Fault Tolerance Measures for *m*-Ary *n*-Dimensional Hypercubes Based on Forbidden Faulty Sets

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Abstract—In this paper, we study fault tolerance measures for *m*-ary *n*-dimensional hypercubes based on the concept of forbidden faulty sets. In a forbidden faulty set, certain nodes cannot be faulty at the same time and this model can better reflect fault patterns in a real system than the existing ones. Specifically, we study the bounds of the minimum vertex cut set for *m*-ary *n*-dimensional hypercubes by requiring each node to have at least *k* healthy neighbors. Our result enhances and generalizes a result by Latifi et al. for binary hypercubes. Our study also shows that the corresponding result based on the traditional fault model (where *k* is zero) tends to underestimate network resilience of large networks such as *m*-ary *n*-dimensional hypercubes.

Index Terms—Fault tolerance, forbidden faulty sets, hypercubes, minimum vertex cut.

1 Introduction

In designing or selecting a network topology for a parallel/distributed system, one fundamental consideration is fault tolerance. Specifically, a system is said to be fault tolerant if it can remain *functional* in the presence of faults (processors and/or communication links). A system is functional as long as there is a nonfaulty communication path between each pair of nonfaulty nodes; that is, the underlying topology of the system remains connected in the presence of faults.

With its numerous attractive features, the hypercube has been one of the dominating topological structures for parallel/distributed systems. A *binary n-dimensional hypercube* (*n*-cube) system [7] consists of exactly 2^n processors (also called nodes) that can be addressed distinctively by *n*-bit binary numbers. Two nodes are directly connected by a link if and only if their binary addresses differ in exactly one bit position. The hypercube structure has been used in many experimental and commercial machines including NCUBE-2, Intel iPSC, and Connection Machines. An *m-ary n-dimensional hypercube* is a direct extension of a binary *n*-dimensional hypercube (also called *n*-cube). It is based on *m* as its radix number system; that is, there are *m* nodes along each dimension. An *m*-ary *n*-dimensional hypercube is a special case of a *generalized hypercube* [1] that has a mixed radix number system.

Traditionally, the edge- and vertex-connectivity have been mainly used for measures of functionality of the system. For example, the minimum number of faulty nodes in an *n*-cube that results in the remaining nodes being disconnected is *n*. However, the probability that all *n* faulty nodes are neighbors of the same node is very small for the following reasons:

1) Some subsets of system nodes are not potentially faulty; this situation is especially true in heterogeneous environments, and
2) If all the nodes have the same failure probability, the probability of all *n* faulty nodes are neighbors of the same node in an *n*-cube is $2^n \Big/ \binom{2^n}{n}$, which is a very small number even for a moderate size of networks.

To compensate for the above shortcoming, several generalized measures of *connectedness* have been proposed ([2], [4]), such as *toughness* and *mean connectivity* of a graph. Esfahanian [5] introduced the concept of *forbidden faulty sets* in which components cannot be faulty at the same time. As a special case of forbidden faulty sets, Latifi et al. [6] studied a model in which each node in an *n*-cube has at least *k* healthy neighbors where $k \leq \lfloor \frac{n}{2} \rfloor$ and $n \geq 3$, and showed that at least $(n-k)2^k$ faulty nodes (these nodes form a *vertex cut set*) are needed to disconnect the remaining *n*-cube.

The use of forbidden faulty set is motivated by the fact that the traditional graph connectivity model cannot correctly reflect network resilience of large systems. The objective of this study is to determine the bound of the size of minimum vertex cut set that can realistically represent the fault tolerance of the *m*-ary *n*-dimensional hypercube. Our results show that the traditional graph connectivity as a fault tolerance metric tends to underestimate network resilience of large networks.

In this paper, we enhance Latifi et al.'s result of minimum vertex cut set for binary hypercubes and generalize the enhanced result to *m*-ary *n*-dimensional hypercubes. More specifically, we find the cardinality of the minimum vertex cut set for a faulty *m*-ary *n*-dimensional hypercube in which each node has at least *k* healthy nodes. Note that the problem of finding the minimum vertex cut set is suspected to be NP-hard as there is no known polynomial algorithm to find a minimum vertex cut for a given graph. Since the *m*-ary *n*-dimensional hypercube contains the binary hypercube as a special case, our results here are one step further toward finding a large class of regular interconnection networks in which minimum vertex cut sets can be successfully determined. To simplify our discussion, we only consider node faults; therefore, only vertex cut sets are considered. Our results can be extended to link faults by addressing edge cut sets. Note that, although our main result here is a generalization and enhancement of Latifi et al.'s result, a different and more involved proving method is introduced in this paper.

In Section 2, basic graph concepts are reviewed together with the graph model of the *m*-ary *n*-dimensional hypercube. Section 3 presents the result on the minimum vertex cut set for a faulty *m*-ary *n*-dimensional hypercube in which each node has at least *k* healthy nodes. Section 4 studies the selection of several parameters that determine network resilience of the *m*-ary *n*-dimensional hypercube and shows relationships between these parameters. Finally, in Section 5, we present our conclusions and future work.

