{3} n^{3/2}.$$

Proof. Let S be a subset of vertices in T_n with $|S| = \left\lceil \sqrt{2n} \right\rceil$. Let T_S be the subtournament of T_n induced by S. Then we can obtain the orientation of all edges in T_S with cost $\binom{\left\lceil \sqrt{2n} \right\rceil}{2}$. Let u be a vertex with the maximum outdegree within the subtournament T_S. By Lemma 2, the outdegree of u within T_S is

$$d^+(T_S, u) \ge \frac{|S| - 1}{2} = \frac{\left\lceil \sqrt{2n} \right\rceil - 1}{2}.$$

Next we can obtain the orientation of the edge between u and each $v \in V(T_n) \setminus S$ with cost $n - \left\lceil \sqrt{2n} \right\rceil$. Then

$$|\Gamma^-(u)| = n - 1 - |\Gamma^+(u)| \le n - 1 - d^+(T_S, u) \le n - \frac{\sqrt{2n}}{2}.$$

To find a king in T_n, by Lemma 1, it suffices to find a king in the subtournament of T_n induced by $\Gamma^-(u)$ with cost at most $f(|\Gamma^-(u)|)$. Thus

$$f(n) \le f(|\Gamma^-(u)|) + \binom{\left\lceil \sqrt{2n} \right\rceil}{2} + n - \left\lceil \sqrt{2n} \right\rceil \le f\left(n - \frac{\sqrt{2n}}{2}\right) + 2n - \frac{\sqrt{2n}}{2}.$$

Now we use induction to prove $f(n) \leq 4\sqrt{2}n^{3/2}/3$ for all $n \geq 1$. It holds for $n = 1$ trivially. Suppose it holds for all cases less than n. Then

$$\begin{aligned} f(n) &\leq f\left(n - \tfrac{\sqrt{2n}}{2}\right) + 2n - \tfrac{\sqrt{2n}}{2} \\ &\leq \tfrac{4\sqrt{2}}{3}\left(n - \tfrac{\sqrt{2n}}{2}\right)^{3/2} + 2n - \tfrac{\sqrt{2n}}{2} \\ &\leq \tfrac{4\sqrt{2}}{3}\left(n - \tfrac{\sqrt{2n}}{2}\right)\left(\sqrt{n} - \tfrac{\sqrt{2}}{4}\right) + 2n - \tfrac{\sqrt{2n}}{2} \\ &< \tfrac{4\sqrt{2}}{3}n^{3/2} \end{aligned}.$$

□

Remark 1. In the actual implementation of the revised-divide-and-conquer algorithm given in Theorem 1, an extra of $\Theta(|S|)$ comparisons in determining the vertex with the maximum outdegree in T_s will be introduced at each recursive call in selecting a pivot vertex. However, since the total number of extra comparisons needed is $O(n)$, the algorithm will have a complexity of at most $4\sqrt{2}n^{3/2}/3 + O(n)$.

Remark 2. We note that if only $\sqrt{2n}/2$ vertices are eliminated in the first stage of divide-and-conquer, a complete subtournament of $\sqrt{2n}/2$ vertices remains. Then we can take advantage of having known all edges within this remaining subtournament to obtain a second complete subtournament of $\sqrt{2n}$ vertices with a cost of only $\binom{\sqrt{2n}}{2} - \binom{\sqrt{2n}/2}{2} \approx 3n/4$ for the second stage of divide-and-conquer. This shows that either more than $\sqrt{2n}/2$ vertices can be eliminated in the first stage of divide-and-conquer or the second stage of divide-and-conquer can be performed with cost less than $2n$. If this is taken into consideration recursively, a much longer and tedious estimate shows that $f(n) \leq 2\sqrt{6}n^{3/2}/3 + o(n^{3/2})$.

In order to prove a lower bound for the worst case cost for finding a king in T_n, we use an adversary argument. Our idea is to design a strategy for the adversary to answer each question. The adversary chooses his/her answers to try to force the algorithm to work hard. Suppose $e_1, e_2, \ldots, e_l$, where $l = f(n)$, is a sequence of edges that we ask the adversary for their orientation. Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_l$ be the answers of the adversary. We define a sequence of digraphs $G_0, G_1, \ldots, G_l$ as follows:

1. Let G_0 be the empty digraph with the same vertex set as T_n.
2. Let $G_i = G_{i-1} + \vec{e}_i$, for each $i = 1, 2, \ldots, l$; that is, G_i is obtained by adding the edge $\vec{e}_i$ to G_{i-1}.

