Polynomial Time and Provably Efficient Network Coding Scheme for Lossy Wireless Networks

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***Abstract*—The network coding problem across multiple unicasts is an open problem. Previously, the capacity region of 2-hop relay networks with multiple unicast sessions and limited feedback was characterized where the coding and decoding nodes are neighbors and packet erasure channels are used. A near-optimal coding scheme that exploits the broadcast nature and the diversity of the wireless links was proposed. However, the complexity of the scheme is hyper–exponential as it requires the knowledge of the packets that are received by any subset of the receivers. In this paper, we provide a polynomial time coding scheme and characterize its performance using linear equations. The coding scheme uses random network coding to carefully mix intra and intersession network coding and makes a linear, not exponential, number of decisions. We also provide a linear programming formulation that uses our 2-hop relay network results as a building block in large lossy multihop networks. Through simulations, we verify the superiority of our proposed schemes over state-of-the-art.**

***Index Terms*—Network coding, lossy wireless networks, 2-hop relay networks, capacity, fairness.**

I. Introduction

Wireless networks have emerged as an integral part of our lives. Therefore, optimizing the performance of such networks is of crucial need. Maximizing the throughput of the users while achieving fairness among them is one of the fundamental performance metrics that have to be optimized. The term *capacity* or capacity region, of wireless networks, can be used in this context [1], as it refers to the set of possible rates that can be achieved by the users simultaneously.

Different works have targeted the characterization of the capacity and the design of different algorithms that can achieve the capacity or portion of it. These approaches can be classified into three main categories. Namely, cross-layer design, intrasession network coding (IANC), and intersession network coding (IRNC). The objective of cross-layer design is to jointly optimize the operations at different layers of the OSI reference model through the use of the queue length information at different nodes [2], [3], [4], [5]. IANC exploits link diversity where links are lossy to maximize the capacity of wireless networks [6]. Intersession network coding (IRNC) [7] mixes packets of different flows to maximize the capacity by exploiting the broadcast nature of wireless links.

Using IANC, intermediate nodes perform coding on the packets of the same flow. In IANC, the source node divides the message it wants to send into batches, each having K packets of the form $P_1, \ldots, P_K$. The source node keeps sending coded packets of the form $\sum_{i=1}^{K} \alpha_i P_i$, where $\alpha_i, \forall i$ is a random coefficient chosen over a finite field of large enough size typically 2^8–2^{16}. Upon receiving a coded packet, the intermediate relay node checks to see if the coded packet is linearly independent to what it has received before. If so, it keeps the coded packet, otherwise it drops the packet. When the destination receives any K linearly independent packets, this means that it can decode all of the packets of the batch. Therefore, it sends a feedback to the source to stop sending from this batch and move to the next batch.

IRNC on the other hand exploits the broadcast nature of wireless links. Take Fig. 1 from [7] as an example; in which we have two opposite direction flows between s_1 and s_2 through r, if the broadcast nature of wireless links is not exploited and assuming that nodes s_1 and s_2 are out of range of each other, we need four transmissions to exchange two packets between nodes s_1 and s_2. The relay node r can exploit the broadcast nature of its output links and reduce the number of transmissions to three using network coding by XORing the two packets, as shown in the figure.

In order to operate the network closer to the capacity, the three approaches have to be used jointly. One can think of cross-layer optimization as an orthogonal approach to the other approaches. Therefore, using IANC jointly and optimally with IRNC provides insights into how to achieve the objective of approaching the capacity of wireless networks. However, the joint IANC and IRNC problem is NP-hard [8] and linear coding is not sufficient for the problem [9]. Nonetheless, one can limit network coding to be in a single-hop [7], where the coding node is a neighbor of the decoding node.

IANC fails to resolve the bottleneck when different flows are using an intermediate node. On the other hand, most of IRNC protocols do not consider lossy links. COPE [7] turns off IRNC when the links have loss rates above 20%. The work in [10] studied single-hop IRNC in lossy wireless networks. The authors in [10] consider only XOR operations, did not optimize overhearing, and treated every packet separately, not as a member of a flow. The authors showed that the problem is #P-complete and provided several heuristics. It is still an open problem how to jointly mix IANC and IRNC. The work in [11] considered the joint design of IANC and IRNC in wireless networks. However, the benefit was marginal and no theoretical analysis or guarantees were provided. In our previous work [12], and in [13], the joint IANC and IRNC in lossy 2-hop relay networks is considered. The work's optimized overhearing, not limitied to XOR, considered flows instead of packets and assumed limited feedback. The capacity region for the problem is characterized using linear equations when the number of sessions is less than three. For more than three sessions, a near-optimal coding scheme is provided and its performance is characterized using linear equations. The complexity of the near-optimal scheme is hyper exponential. Even though a near-optimal scheme is found, different problems are still open and need investigations.