2 Notation and Preliminaries

The interconnection of a set of processors can be adequately represented by a *simple graph* $G = (V, E)$, where each vertex (also node) $u \in V$ represents a processor, and each edge (also link) $(u, v) \in E$ represents a link between the vertices *u* and *v*. Two linked processors can directly access each other and are called *neighbors*. In the following, we only review concepts that are used in this paper. For other graph-related concepts, the reader may refer to a standard book on graph theory such as [3].

A graph G' is a *subgraph* of *G* (written $G' \subseteq G$), if $V(G') \subseteq V(G)$ and $E(G') = \{(u, v) \mid (u, v) \in E(G), u, v \in V(G')\}$. Suppose that *F* is a nonempty subset of $V(G)$, the *induced subgraph* $G - F$ is a subgraph of *G* such that vertex set $V(G - F) = V(G) - F$ and edge set $E(G - F) = \{(u, v) \mid (u, v) \in E(G), u, v \in V(G) - F\}$. The *vertex cut* of *G* is a subset *F* of $V(G)$ such that the induced subgraph is disconnected.

The *neighbor set* of vertex *u* is the subset of $V(G)$ in which each vertex is connected to vertex *u*, denoted by $N(u \mid G)$, where *u* may

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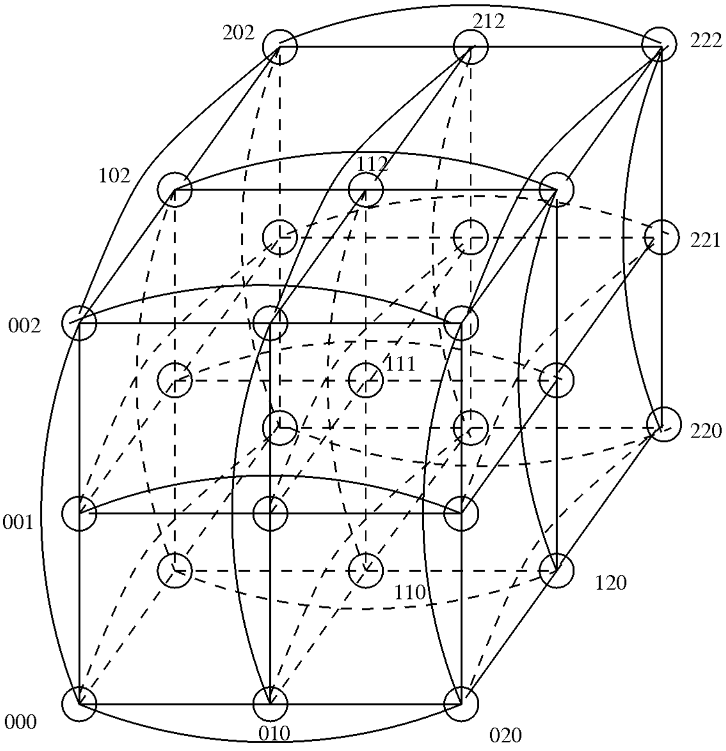

Fig. 1. A $G(3, 3)$.

or may not be an element of $V(G)$. $N(V'|G) = \bigcup_{u \in V'} N(u|G)$ is called the *neighbor set* of subset V' in G.

The *vertex degree* $d(u \mid G)$ of a vertex u in G is the number of edges of G incident with u. For a simple graph G, $d(u \mid G) = |N(u \mid G)|$. The *minimum-vertex-degree*, $d_{min}(G)$, of G is defined as the minimum *vertex degree* in G, i.e., $min\{d(u \mid G)\}$.

DEFINITION 1. *A* forbidden faulty set *in a given graph G, S, is a subset of V(G) that cannot be faulty simultaneously.*

Note that if S is a forbidden faulty set, then any set that contains S as its subset is also a forbidden faulty set. If a subset of $V(G)$ is not a forbidden set, it is also called a *feasible faulty set* in G.

DEFINITION 2. Feasible-vertex-connectivity *of a graph G is the minimum cardinality* $|F|$ *such that the graph* $G - F$ *is a disconnected graph, where F is a feasible fault set. F is also called a* minimum feasible vertex cut set.

In general, there are many ways to define a forbidden (feasible) faulty set depending on the topology of the system, application environment, statistical analysis of fault patterns, and distribution of fault-free nodes. The following defines a feasible faulty set based on the number of healthy neighbors of each node.

DEFINITION 3. *A feasible faulty set in a given graph G is called* k-neighbor-feasible *if each healthy node in G has at least k healthy neighbors and there is at least one node that has exactly k healthy neighbors.*

Obviously, the regular fault model is a special case of feasible faulty set, that is, it is 0-neighbor feasible.

The *generalized hypercube* interconnection [1] is based on a mixed radix number system (as opposed to binary numbers used in regular binary hypercubes) and this technique results in a variety of hypercube structures for a given number of processors, depending on the desired diameter of the network. An *m-ary n-dimensional hypercube*, $G(n, m)$, is a special generalized hypercube with m as its fixed radix number. Each node corresponds to an n-vector address $(a_n, a_{n-1}, \cdots, a_1)$, where $0 \le a_i \le m - 1$. Node connections in $G(n, m)$ are defined as follows: Two nodes are linked by an edge if they differ in exactly one coordinate. Fig. 1 shows a $G(3, 3)$, which is a three-cube with three nodes along each dimension.