Suppose v_i and w_i are the two vertices incident to e_i. We design the following strategy for the adversary to determine his/her answer to the question about the orientation of e_i:

1. Let $v_i \to w_i$ if $d^+(G_{i-1}, v_i) < d^+(G_{i-1}, w_i)$;

2. Let $w_i \to v_i$ if $d^+(G_{i-1}, v_i) > d^+(G_{i-1}, w_i)$;

3. Orientate e_i arbitrarily if $d^+(G_{i-1}, v_i) = d^+(G_{i-1}, w_i)$.

We call the above strategy the Pro-Small-Outdegree-Strategy, and call G_l the digraph constructed by using the Pro-Small-Outdegree-Strategy. From now on we use $d(v)$ to denote $d^+(G_l, v)$ if there is no confusion from the context.

Lemma 3 *Suppose v is a vertex in G_l constructed by using the Pro-Small-Outdegree-Strategy. Then, for any $S \subseteq \Gamma^+(v)$,*

$$\sum_{w \in S} d(w) \geq \binom{|S|}{2}.$$

Proof. We may order the vertices of $\Gamma^+(v) = \{w_i : 1 \leq i \leq d(v)\}$ according to the ordering of the questions whose answers are of the type "$v \to w$". Then $d(w_i) \geq i - 1$ by the Pro-Small-Outdegree-Strategy. Thus

$$\sum_{w \in S} d(w) \geq \sum_{1 \leq i \leq |S|} (i-1) = \binom{|S|}{2}.$$

□

Lemma 4 *Suppose v is a vertex in G_l constructed by using the Pro-Small-Outdegree-Strategy. Then, for any $S \subseteq \Gamma^{++}(v)$,*

$$\sum_{w \in S} d(w) \geq d(v) \cdot \binom{|S|/d(v)}{2}.$$

Proof. Let $\Gamma^+(v) = \{w_i : 1 \leq i \leq d(v)\}$. Since $S \subseteq \Gamma^{++}(v)$, we can partition S into pairwise disjoint sets as follows:

1. $S = \cup_{i=1}^{d(v)} S_i$; (It is possible that some S_i may be empty.)

2. $S_i \subseteq \Gamma^+(w_i)$ for each i, $1 \leq i \leq d(v)$.

(Note that such a partition of S may not be unique since a vertex in S could be dominated by more than one vertex in $\Gamma^+(v)$.) By Lemma 3,

$$\sum_{w \in S} d(w) = \sum_{i=1}^{d(v)} \sum_{w \in S_i} d(w) \geq \sum_{i=1}^{d(v)} \binom{|S_i|}{2} \geq d(v) \cdot \binom{\sum |S_i|/d(v)}{2} = d(v) \cdot \binom{|S|/d(v)}{2},$$

where the last inequality holds since the function $\binom{x}{2}$ is concave upward. $\square$

Theorem 2 *No algorithm can find a king in T_n with cost less than $3\sqrt[3]{2}(n-1)^{4/3}/4 - 3(n-1)/2$ in the worst case.*

Proof. Suppose G_l ($l = f(n)$) is the digraph constructed by using the Pro-Small-Outdegree-Strategy. Let v be a king in G_l. Then $V(G_l) = \{v\} \cup \Gamma^{+}(v) \cup \Gamma^{++}(v)$ and so $|\Gamma^{++}(v)| = n - d(v) - 1$. By Lemmata 3 and 4,

$$\begin{aligned} f(n) &= l = |E(G_l)| = d(v) + \textstyle\sum_{w\in\Gamma^{+}(v)} d(w) + \sum_{w\in\Gamma^{++}(v)} d(w) \\ &\geq d(v) + \binom{d(v)}{2} + d(v)\cdot\binom{(n-d(v)-1)/d(v)}{2} \\ &> \tfrac{1}{2}\left(\tfrac{(n-1)^2}{d(v)} + (d(v))^2 - 3n + 3\right) \\ &\geq \tfrac{3}{4}\sqrt[3]{2}(n-1)^{4/3} - \tfrac{3}{2}(n-1), \end{aligned}$$

where the last inequality follows from minimizing the function $(n-1)^2/x + x^2$. $\square$

By combining Theorems 1 and 2, we conclude that the worst case asymptotic cost for finding a king in T_n is $\Omega(n^{4/3})$ and $O(n^{3/2})$.