In this work, we tackle some of these problems. The main contribution of this work is two-fold. (i) We develop a polynomial time coding scheme for the 2-hop relay network problem that makes a linear number of decisions and uses random network coding. We characterize the performance of the polynomial time scheme using linear constraints in terms of the links delivery ratios. (ii) Since achieving the full capacity region of multihop wireless networks is an open problem, we formulate an achievable rate region for general lossy multihop networks when using any achievable scheme for 2-hop relay networks as a building block. The formulation is also in terms of linear equations. We evaluate the effectiveness of our schemes in lossy wireless networks by simulating the different linear equations.

The rest of the paper is organized as follows: In Section II, we describe our settings. We then present the polynomial time algorithm in Section III. We provide an extension to the multihop networks case in Section IV and present our simulation results in Section V. We conclude the paper in Section VI.

II. The Setting

In an N-session 2-hop relay network, we have N sessions, each with a source and destination pair where source s_i would like to send packets to destination d_i, $\forall i \in \{1,\ldots,N\}$ with the help of the relay node r. Figure 2(a) represents a 2-session 2-hop relay network. PEC in the figure stands for *packet erasure channel* such that the sent packet by the source of the PEC is received by any subset of the receivers of the PEC. Each of s_i and r can use the corresponding PEC n times, respectively. Each of s_i would like to send $n \times R_i$ packets and we are interested in the largest achievable rate vector $(R_1,\ldots,R_N)$ that guarantees decodability of the packets sent by s_i at $d_i, \forall i$ with close-to-1 probability for sufficiently large n and finite field size. In this paper, we use p_{uv} to represent the delivery ratio of link (u,v), and we use X_i to represent the set of symbols representing the set of packets sent by node s_i.

To model the "reception report" suggested by practical implementations, we enforce the following sequential, round-based feedback schedule: Each of $s_i, \forall i \in \{1,\ldots,N\}$ transmits n symbols, respectively. After the transmission of $N \times n$ symbols, N reception reports are sent from $d_1,\ldots,d_N$, back to the relay r so that r knows which packets have successfully arrived at which destinations. After the reception reports, no further feedback is allowed and the relay r has to make its own decision of how to use the available n PEC usages to guarantee decodability at the destinations. In our setting, we also assume that the success probability parameters of all PECs and all coding operations are known to all nodes. The only unknown parts are the values of the sent packets by $s_i, \forall i$.

For the purpose of illustration, a simplified network setting is also depicted in Fig. 2(b), in which the packets sent by s_i will not be overheard by the 2-hop-away destination d_i. For future reference, we say Fig. 2(a) admits OpR, as the packets can be overheard by the two-hop destinations while Fig. 2(b) does not admit OpR. Without the loss of generality, we assume that $p_{rd_i} \geq p_{rd_{i+1}}, \forall i$, which can be achieved by relabelling the sessions.

For convenience, we use $nR_i^{I^cL}$ to denote the number of packets received by the destination nodes of the sessions in the set L and not received by the destination nodes of the sessions in the set I after node s_i sends n packets. We also use $X_i^{I^cL}$ to refer to the set of symbols representing these packets. For example, $nR_2^{1^c3}$ is the number of packets not received by d_1 and received by d_3 when node s_2 sends n packets. Also, $X_2^{1^c3}$ is used to denote the set of the symbols representing these packets. We also use X_i to refer to the symbols representing the packets sent by node s_i. Table I summarizes the symbols used for the 2-hop relay network results used in Sections III.

III. Low Complexity Algorithm

In this section, we provide a low complexity coding scheme to be implemented at the relay node. The scheme is described in Algorithm 1: After the sources send their

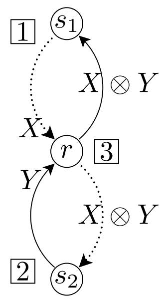
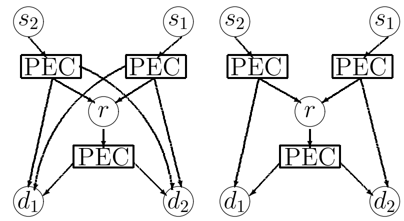


Fig. 1. A 2-flow network.



(a) With OpR (b) Without OpR

Fig. 2. Illustration of 2-session relay networks.

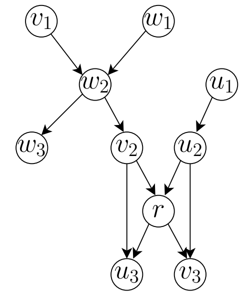


Fig. 3. Sample network.

Symbol	Definition
N	Number of sessions
i^c	Not session i
n	Number of time slots node r is scheduled
R_i	Rate of session i
X_i	Symbols representing the set of packets for session i
s_i	Source of session i
d_i	Destination of session i
r	Relay node
$t(u)$	Fraction of the time node u is scheduled
p_{uv}	Delivery rate between nodes u and v
$X_i^{I^c L}$	The packets sent by s_i and overheard by d_j, $\forall j \in L$. and not overheard by d_k, $\forall k \in I$.
$R_i^{I^c L}$	The rate of the packets represented by $X_i^{I^c L}$.