PROPERTY 1. $|V(G(n, m))| = m^n$, $d(u \mid G(n, m)) = n(m-1)$, *and* $d_{min}(G(n, m)) = n(m-1)$.

PROPERTY 2. *Suppose that* $G_1(n-1, m), G_2(n-1, m), \cdots, G_m(n-1, m)$ *is a partition of* $G(n, m)$ *along a dimension, say l, then, for each vertex* $u \in V(G_i(n-1, m))$, $1 \le i \le m$, *there exists* $m - 1$ *and only* $m - 1$ *neighboring nodes which do not belong to* $G_i(n-1, m)$; *and there is only one neighbor of u in each* $G_j(n-1, m)$, $i \ne j$.

3 FAULT TOLERANCE OF *m*-ARY *n*-DIMENSIONAL HYPERCUBES

In this section, we determine the *feasible-vertex-connectivity* of a given $G(n, m)$, where the feasible faulty set is *k-neighbor-feasible*. To obtain this result, we first show in Theorem 1 the relationship between the *minimum-vertex-degree* of a given subgraph of $G(n, m)$ and the size of this subgraph. We use the following notation: The quotient of n divided by m is $\lfloor n/m \rfloor$, where m and n are positive integers. The remainder of this division is m mod n, which is $m - \lfloor n/m \rfloor$ for $m \ne 0$.

THEOREM 1. *Let* G' *be a subgraph of* $G(n, m)$. *If* $d_{min}(G') = k$, *then*

$$|V(G')| \ge m^d (s + 1), \tag{1}$$

where $d = \left\lfloor \frac{k}{m-1} \right\rfloor$ *and* $s = k \bmod (m-1)$.

PROOF. In (1), we have $k = (m-1)d + s$, where $0 \le s < m - 1$, and it is called a *standard expression* of k. We prove (1) by induction on k.

Basis. When $0 \le k \le m - 2$, i.e., $d = 0$ and $s = k$. We have $|V(G')| \ge |\{u\}| + d(u \mid G') = 1 + d_{min}(G') \ge 1 + k = 1 + s$. Theorem 1 clearly holds.

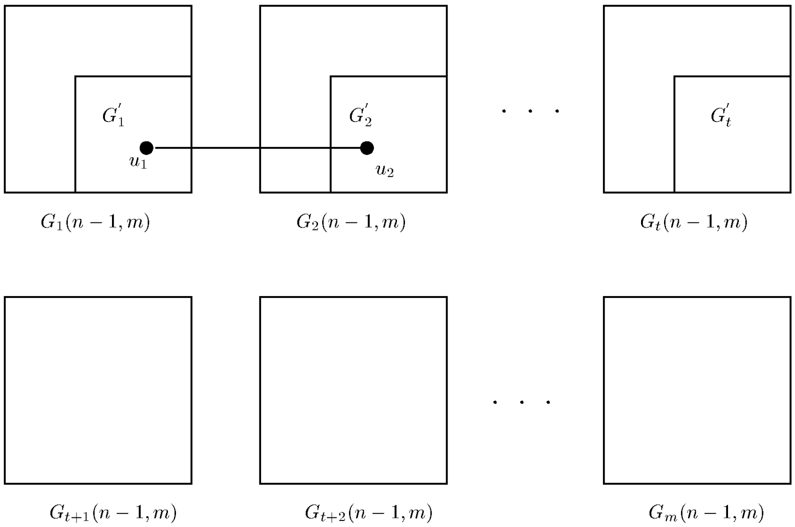

Fig. 2. A partition of $G(n, m)$ and G' along the lth dimension.

Inductive step. We assume that Theorem 1 holds for $k < p - 1$, where p ($\geq m - 1$) is a constant. (Because, in *Basis*, we have shown that Theorem 1 holds when $0 \leq k < m - 1$.) We now show that if G' is a subgraph of G and $d_{min}(G') = p$, then the following inequality holds:

$$|V(G')| \geq m^{d'}(s' + 1), \tag{2}$$

where $d' = \left\lfloor \frac{p}{m-1} \right\rfloor$ and $s' = p \bmod (m - 1)$.

Let $p - 1 = (m - 1)d + s$, where $0 \leq s < m - 1$. It is easy to verify that (2) can be rewritten as

$$|V(G')| \geq m^{d}(s + 2) \quad (0 \leq s < m - 1). \tag{3}$$

To prove (3) using the inductive step, we partition subcubes G and G' as follows: Because $d_{min}(G') = p > 0$, there must exist two vertices, say u_1 and u_2 in G', which are connected by an edge; based on the definition of $G(n, m)$, their addresses differ in exactly one dimension. Without loss of generality, suppose they differ in the lth dimension. We partition the given $G(n, m)$ into m m-ary $(n - 1)$-dimensional hypercubes along the selected dimension l and these cubes are denoted as $G_1(n - 1, m)$, $G_2(n - 1, m)$, $\cdots$, $G_m(n - 1, m)$ (see Fig. 2). Without causing confusion, they can also be denoted as $G_1, G_2, \cdots, G_m$, respectively. Subgraph G' itself is also partitioned into t ($2 \leq t \leq m$) subgraphs along the lth dimension. Without loss of generality, let $G_1', G_2', \ldots, G_t'$ be such subgraphs, and $G_i' \subset G_i$ ($0 \leq i \leq t$). Meanwhile, $u_1 \in V(G_1')$ and $u_2 \in V(G_2')$. Fig. 2 shows the partition of $G(n, m)$ and G' and the relationships between G_is and G_i's.