3 Worst case cost for finding a sorted sequence of kings

If one uses the divide-and-conquer algorithm introduced in Section 1 to search for a sorted sequence of kings in T_n, the worst case happens when either $\Gamma^{+}(u) = \emptyset$ or $\Gamma^{-}(u) = \emptyset$ at each stage of divide-and-conquer. Therefore, in order to avoid the worst case, one should carefully choose a pivot vertex u with both $\Gamma^{+}(u)$ and $\Gamma^{-}(u)$ large enough at each stage of divide-and-conquer. Similar to the proof of Theorem 1, we can first use n questions to obtain the orientation of all edges in a subtournament of roughly $\sqrt{2n}$ vertices. (Again note that this part is the sacrifice of the average performance.) We choose a vertex with the median outdegree in this subtournament as the pivot vertex u to apply Lemma 1. Then, by Lemma 2, each of $\Gamma^{+}(u)$ and $\Gamma^{-}(u)$ contains at least $\sqrt{2n}/4$ vertices.

Theorem 3 *One can find a sorted sequence of kings in T_n with cost at most $(8\sqrt{2})n^{3/2}/3 + O(n^{5/4})$; that is,*

$$g(n) \leq \frac{8\sqrt{2}}{3}n^{3/2} + cn^{5/4},$$

for some constant c, for example, one may choose $c = 25$.

Proof. Let S be a subset of vertices in T_n with $|S| = \left\lceil \sqrt{2n} \right\rceil + 2$. Let T_S be the subtournament of T_n induced by S. Then we can obtain the orientation of all edges in T_S with cost $\binom{\left\lceil \sqrt{2n} \right\rceil + 2}{2}$. Let $d_1^+(T_S) \le d_2^+(T_S) \le \ldots \le d_{|S|}^+(T_S)$ be the outdegree sequence of vertices within T_S. Let u be a vertex in T_S such that $d^+(T_S, u) = d_t^+(T_S)$, where $t = \left\lceil \left\lceil \sqrt{2n} \right\rceil / 2 \right\rceil + 1$. (That is, u is a vertex with the median outdegree in T_S.) By Lemma 2, we have

$$d^+(T_S, u) \ge \frac{t-1}{2} \ge \frac{\sqrt{2n}}{4}$$

and

$$d^-(T_S, u) = |S| - 1 - d^+(T_S, u) \ge |S| - 1 - \frac{|S| + t - 2}{2} \ge \frac{\sqrt{2n}}{4}.$$

Next we can obtain the orientation of the edge between u and each $v \in V(T_n) \setminus S$ with cost $n - 2 - \left\lceil \sqrt{2n} \right\rceil$. Then

$$\frac{\sqrt{2n}}{4} \le d^+(T_S, u) \le |\Gamma^+(u)| \le n - 1 - d^-(T_S, u) \le n - \frac{\sqrt{2n}}{4}$$

and similarly

$$\frac{\sqrt{2n}}{4} \le |\Gamma^-(u)| \le n - \frac{\sqrt{2n}}{4}.$$

To find a sorted sequence of kings in T_n, by Lemma 1, it suffices to find sorted sequences of kings in both subtournaments of T_n induced by $\Gamma^+(u)$ and by $\Gamma^-(u)$, respectively, with total cost at most $g(|\Gamma^+(u)|) + g(|\Gamma^-(u)|)$. Thus

$$\begin{aligned} g(n) &\le g(|\Gamma^+(u)|) + g(|\Gamma^-(u)|) + \binom{\left\lceil \sqrt{2n} \right\rceil + 2}{2} + n - 2 - \left\lceil \sqrt{2n} \right\rceil \\ &\le g(|\Gamma^+(u)|) + g(|\Gamma^-(u)|) + 2n + \tfrac{3\sqrt{2n}}{2}. \end{aligned}$$