TABLE I
SUMMARY OF THE SYMBOLS USED FOR 2-HOP RELAY NETWORK.

Algorithm 1 Low Complexity Algorithm

$W_0 \leftarrow 0$;
for $i \leftarrow 1$ **to** N **do**
  if $i = N$ **then**
    $W_i = n$
  else
    $W_i \leftarrow n[\sum_{j:j<i} \frac{1}{p_{rd_j}} (R_j^{(i+1)^c} + \sum_{k:i>k>j} R_j^{k^c k+1 \dots i+1})]$
  end if
  Perform random linear network coding RLNC on the packets that belong to the following sets $\{X_1^{2\dots i}, \dots, X_{i-1}^i, X_i\}$
  Send the coded packets in $(W_i - W_{i-1})$ time slots.
end for

packets, the relay node receives one feedback packet from every destination so that it acquires the knowledge of the overheard packets. Based on this knowledge, the relay node finds the packets that belong to each set of the following: $\{X_1^{2\dots i}, \dots, X_{i-1}^i, X_i\}, \forall i \in \{1, \dots, N\}$, which we call the i-th collection of sets. For every i, the relay node performs random linear network coding RLNC on the packets of the i-th collection of sets and send the coded packets in $n \times (W_i - W_{i-1})$ time slots. We call these coded packets the i-th batch B_i. Also, W_i, $\forall i$ are auxiliary variables that we will use in the proofs. Note that we here mix both IANC and IRNC by coding packets of the same session and others among different sessions. The following theorem characterizes the performance of Algorithm 1:

Theorem 1: The following set of rates can be achieved by the coding scheme in Algorithm 1:

$$R_i + \sum_{j:j>i} R_j^{i^c} + \sum_{j:j<i} \frac{p_{rd_i}}{p_{rd_j}} (R_j^{i^c} + \sum_{k:i>k>j} R_j^{k^c k+1 \dots i}) \le p_{rd_i}, \forall i \quad (1)$$

The proof of this theorem is provided in the appendix.

Note that Algorithm 1 has a running time of $O(N)$.

The following corollary can be obtained from the examples above and from [12].

Corollary 1: When there are two sessions i.e. $N = 2$, Algorithm 1 achieves the capacity region of the network.

If we assume that the erasure patterns through the links are independent, we can characterize the performance of Algorithm 1 using a linear program where the only variables are $R_i, \forall i$, according to the following corollary:

Corollary 2: The achievable rate of Algorithm 1 can be represented by the following linear program that only requires the knowledge of the delivery rates of the links

$$R_i + \sum_{j:j>i} [R_j - p_{s_j d_i}]^+ + \sum_{j:j<i} \frac{p_{rd_i}}{p_{rd_j}} [R_j - \prod_{k:j<k\le i} p_{s_j d_k}]^+ \le p_{rd_i}.$$

Here, $[.]^+$ is a projection on $[0, \infty]$.

Proof: To maximize overhearing, the term R_{ji^c} in (1) can be rewritten as $[R_j - p_{s_j d_i}]^+$. The term $(R_j^{i^c} + \sum_{k:i>k>j} R_j^{k^c k+1 \dots i})$ in (1) can be rewritten as $R_j - R_j^{j+1 \dots i}$. Assuming that the channels are i.i.d, we have:

$$R_j - R_j^{j+1 \dots i} = [R_j - \prod_{k:j<k\le i} p_{s_j d_k}]^+,$$

which completes the proof. ■

When flexible scheduling is used. i.e. node u is scheduled for $t(u)$ fraction of the time, Algorithm 1 can be modified to Algorithm 2.

Algorithm 2 Flexible Scheduling Algorithm

$W_0 \leftarrow 0$
for $i \leftarrow 1$ **to** N **do**
  if $i = N$ **then**
    $W_i = n$
  else
    $W_i \leftarrow n[\sum_{j:j<i} \frac{1}{p_{rd_j}}(R_j^{(i+1)^c} + \sum_{k:i>k>j} R_j^{k^c k+1 \ldots i+1}]$
  end if
  Perform RNC on the packets that belong to the following sets $\{X_1^{2\ldots i}, \ldots, X_{i-1}^{i}, X_i\}$
  Send the coded packets in $t(r) \times (W_i - W_{i-1})$ time slots.
end for

Corollary 3: When flexible scheduling is used. i.e. node u is scheduled for $t(u)$ fraction of the time, the achievable rate of Algorithm 2 is

$$R_i + \sum_{j:j>i} [R_j - t(s_j) p_{s_j d_i}]^+ + \sum_{j:j<i} \frac{p_{rd_i}}{p_{rd_j}} [R_j - t(s_j) \prod_{k:j<k\leq i} p_{s_j d_k}]^+ \leq t(r) p_{rd_i}.$$