Based on the above partition and Property 2, the node degree of u in G_i' can be calculated as follows:

$$\begin{aligned} d\left(u \mid G_i'\right) &= \left|N\left(u \mid G_i'\right)\right| = \left|N\left(u \mid G'\right)\right| - \sum_{j \neq i}\left|N\left(u \mid G_j'\right)\right| \\ &= d\left(u \mid G'\right) - (t - 1). \end{aligned}$$

Therefore,

$$\begin{aligned} d_{min}\left(G_i'\right) &= min_{u \in G_i'}\left\{d\left(u \mid G_i'\right)\right\} \geq min_{u \in G'}\left\{d\left(u \mid G'\right) - (t - 1)\right\} \\ &= d_{min}\left(G'\right) - (t - 1) = p - (t - 1) \qquad (4) \\ &= \left[(m - 1)d + s + 1\right] - (t - 1) = (m - 1)d + (s - t + 2). \qquad (5) \end{aligned}$$

With (4), we can determine $\left|G_i'\right|$ for each $i = 1, 2, \cdots, t$, using the inductive assumption. Then, $|G'|$ can be derived by summarizing all these $\left|G_i'\right|$s. In order to use the inductive assumption to find out $\left|G_i'\right|$, we need to convert (4) to a standard expression, i.e., by ensuring $0 \leq s - t + 2 < m - 1$ in (4). To do so, we consider the following two cases:

1) If $2 \leq t \leq s + 2 \leq m$, then $0 \leq s - t + 2 < m - 1$, thus,

$$d_{min}\left(G_i'\right) \geq (m - 1)d + (s - t + 2)$$

is a standard expression. Since $(m - 1)d + (s - t + 2) < p$, based on the inductive assumption, we have

$$\begin{aligned} \left|V(G')\right| &= \sum_{i=1}^{t}\left|V\left(G_i'\right)\right| \geq \sum_{i=1}^{t} m^{d}\left[(s - t + 2) + 1\right] \\ &= m^{d}\left[t(s - t + 3)\right]. \end{aligned}$$

To determine the minimum value of $|V(G')|$, we need to find out the minimum value for function $f(t) = t(s - t + 3)$ defined on $[2, s + 2]$. Since $f(t)$ is a parabolic function, the minimum value is either $f(2)$ or $f(s+2)$. Because $f(2) - f(s + 2) = 2(s + 1) - (s + 2) = s \geq 0$, $f(t)$ has a minimum value $f(s + 2) = s + 2$. We conclude that $|V(G')| \geq m^{d}[t(s - t + 2)] = m^{d}(s + 2)$. Theorem 1 holds in this case.

2) If $s + 3 \leq t \leq m$, then $0 \leq m + s - t + 1 < m - 1$. Note that $p \geq m - 1$, we must have $d > 0$. Therefore, the standard expression of inequality (4) is

$$d_{min}(G_i') \geq (m - 1)(d - 1) + (m + s - t + 1) < p.$$

With the inductive assumption,

$$\left|V\left(G_i'\right)\right| \geq m^{d-1}\left[(m + s - t + 1) + 1\right],$$

we have

$$\begin{aligned} \left|V(G')\right| &= \sum_{i=1}^{t}\left|V\left(G_i'\right)\right| \geq \sum_{i=1}^{t} m^{d-1}\left[(m + s - t + 1) + 1\right] \\ &= m^{d-1}\left[t(m + s - t + 2)\right]. \end{aligned}$$

To determine the minimum value of $|V(G')|$, we need to find out the minimum value of function $g(t) = t(m + s - t + 2)$ defined on $[s + 3, m]$. Again, the minimum value is either $g(s + 3)$ or $g(m)$. Because $g(s + 3) - g(m) = (s + 3)(m - 1) - m(s + 2) = m - s - 2 > 0$, $g(m) = m(s + 2)$ is the minimum value. Therefore, $|V(G')| \geq m^{d-1}g(t) \geq m^{d-1}g(m) = m^{d-1}[m(s+2)] = m^d(s+2)$. Theorem 1 holds in this case. □

COROLLARY 1. *Let G' be a subgraph n-cube. If $d_{min}(G') = k$, then* $|V(G')| \geq 2^k$.

PROOF. Substituting $m = 2$ in Theorem 1, we have $d = \left\lfloor \frac{k}{m-1} \right\rfloor = k$ and $s = k \bmod (m-1) = 0$. Thus, $|V(G')| \geq m^d(s+1) = 2^k$. □

The following Theorem 2 determines the lower bound for the cardinality of a k-neighbor-feasible faulty vertex cut set, that is, the lower bound for the feasible-vertex-connectivity of $G(n, m) - F$.

THEOREM 2. *If F is a k-neighbor-feasible faulty vertex cut set of $G(n, m)$ and $k \leq (n-2)(m-1)$, then*

$$|F| \geq m^d[(n-d-1)(m-1)(s+1) + (m-s-1)], \tag{6}$$

where $d = \left\lfloor \frac{k}{m-1} \right\rfloor$, $s = k \bmod (m-1)$.