Now we use induction to prove $g(n) \le h(n)$, where $h(n) = 8\sqrt{2}n^{3/2}/3 + 25n^{5/4}$, for all $n \ge 1$. It holds for $n = 1$ trivially. Suppose it holds for all cases less than n. Then

$$g(|\Gamma^+(u)|) + g(|\Gamma^-(u)|) \le h(|\Gamma^+(u)|) + h(|\Gamma^-(u)|) \le h\left(\frac{\sqrt{2n}}{4}\right) + h\left(n - \frac{\sqrt{2n}}{4}\right),$$

where the last inequality holds since the function $h(x) = 8\sqrt{2}x^{3/2}/3 + 25x^{5/4}$ is concave upward. Thus

$$\begin{aligned} g(n) &\le h\left(\tfrac{\sqrt{2n}}{4}\right) + h\left(n - \tfrac{\sqrt{2n}}{4}\right) + 2n + \tfrac{3\sqrt{2n}}{2} \\ &\le \tfrac{8\sqrt{2}}{3}\left(\tfrac{\sqrt{2n}}{4}\right)^{3/2} + 25\left(\tfrac{\sqrt{2n}}{4}\right)^{5/4} + \tfrac{8\sqrt{2}}{3}\left(n - \tfrac{\sqrt{2n}}{4}\right)\left(\sqrt{n} - \tfrac{\sqrt{2}}{8}\right) \\ &\quad + 25\left(n - \tfrac{\sqrt{2n}}{4}\right)\left(\sqrt[4]{n} - \tfrac{\sqrt{2}}{16\sqrt[4]{n}}\right) + 2n + \tfrac{3\sqrt{2n}}{2} \\ &< \tfrac{8\sqrt{2}}{3}n^{3/2} + 25n^{5/4} \end{aligned}$$

$\square$

Remark 3: In the actual implementation of the revised-divide-and-conquer algorithm given in Theorem 3, an extra of $\Theta(|S|)$ comparisons will be introduced at each recursive call to select a median. The median can be determined using a linear selection algorithm [2]. It is still unknown exactly how many comparisons are needed to determine the median. Dor and Zwick [3] showed that the upper bound is slightly less than $2.95|S|$ and the lower bound is slightly more than $2|S|$. Again, since the total number of extra comparisons needed is $O(n)$, the revised-divide-and-conquer algorithm will have a complexity of at most $8\sqrt{2}n^{3/2}/3 + 25n^{5/4} + O(n)$.

Theorem 4 *No algorithm can find a sorted sequence of kings in T_n with cost less than $\sqrt{3}(n-1)^{3/2}/3 - 3n/2$ in the worst case.*

Proof. Suppose G_l ($l = g(n)$) is the digraph constructed by using the Pro-Small-Outdegree-Strategy. Suppose $u_1, u_2, \ldots, u_n$ form a sorted sequence of kings in G_l. Since $g(n) = l = |E(G_l)|$, it suffices to prove $|E(G_l)| \geq \sqrt{3}(n-1)^{3/2}/3 - 3n/2$. The proof is split into two cases:

Case 1: Suppose $d(u_i) \geq \sqrt{3(n-i)}/2$ for all i, $1 \leq i \leq n$. Then

$$|E(G_l)| = \sum_{i=1}^{n} d(u_i) \geq \frac{1}{2}\sum_{i=1}^{n}\sqrt{3(n-i)} \geq \frac{1}{2}\int_0^{n-1}\sqrt{3x}\ dx = \frac{\sqrt{3}}{3}(n-1)^{3/2}.$$