If the 2-hop relay network admits OpR, the relay network can perform coding on only the vectors in the complementary space of the vectors directly received by the intended receiver, which gives us the following corollary:

Corollary 4: If the 2-hop relay network admits OpR, the achievable rate of algorithm 2 is:

$$R_i + \sum_{j:j>i} [R_j - t(s_j)(p_{s_j d_j} + (1 - p_{s_j d_j}) p_{s_j d_i})]^+ + \sum_{j:j<i} \frac{p_{rd_i}}{p_{rd_j}} [R_j - t(s_j)(p_{s_j d_j} + (1 - p_{s_j d_j}) \prod_{j<k\leq i} p_{s_j d_k})]^+ \leq t(s_1) p_{s_1 d_1} + t(r) p_{rd_i}$$

IV. Extensions to The Multihop Case

In this section, we build on the 2-hop relay network results to study the throughput and fairness benefits of using IANC and IRNC jointly in multihop wireless networks. We provide a linear programming formulation of an achievable rate region that uses the 2-hop relay networks results as building blocks.

We consider a general multihop wireless network represented by a graph $G = (V, E)$, where V is the set of vertices representing the nodes and E is the set of edges representing the links between the nodes. Transmission by a node can be overheard by multiple nodes, which we model by a hyperarc (u, J), where u is the transmitter and J is a subset of the set of direct receivers. There are N sessions in the network. For every session i, the source node s_i wants to send packets at rate R_i to the session's destination node d_i over possibly multiple intermediate nodes. We use $\mathbb{P}(i)$ to refer to the path used for session i. For every node u on path $\mathbb{P}(i)$, $V_1(u,i)$ ($U_1(u,i)$) represents the next-hop (previous-hop) node, respectively, on that path and $V_2(u,i)$ ($U_2(u,i)$) represents the next-hop (previous-hop), respectively, node of $V_1(u,i)$ ($U_1(u,i)$) on $\mathbb{P}(i)$.

Symbol	Definition
u, v	Intermediate nodes
$\mathbb{P}(i)$	Path used for session i
$V_1(u,i)$	Next hop node of u on $\mathbb{P}(i)$
$V_2(u,i)$	Next hop node of $V_1(u,i)$ on $\mathbb{P}(i)$
$U_1(u,i)$	Previous hop node of u on $\mathbb{P}(i)$
$U_2(u,i)$	Previous hop node of $U_1(u,i)$ on $\mathbb{P}(i)$
$y_{uv}(i)$	Rate of IANC packets for session i through link (u,v)
$y'_{uv}(i)$	Same as $y_{uv}(i)$, but for joint IANC and IRNC packets
$t(u,i)$	Fraction of the time node u is scheduled to send IANC packets of session i
$t'(u)$	Fraction of the time node u is scheduled to send joint IANC and IRNC packets
P_u^J	The probability that packet sent by node u is overheard by any node in J
$c_k(u)$	The k-th session that uses node u.
$\gamma_{u,v}(i)$	By definition equals to $\frac{y_{uv}(i)}{p_{uv}}$.

TABLE II
Summary of the symbols used in the multihop case.

We assume that every sent packet is either a packet formed by performing IANC for the packets of one session or joint IRNC and IANC for different sessions' packets. This includes the case of sending non-coded packets as a special case. We use $y_{uv}(i)$ to represent the rate of linearly independent IANC packets for session i that are sent by node u and can be decoded by node v, only if node v is d_i, or can be forwarded by v. Symbol $y'_{uv}(i)$ represents the same as $y_{uv}(i)$, but for packets with joint IANC and IRNC. The fraction of time node u is scheduled for sending session i IANC coded packets is represented by $t(u,i)$. $t'(u)$ represents the fraction of time node u is scheduled to send joint IANC and IRNC packets. Symbol $t(u)$ represents the fraction of time node u is scheduled. Table II represents the symbols used for the multihop case. To avoid the use of complex notations, we use in this section $y_{uv}, y'_{uv}, \gamma_{uv}, V_1(u), V_2(u), U_1(u), U_2(u)$ to represent $y_{uv}(i), y'_{uv}(i), \gamma_{uv}(i), V_1(u,i), V_2(u,i), U_1(u,i)$, $U_2(u,i)$, respectively.