PROOF. The fact that F is a k-neighbor-feasible faulty cut set of $G(n, m)$ implies $d_{min}(G(n, m) - F) = k$. We prove this theorem by induction on k.

Basis: When $k = 0$, $d = 0$ and $s = 0$. We only need to show $|F| \geq m^0[(n-0-1)(m-1)(0+1) + (m-0-1)] = n(m-1)$. Based on Property 1 of the m-ary n-dimensional hypercube, the *vertex-connectivity* of $G(n, m)$, $d_{min}(G(n, m)) = n(m-1)$, that is, we need to remove at least $n(m-1)$ vertices to disconnected a vertex from $G(n, m)$. Based on the definition of $G(n, m)$, it needs to remove more vertices to disconnect more than one vertex. Theorem 2 clearly holds.

Inductive step. Assuming Theorem 2 holds for $k \leq p-1$, we now show that if F is a k-neighbor-feasible faulty vertex cut set, where $k = p$, then the following inequality holds:

$$|F| \geq m^{d'}\left[(n-d'-1)(m-1)(s'+1) + (m-s'-1)\right], \tag{7}$$

where $d' = \left\lfloor \frac{p}{m-1} \right\rfloor$, $s' = p \bmod (m-1)$.

Let $p - 1 = (m-1)d + s$, where $0 \leq s < m-1$. Equation (6) can be rewritten as:

$$|F| \geq m^d[(n-d-1)(m-1)(s+2) + (m-s-2)]. \tag{8}$$

Our next step is to randomly select a subgraph G' of $G(n, m)$ and to find out the cardinality of a minimum feasible faulty vertex cut set F needed to disconnect G' from $G(n, m) - F$. Based on the definition of *k-neighbor-feasible*, we have $d_{min}(G') \geq d_{min}(G(n, m) - F) = p > 0$. Thus, there must exist two neighboring vertices, say u_1 and u_2, in $V(G')$, and their addresses differ in exactly one coordinate, say l. We partition $G(n, m)$ into m m-ary $(n-1)$-dimensional hypercubes $G_1(n-1, m)$, $G_2(n-1, m)$, $\cdots$, $G_m(n-1, m)$ along the lth dimension (see Fig. 3). Again, they can also be denoted as $G_1, G_2, \cdots, G_m$, respectively. Without loss of generality, we assume that, along the lth dimension, the disconnected subgraph G' is partitioned into t subgraphs, $G'_1, G'_2, \ldots, G'_t$, such that $G'_i \subset G_i$, for $1 \leq i \leq t$, and the feasible vertex cut set F is partitioned into m subsets $F_1, F_2, \cdots, F_m$, such that $F_i \subseteq V(G_i)$, for $1 \leq i \leq m$. Meanwhile, $u_1 \in V(G'_1)$ and $u_2 \in V(G'_2)$.

First, we estimate the cardinality of F_j, $t+1 \leq j \leq m$. Since G' is separated from G, F_j contains at least all the neighboring vertices of G' in G_j. Therefore,

$$F_j \supseteq \bigcup_{i=1}^{t} N\left(G'_i \mid G_j\right) \supseteq N\left(G'_c \mid G_j\right),$$

where c is a randomly selected integer from $[1, 2, \cdots, t]$. The above approximation is based on the fact that nodes in set $N(G'_i|G_j)$ for different i may share the same neighbor in G_j. Therefore,

$$\left|F_j\right| \geq \left|N\left(G'_c \mid G_j\right)\right| = \left|V\left(G'_c\right)\right|.$$

Using the above approximation for $|F_j|$, we have,

$$\begin{aligned} |F| &\geq \sum_{i=1}^{t}\left|F_i\right| + \sum_{j=t+1}^{m}\left|F(j)\right| = \sum_{i=1}^{t}\left|F_i\right| + \sum_{j=t+1}^{m}\left|V\left(G'_c\right)\right| \\ &= \sum_{j=1}^{t}\left|F_i\right| + (m-t)\left|V\left(G'_c\right)\right|. \end{aligned} \tag{9}$$

We use the result of Theorem 1 and the inductive assumption of Theorem 2 to determine $|G'_c|$ and $|F_i|$, respectively. Using the result of (4) (in the proof of Theorem 1), we have

$$d_{min}\left(G'_c\right) \geq (m-1)d + (s-t+2). \tag{10}$$

We consider the following two cases:

1) If $2 \leq t < s + 1 \leq m$, then $0 \leq s - t + 2 < m - 1$, thus,

$$d_{min}\left(G'_c\right) \geq (m-1)d + (s-t+2)$$

is a standard expression of (9). Based on Theorem 1,

$$\left|V\left(G'_c\right)\right| \geq m^d(s-t+3). \tag{11}$$

The same result of (9) applies to other G'_is, $1 \leq i \leq t$, i.e., $d_{min}(G'_i) \geq (m-1)d + (s-t+2)$. Because the selection of G' in G (and, hence, G'_i in G_i) is random, each node in G_i has at least $(m-1)d + (s-t+2)$ healthy nodes. Based on Definition 3, F_i is at least an $[(m-1)d + (s-t+2)]$-neighbor-feasible faulty vertex cut set in graph G_i. Since $(m-1)d + (s-t+2) < p$, using the inductive assumption of Theorem 2, we have

$$|F_i| \geq m^d[(n-d-2)(m-1)(s-t+3) + (m-s+t-3)]. \tag{12}$$

Applying (10) and (11) to inequality (8), we have

$$\begin{aligned} |F| &\geq \sum_{i=1}^{t} m^d\left[(n-d-2)(m-1)(s-t+3) + (m-s+t-3)\right] \\ &\quad + (m-t)\left(m^d(s-t+3)\right) \\ &= m^d\left[t\left[(n-d-2)(m-1)(s-t+3) + (m-s+t-3)\right]\right. \\ &\quad \left. + (m-t)(s-t+3)\right]. \end{aligned} \tag{13}$$