Case 2: Suppose $d(u_i) < \sqrt{3(n-i)}/2$ for some i, $1 \leq i \leq n$. Let t be the smallest i satisfying $d(u_i) < \sqrt{3(n-i)}/2$. Let $S_1 = \Gamma^+(u_t) \cap \{u_i : i \geq t+1\}$ and $S_2 = \Gamma^{++}(u_t) \cap \{u_i : i \geq t+1\}$. By the definition of a sorted sequence of kings, we know that S_1 and S_2 form a disjoint partition of $\{u_i : i \geq t+1\}$ and, hence, $|S_2| = n - t - |S_1|$. Since $|S_1| \leq d(u_t) < \sqrt{3(n-t)}/2$, by Lemma 4,

$$\begin{aligned}
\textstyle\sum_{i=t+1}^{n} d(u_i) &\geq \textstyle\sum_{u_i \in S_2} d(u_i) \\
&\geq d(u_t) \cdot \binom{(n-t-|S_1|)/d(u_t)}{2} \\
&\geq \tfrac{1}{2}\left(\tfrac{(n-t)^2}{d(u_t)} - \tfrac{2n|S_1|}{d(u_t)} - n\right) \\
&\geq \tfrac{1}{2}\left(\tfrac{2(n-t)^2}{\sqrt{3(n-t)}} - 3n\right) \\
&= \tfrac{\sqrt{3}}{3}(n-t)^{3/2} - \tfrac{3}{2}n.
\end{aligned}$$

Thus

$$
\begin{aligned}
|E(G_l)| &\geq \textstyle\sum_{i=1}^{t-1} d(u_i) + \sum_{i=t+1}^{n} d(u_i) \\
&\geq \tfrac{1}{2} \textstyle\sum_{i=1}^{t-1} \sqrt{3(n-i)} + \tfrac{\sqrt{3}}{3}(n-t)^{3/2} - \tfrac{3}{2}n \\
&\geq \tfrac{\sqrt{3}}{2} \textstyle\int_{n-t}^{n-1} \sqrt{x}\ dx + \tfrac{\sqrt{3}}{3}(n-t)^{3/2} - \tfrac{3}{2}n \\
&= \tfrac{\sqrt{3}}{3}(n-1)^{3/2} - \tfrac{3}{2}n.
\end{aligned}
$$

□

By combining Theorems 3 and 4, we conclude that the worst case asymptotic cost for finding a sorted sequence of kings in T_n is $\Theta(n^{3/2})$.

4 Connection with median order

A sorted sequence of kings may also be viewed as a weak approximation for ranking players in a tournament. In general, the *tournament ranking problem* [8] is a difficult one without applausive solution. Suppose $u_1, u_2, \ldots, u_n$ is a ranking of players such that u_i is ranked in the ith place. For any pair of players u_i, u_j with $i < j$, a *happiness* is an outcome that u_i beats u_j while an *upset* is an outcome that u_j beats u_i. A ranking strategy introduced in [10] is to minimize the number of total upsets. A *median order* of a tournament is defined as a ranking of players with the minimum number of total upsets. Similarly, a median order of a digraph can be defined as an ordered list of vertices which induces an acyclic digraph with the maximum number of edges. It is known that determining a median order of a digraph is NP-complete, and that the complexity for determining a median order for a tournament is still unknown [1]. Now suppose $u_1, u_2, \ldots, u_n$ form a median order for T_n. Havet and Thomassé [4] showed that u_1 is a king for T_n. By the definition of a median order, it is easy to see that $u_i, u_{i+1}, \ldots, u_n$ form a median order for the subtournament induced by $\{u_j : i \leq j \leq n\}$ for each $i \leq n$. These facts reveal the following connection between a median order and a sorted sequence of kings in a tournament.

Theorem 5 *Any median order in a tournament is a sorted sequence of kings.*

Theorem 5 suggests that one does not have to check all $n!$ possible orderings of vertices in order to find a median order of T_n. Instead, one may narrow the search within all sorted sequences of kings.

5 Conclusion

In this paper, we have shown that the worst case asymptotic cost for finding a sorted sequence of kings in T_n is $\Theta(n^{3/2})$. We have also shown that the worst case asymptotic cost for finding a king in T_n is $\Omega(n^{4/3})$ and $O(n^{3/2})$. The lower bounds are derived by using an adversary argument called Pro-Small-Outdegree-Strategy proposed in this paper. In addition, we have provided a revised-divide-and-conquer algorithm that finds a sorted sequence of kings (including a king) with a cost of $\Theta(n^{3/2})$ in the worst case. It is still an open problem on exactly how many binary questions are needed in the worst case to determine a sorted sequence of kings in a tournament. Also it would be interesting to know the worst case asymptotic cost for finding a king in a tournament.

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