Using IANC only and assuming we do not have specified paths, the linear constraints that specify the capacity region are as follows:

$$\sum_{v:u\neq v} y_{vu} - \sum_{v:u\neq v} y_{uv} \leq \begin{cases} -R_i & u = s_i \\ 0 & \text{Else}, \end{cases} \quad (2)$$

$$\forall i, \forall u \in E \backslash d_i \quad (3)$$

$$\sum_{v:v\in J} y_{uv} \leq t(u,i) p_u^J, \qquad \forall (u,J), \forall i, \quad (4)$$

where p_u^J is the probability that any node in J receives the packet. The constraints in (2) represent balance equations such that the total received linearly independent packets, and the total generated packets at a node, should be at most equal to the totally sent linearly independent ones. Constraint (4) states that for any set of nodes that can receive the sent packets by a specific node, the total number of linearly independent packets per unit time that these nodes can forward equals to the probability that any one of these nodes received the packet which is p_u^J. If the paths are not specified, the solution of ((2)-(4)) will result in a back-pressure algorithm, which has bad delay performance and might not converge to the optimal solution, as noted in [14]. Therefore, in the following, we study the case of specified paths. The formulation becomes:

$$\begin{aligned} & y_{U_1(u)u} + y_{U_2(u)u} \\ & -y_{uV_1(u)} - y_{uV_2(u)} \leq \begin{cases} -R_i & u = s_i \\ 0 & \text{Else}, \end{cases} \\ & \forall i, \forall u \in E \backslash d_i, \end{aligned} \quad (5)$$

$$y_{uv} \leq t(u,i) p_{uv}, \forall i, \qquad \forall u,v \in \mathbb{P}(i) \quad (6)$$

$$\begin{aligned} y_{uV_1(u)} + y_{uV_2(u)} \leq t(u,i)(&p_{uV_1(u)} + p_{uV_2(u)} \\ &- p_{uV_1(u)} p_{uV_2(u)}) \forall u \in \mathbb{P}(i) \end{aligned} \quad (7)$$

The above formulation can be obtained from ((2)-(4)) by noting that the only hyperarc for node u through the path for session i is the one with the receivers being $V_1(u)$ and $V_2(u)$. This modeling agrees with practical implementations of IANC, as in [6], [15], which state that overhearing of a node transmission over a path happens only for one and two hop away nodes.

As it is hard to jointly use IANC and IRNC in multihop networks, we provide a restriction that allows us to use the 2-hop relay networks results as a building block in large multihop networks. The restriction is that joint IANC and IRNC opportunities are limited to be in the form of 2-hop relay networks, where the coding node is the relay node and the set of decoding nodes is a subset of the next-hop nodes of the relay node. Therefore, the general lossy multihop network can be decomposed into a superposition of IANC alone traffic and joint IANC and IRNC traffic in 2-hop relay networks.

The assumption we have, has the following implications on the capacity region: **(1)** In Fig. 3, if there are two sessions, one of them goes through the path $v_1w_2v_2rv_3$, the other one goes through $u_1u_2ru_3$, and joint IANC & IRNC can happen at node r. Due to our restriction, we assume there is no side information from w_2 to u_3, nor from u_1 to v_3. If such side information exists, we ignore it. **(2)** If in the same figure there is another session that goes through $w_1w_2w_3$, and node w_2 is acting as a relay node for performing joint IANC & IRNC between this session and the session that goes through $v_1w_2v_2rv_3$, and the joint IANC and IRNC packets are overheard by r, node r will deal with these packets as useless and drop them.

Under the assumption that the channels are independent, for N-sessions 2-hop relay networks, let $\boldsymbol{t} \triangleq [t(r), t(s_1), \ldots, t(s_N)]$, $\mathbf{p_s} \triangleq [p_{s_1r}, \ldots, p_{s_Nr}]$, $\mathbf{p_d} \triangleq [p_{rd_1}, \ldots, p_{rd_N}]$, $\mathbf{p_{sd}} \triangleq [\mathbf{p_{s_1d}}, \ldots, \mathbf{p_{s_Nd}}]$, and $\mathbf{p_{s_ld}} \triangleq [p_{s_ld_1}, \ldots, p_{s_ld_{l-1}}, p_{s_ld_{l+1}}, \ldots, p_{s_ld_N}]$. In the case when d_l cannot overhear s_l, we use

$$\begin{aligned} &\text{CapA}(\boldsymbol{t}, \mathbf{p_{sd}}, \mathbf{p_s}, \mathbf{p_d}) = \Big\{ (R_1, \ldots, R_N) : \\ &\text{The rates } R_1, \ldots, R_N \text{ satisfy the linear programming} \\ &\text{constraints for the capacity region } A \Big\}. \end{aligned}$$

A is any achievable rate region for the 2-hop relay network that uses IANC and IRNC jointly. It could be the optimal capacity region in [13], or an approximation of it, as in this paper. For example, when $N = 2$, the rates that satisfy

$$\begin{aligned} R_1 &\leq \min(t(s_1)p_{s_1r}, t(r)p_{rd_1} - (R_2 - t(s_2)p_{s_2d_1})^+) \\ R_2 &\leq \min(t(s_2)p_{s_2r}, t(r)p_{rd_2} \\ &\qquad - (R_1 - t(s_1)p_{s_1d_2})^+ \frac{p_{rd_2}}{p_{rd_1}}) \end{aligned}$$

belong to CapA, where A is the optimal capacity region.