To determine the minimum value of $|F|$, we define a function $f(t) = t[(n-d-2)(m-1)(s-t+3) + (m-s+t-3)] + (m-t)(s-t+3)$ on $[2, s+2]$. Based on the fact that $k = p \leq (m-1)(n-2)$, we have $n - 2 \geq p/(m-1) = [(m-1)d + s + 1]/(m-1)\} > d$, that is, $(n-d-2) > 0$. We consider the following two subcases:

a) If $(n-d-2)(m-1) > 2$, $f(t)$ is a parabolic function, thus either $f(2)$ or $f(s+2)$ is the minimum value of $f(t)$. It is easy to verify that $f(2) = 2[(n-d-2)(m-1)(s+1) + (m-s-1)] + (m-2)(s+1)$ and $f(s+2) = (s+2)[(n-d-2)(m-1) + (m-1)] + (m-s-2)$ and, hence, $f(2) - f(s+2) = [(n-d-2)(m-1) - 2]s - s \geq 0$. We conclude that $f(s+2)$ is the minimum value of $f(t)$ in $[2, s+2]$, therefore, $f(t) \geq f(s+2) = (n-d-1)(m-1)(s+2) + (m-s-2)$.

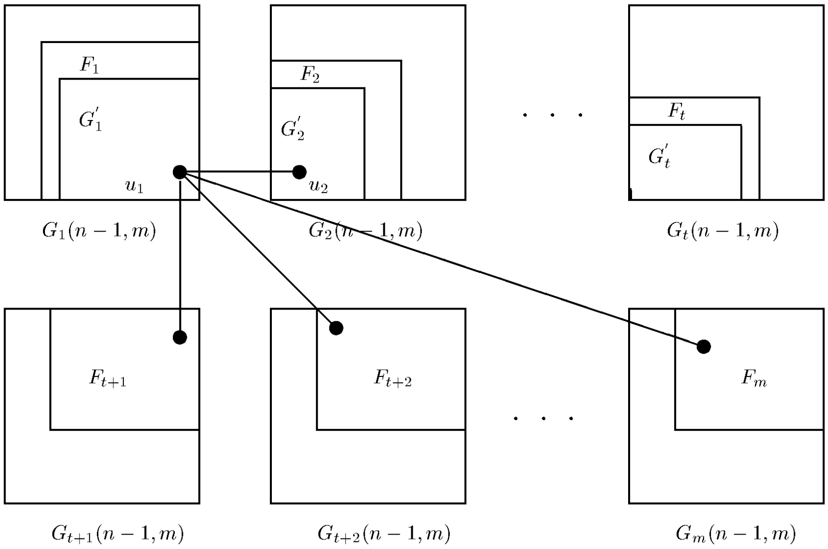

Fig. 3. The partition of G', F, and $G(n, m)$ along the l th dimension.

b) If $(n - d - 2)(m - 1) = 2$ or 1, then there are only four possible selections of (m, s, t): (2, 0, 2), (3, 0, 2), (3, 1, 3), (3, 1, 2). Each of them either meets the condition $t = s + 2$ or has the corresponding $f(t)$ value the same as $f(s+2)$. Therefore, $f(t) = f(s + 2) = (n - d - 1)(m - 1)(s + 2) + (m - s - 2)$.

Combining results of the above two subcases, we have $f(t) \geq f(s + 2) = (n - d - 1)(m - 1)(s + 2) + (m - s - 2)$ when $2 \leq t \leq s + 2$. Therefore, we induce that

$$|F| \geq m^d f(t) \geq m^d[(n - d - 1)(m - 1)(s + 2) + (m - s - 2)].$$

Hence, Theorem 2 holds when $2 \leq t \leq s + 2$.

2) If $s + 3 \leq t \leq m$, then $0 \leq m - t + s + 1 < m - 1$; thus, $d_{min}(G_c') \geq (m-1)(d-1) + (m-t+s+1)$ is a standard expression of (9). Based on Theorem 1,

$$\left|V\left(G_c'\right)\right| \geq m^{d-1}(m-t+s+1+1) = m^{d-1}(m-t+s+2). \quad (14)$$

Using the same argument for Case 1, F_i is at least an $[(m - 1)(d - 1) + m - t + s + 1]$-feasible-neighbor faulty vertex cut set in graph $G_i(n - 1, m)$, where $1 \leq i \leq t$. Based on the inductive assumption of Theorem 2, we have,

$$\begin{aligned}\left|F_i\right| &\geq m^{d-1}\Big[(n-1-(d-1)-1)(m-1)(m-t+s+1+1) \\ &\quad + \big(m-(m-t+s+1)-1\big)\Big] \\ &= m^{d-1}\Big[(n-d-1)(m-1)(m-t+s+2)+(t-s-2)\Big]. \quad (15)\end{aligned}$$