When d_l can overhear s_l, and if d_l forwards $\gamma_l p_{s_ld_l}$ linearly independent symbols of the overheard packets, or decodes them if it is the last destination of the packets, joint IRNC and IANC should happen for the symbols in the complementary spaces of the forwarded or decoded symbols. Letting $\boldsymbol{\gamma} \triangleq [\gamma_1, \ldots, \gamma_N]$ and $\mathbf{p'_{sd}} \triangleq [p_{s_1d_1}, \ldots, p_{s_Nd_N}]$, we use

$$\begin{aligned} &\text{Cap}'\text{A}(\boldsymbol{\gamma}, \boldsymbol{t}, \mathbf{p'_{sd}}, \mathbf{p_{sd}}, \mathbf{p_s}, \mathbf{p_d}) = \\ &\big\{ (R'_1 = R_1 - \gamma_1 p_{s_1;d_1}, \ldots, R'_N = R_N - \gamma_N p_{s_N;d_N}) : \\ &\text{The rates } R_1, \ldots, R_N \text{ satisfy the linear programming} \\ &\text{constraints with OpR when } d_l \text{ can overhear } s_l. \big\}. \end{aligned}$$

For example, when N=2, any (R'_1, R'_2) that satisfy the following constraints belong to $\text{Cap}'\text{A}(\boldsymbol{\gamma}, \boldsymbol{t}, \mathbf{p'_{sd}}, \mathbf{p_{sd}}, \mathbf{p_s}, \mathbf{p_d})$, where A is the optimal capacity region.

$$R'_1 \le \min(Y_1, t(r)p_{rd_1} - (R_2 - Z_1)^+)$$
$$R'_2 \le \min(Y_2, t(r)p_{rd_2} - (R_1 - Z_2)^+)\frac{p_{rd_2}}{p_{rd_1}}),$$

where Y_1, Y_2, Z_1, Z_2 satisfy the following:

$$Y_1 \le t(s_1)(p_{s_1r} + p_{s1d_1} - p_{s_1d_1}p_{s_1r}) - \gamma_1 p_{s_1d_1}.$$
$$Y_1 \le t(s_1)p_{s_1r}.$$
$$Y_2 \le t(s_2)(p_{s_2r} + p_{s2d_2} - p_{s_2d_2}p_{s_2r}) - \gamma_2 p_{s_2d_2}$$
$$Y_2 \le t(s_2)p_{s_2r}.$$
$$Z_1 \le t(s_2)(p_{s_2d_1} + p_{s2d_2} - p_{s_2d_1}p_{s_2d_2})$$
$$Z_1 \le t(s_1)(p_{s_1d_2} + p_{s1d_1} - p_{s_1d_2}p_{s_1d_1})$$

Using random network coding and when considering the symbols directly received from s_l by any d_m or r, any two symbols related to two different received packets are linearly independent. Therefore, using the feedback, the relay will be able to know the coefficients related to the received packets by its next-hop nodes to generate packets with coefficients in their complementary space. The following linear equations represent an achievable rate region that uses joint IANC and IRNC:

$$y_{U_1(u)u} + y_{U_2(u)u} + y'_{U_1(u)u}$$
$$-y_{uV_1(u)} - y_{uV_2(u)} - y'_{uV_1(u)} \le \begin{cases} -R_i & u = s_i \\ 0 & \text{Else,} \end{cases}$$
$$\forall i, \forall u \in E \backslash d_i, \tag{8}$$
$$y_{uv} = \gamma_{uv}p_{uv} \le t(u,i)p_{uv}, \forall i, \qquad \forall u, v \in \mathbb{P}(i) \tag{9}$$
$$y_{uV_1(u)} + y_{uV_2(u)} \le t(u,i)(p_{uV_1(u)} + p_{uV_2(u)} - p_{uV_1(u)}p_{uV_2(u)}), \forall u \in \mathbb{P}(i). \tag{10}$$

$$(y'_{uV_1(u,1)}(1), \dots, y'_{uV_1(u,k)}(k)) \in \mathrm{Cap}'\mathrm{A}(\boldsymbol{\gamma}^u, \boldsymbol{t}^u, \mathbf{p'^u_{sd}}, \mathbf{p^u_{sd}}, \mathbf{p^u_s}, \mathbf{p^u_d}). \tag{11}$$

Here, k is the number of sessions intersecting at node u. To avoid the complex notations, we assume that these sessions are $1, \dots, k$. Also, we have
$\boldsymbol{\gamma}^u \triangleq [\gamma_{U_1(u,1)V_1(u,1)}(1), \dots, \gamma_{U_1(u,k)V_1(u,k)}(k)]$,
$\boldsymbol{t}^u \triangleq [t'(u), t(U_1(u,1),1), \dots, t(U_1(u,k),k)]$,
$\mathbf{p'^u_{sd}} \triangleq [p_{U_1(u,1)V_1(u,1)}, \dots, p_{U_1(u,k)V_1(u,k)}]$,
$\mathbf{p^u_{s_l d}} \triangleq [p_{U_1(u,l)V_1(u,1)}, \dots, p_{U_1(u,l)V_1(u,l-1)}, p_{U_1(u,l)V_1(u,l+1)}, \dots, p_{U_1(u,l)V_1(u,k)}]$,
$\mathbf{p^u_{sd}} \triangleq [\mathbf{p^u_{s_1d}}, \dots, \mathbf{p^u_{s_kd}}]$,
$\mathbf{p^u_s} \triangleq [p_{U_1(u,1)u}, \dots, p_{U_1(u,k)u}]$,
and $\mathbf{p^u_d} \triangleq [p_{uV_1(u,1)}, \dots, p_{uV_1(u,k)}]$.