Applying (13) and (14) to inequality (8), we have

$$\begin{aligned}|F| &\geq \sum_{i=1}^{t}\left(\left|F_i\right|\right) + (m-t)\left|V\left(G_c'\right)\right| \\ &\geq m^{d-1}\Big[t\big[(n-d-1)(m-1)(m-t+s+2)+(t-s-2)\big] \\ &\quad + (m-t)(m-t+s+2)\Big].\end{aligned}$$

To determine the minimum value of $|F|$, we define a function $g(t) = t[(n - d - 1)(m - 1)(m - t + s + 2) + (t - s - 2)] + (m - t)(m - t + s + 2)$ on $[s + 3, m]$. Based on the facts that $(n - d - 2) > 0$ and $s + 3 \leq t \leq m$, i.e., $m \geq 3$, we have $(n - d - 1)(m - 1) > 2$. Therefore, the function of $f(t)$ is a parabolic function, and the minimum value is either $f(s + 3)$ or $f(m)$. It can be verified that $g(s + 3) = (s + 3)[(n - d - 1)(m - 1)(m - 1) + 1] + (m - s - 3)(m - 1)$ and $g(m) = m[(n - d - 1)(m - 1)(s + 2) + (m - s - 2)] + 0$ and, hence, $g(s + 3) - g(m) = (n - d - 1)(n - 1)(m - s - 3) - 2(m - s - 3) > m - s - 3 \geq 0$. Therefore, $g(t) \geq g(m)$, and we have $|F| \geq m^{d-1}g(t) \geq m^{d-1}g(m) = m^d[(s + 2)(n - d - 1)(m - 1) + (m - s - 2)]$, where $s + 3 \leq t \leq m$. Theorem 2 holds. □

COROLLARY 2. *If F is an arbitrary k-neighbor-feasible faulty vertex cut set of an n-cube, where $0 \leq k \leq n - 2$, then $|F| \geq (n - k)2^k$.*

PROOF. Substituting $m = 2$ in Theorem 2, we have $d = \left\lfloor \frac{k}{m-1} \right\rfloor = k$ and $s = k \bmod (m - 1) = 0$. The constraint on k, $k \leq (n - 2)(m - 2)$, is reduced to $k \leq n - 2$. Thus,

$$\begin{aligned}|F| &\geq m^d\big[(n-d-1)(m-1)(s+1)+(m-s-1)\big] \\ &= 2^k\big[(n-k-1)(2-1)(0+1)+2-0-1\big] = (n-k)2^k.\end{aligned}$$

□

The above corollary replies that $n \geq 2$ because $0 \leq k \leq n - 2$. Also, Corollary 2 is an enhancement of the main result from Latifi et al. [6]. Recall that the result in [6] states as follows: If F is an arbitrary k-neighbor-feasible vertex cut set of an n-cube, where $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $n \geq 3$, then $|F| \leq (n - k)2^k$. First of all, this result cannot be used for two-cubes. Our result show that it is possible for $n = 2$ and $k = 0$. For example, we can remove $(n - k)2^k = (4 - 2)2^2 = 2$ nodes, say 01 and 10, to disconnect the remaining nodes 00 and 11. Also, the condition $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$ is too restrictive. For example, when $n = 5$, the maximum k is two based on the result in [6]. Using our result, it is possible to select $k = 3$ (the condition $k \leq n - 2$ still holds). We can remove $(n - k)2^k = (5 - 3)2^3 = 16$ nodes to disconnect a five-cube and each of the remaining vertices has a vertex degree of at least three. This can be done by removing three-dimensional subcubes ***01 and ***10 to disconnect three-dimensional subcube ***00 and ***11, where * is either 0 or 1. Therefore, our result here is not only a generalization but also an enhancement of the result in [6].

4 DISCUSSION

In general, the degree of fault tolerance of a parallel/distributed system is determined by a combination of parameters, such as vertex degree, bisection width, minimum vertex cut set, etc. In addition, the failure probability of each component is also an important factor. In this section, we study the effects of k, n, and m on the size of minimum k-neighbor-feasible vertex cut of various configurations of $G(n, m)$, by considering the the following three cases:

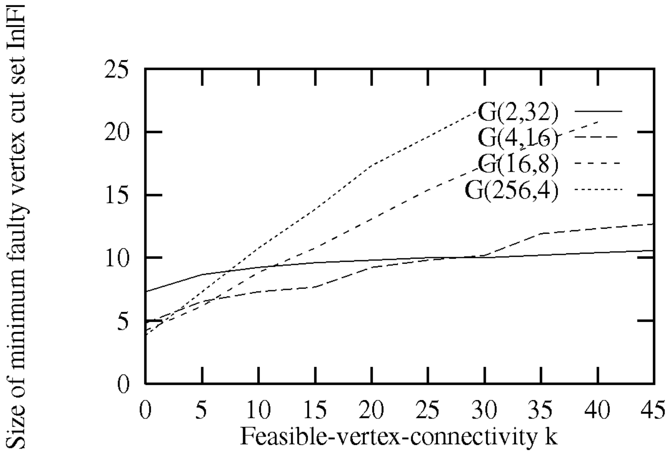

Fig. 4. The size of the minimum *k-neighbor-feasible* vertex cut set in $G(n, m)$s with 2^{32} nodes.