For session i, any node u has three different kinds of incoming packets and three different kinds of outgoing packets. The incoming packet types are IANC packets received from a previous hop with rate $y_{U_1(u)u}$, IANC packets overheard from a two-hop away node with rate $y_{U_2(u)u}$, and joint IANC and IRNC packets received from a previous hop with rate $y'_{U_1(u)u}$. Note that due to the restriction we have, joint IANC and IRNC packets overheard from two-hop away nodes are dropped. The outgoing packets can also be classified as joint IRNC and IANC packets with rate $y'_{uV_1(u)}$, IANC packets that are received and used by the next-hop with rate $y_{uV_1(u)}$, and IANC packets that are overheard and used by the next two-hop away nodes with rate $y_{uV_2(u)}$.

The constraints in (8) state that at every node, and for every session, the total incoming traffic at a node should be equal to the total outgoing traffic. The constraints (9)-(10) are for IANC and are the same as in the previous section. Constraints (11) specify the joint IANC and IRNC rate at node u by treating it as a relay node in a 2-hop relay network. Due to the restriction we have, at node u, only the incoming IANC traffic from a previous hop can be used for joint IANC and IRNC at node u. This is reflected in the formulation by using $\boldsymbol{t}^u$ as the second argument of $\mathrm{Cap}'\mathrm{A}$, which only contains the IANC scheduling frequency of the previous-hop nodes of node u. Since $\gamma_{U_1(u,l)V_1(u,l)}(l)p_{U_1(u,l)V_1(u,l)}$ is the rate of the IANC packets for session l that are sent by node $U_1(u,l)$, overheard by the node $V_1(u,l)$, and used by that node, the first argument in $\mathrm{Cap}'\mathrm{A}$ states that joint IANC and IRNC is performed in the complementary space of the symbols related to these packets.

V. Simulations

In this section, we present several simulation results to show the effectiveness of our approximation scheme in 2-hop relay networks, and show the improvement that joint IANC and IRNC schemes can provide for the multihop case.

We start from a unit circle with the relay r placed at the center. We then uniformly at random place N source nodes s_i, and N destination nodes d_i, in the circle (see Fig. 5). The only condition we impose is that for each (s_i, d_i) pair, d_i must be in the 90-degree pie area opposite to s_i (see Fig. 5). For each randomly constructed network, we use the Euclidean distance between each node to determine the overhearing probability. More explicitly, for any two nodes separated by distance D, we use the Rayleigh fading model to decide the overhearing probability $p = \int_{T^*}^{\infty} \frac{2x}{\sigma^2} e^{-\frac{x^2}{\sigma^2}} dx$, where we choose $\sigma^2 \triangleq \frac{1}{(4\pi)^2 D^\alpha}$, the path loss order $\alpha = 2.5$, and the decodable SNR threshold $T^* = 0.06$.

We assume that the overhearing event is independent among different receivers.

For each randomly generated network, we compute the overhearing probabilities and use the corresponding linear constraints on the time-sharing variables t's and the rate variables R's to compute the achievable rate of each scheme.

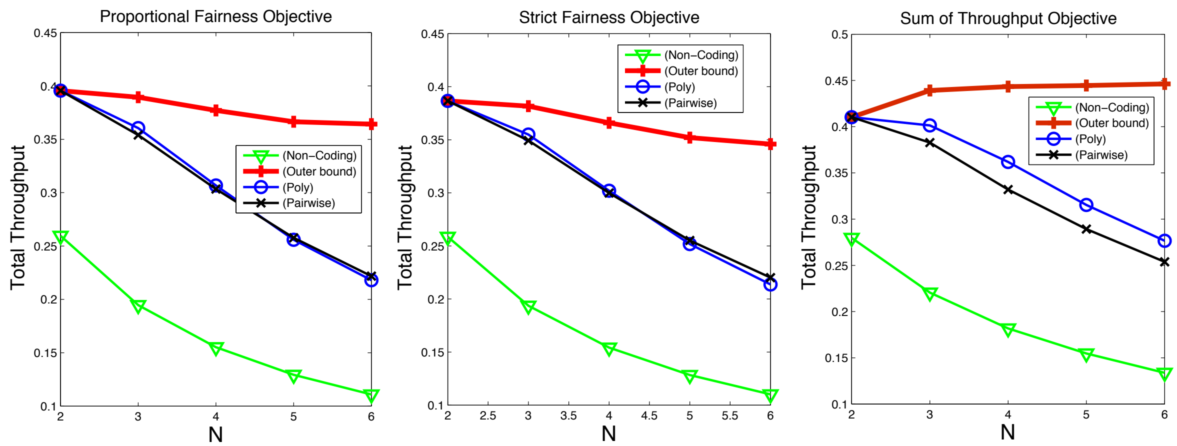


Fig. 4. Average throughput results for 1000 topologies with different objective functions when using no OpR and round robin scheduling.