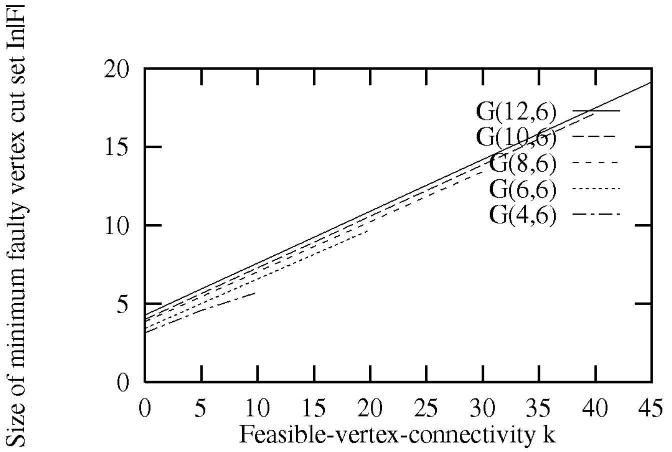

Fig. 5. The size of the minimum *k-neighbor-feasible* vertex cut set in $G(n, m)$ with a fixed value of m ($m = 6$).

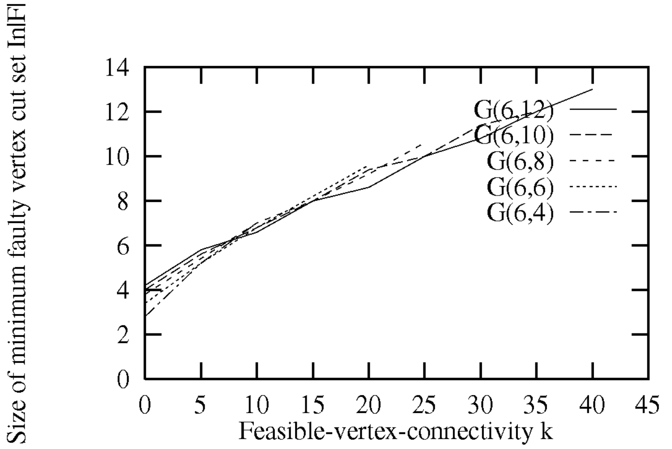

Fig. 6. The size of the minimum *k-neighbor-feasible* vertex cut set in $G(n, m)$ with a fixed value of n ($n = 6$).

1) The number of vertices $N = m^n$ is fixed, in our case, we select $N = 2^{32}$,
2) The number of dimensions n is fixed, but the number of vertices m in each dimension varies, and
3) The number of vertices m in each dimension is fixed, but the number of dimensions n varies.

Fig. 4 shows results which compare different selections of m and n in $G(n, m)$ under the same number of nodes $N = 2^{32}$. Fig. 5 shows results which compare different m-ary n-dimensional hypercubes under a fixed number of dimensions, but with the different number of nodes on each dimension. Fig. 6 shows results which compare different m-ary n-dimensional hypercubes under a fixed number of nodes on each dimension, but with the different number of dimensions. In all these figures, we use a regular scale for k (feasible-vertex-connectivity) and a logarithmic scale (ln) for $|F|$ (the size of minimum faulty vertex cut set). For each case, the size of minimum vertex cut set increases monotonically as k increases, that is, when $k = 0$, the corresponding $|F|$ is the smallest.

In Fig. 4, for a $G(n, m)$ with a small dimension (n), its $|F|$ is smaller than the one with a larger dimension when k is small. However, this situation reverses for large k. Specifically, for any two $|F|$ curves, there is one and only one cross point between these two curves. That is, for any two configurations in Fig. 4, $G(n, m)$ and $G(n', m')$, with the same number of nodes and $n < n'$, we can always find a value c such that, when $k < c$, $|F|$ of $G(n, m)$ is smaller than the one for $G(n', m')$, and the situation reverses when $k > c$.

When the number of nodes along each dimension (m) is a fixed value in $G(n, m)$ (see Fig. 5), the larger n is the larger the corresponding value of $|F|$ becomes. This fact can be easily observed from the expression for F in Theorem 2. However, when the number of dimensions (n) is a fixed value in $G(n, m)$ (see Fig. 6), a large m may or may not result in a large $|F|$. This is because a large m value may or may not generate a large d value, defined as $d = \left\lfloor \frac{k}{m-1} \right\rfloor$ in F.

5 Conclusion

In this paper, we have determined the conditional connectivity for the m-ary n-dimensional hypercube, $G(n, m)$, by requiring each node to have at least k healthy neighbors. By this, we have extended and enhanced a result by Latifi et al. for the binary hypercube. Although we have obtained the results for the conditional connectivity in the m-ary n-dimensional hypercube, we still need to analyze quantitatively the effect of n and m in $G(n, m)$, where the total number of nodes m^n is fixed. In addition, the successful application of our results depends on a reasonably good estimation of k in the *k-vertex-connectivity* condition. Another interesting work is to develop a model to define a forbidden faulty set based on application environment, network component reliability, network topology, and statistics related to fault patterns.

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