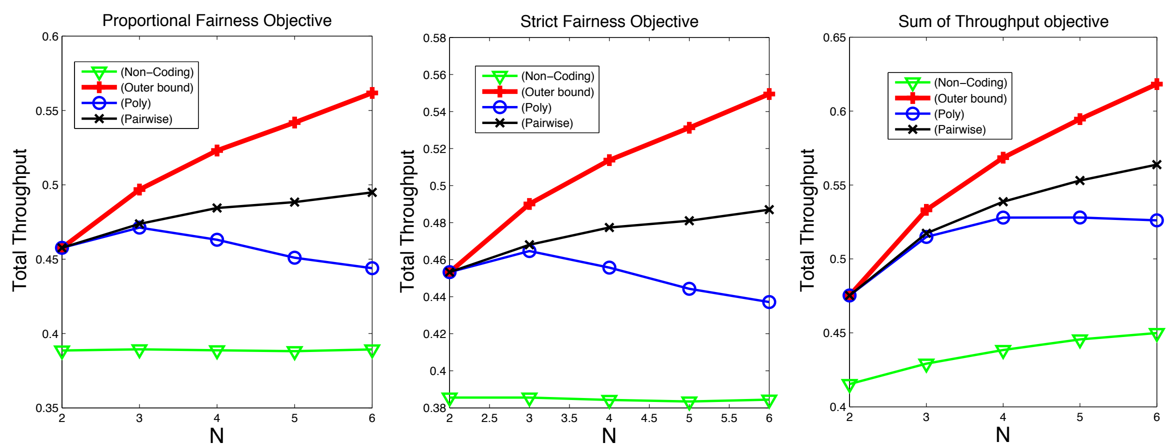


Fig. 6. Average throughput results for 1000 topologies with different objective functions when using no OpR and optimal scheduling weights.

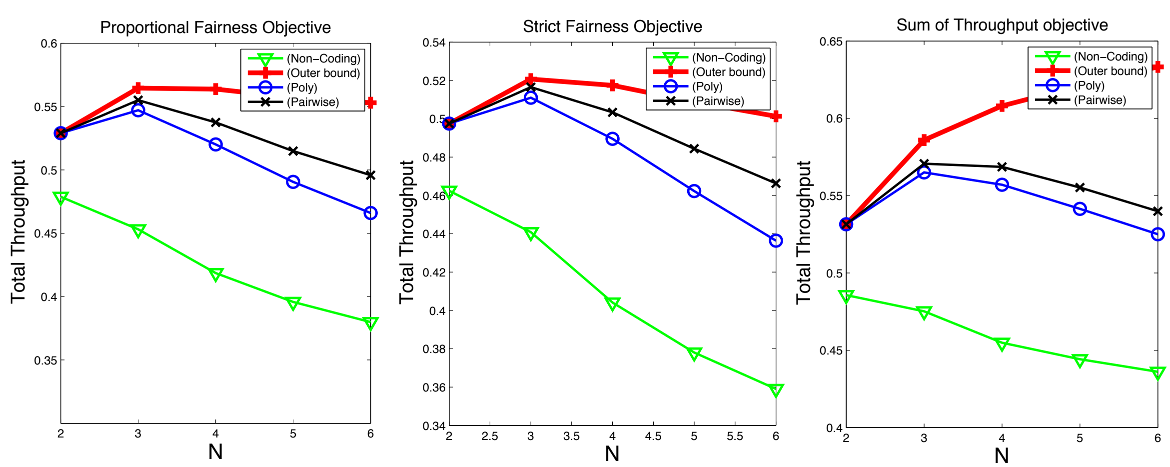


Fig. 7. Average throughput results for 1000 topologies with different objective functions when using OpR and round robin scheduling.

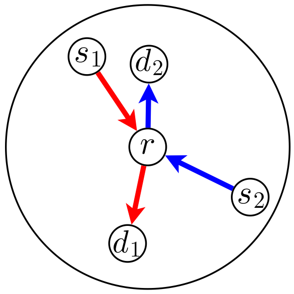


Fig. 5. Simulation settings.

Given a randomly generated network, the achievable sum rates are computed for all the schemes. We then repeat this computation for 1000 randomly generated networks. Let $\zeta^*_{\text{scheme},k}$ denote the achievable sum rate of the given scheme for the k-th randomly chosen topology. We are interested in the following two performance metrics: The average sum rate over 1000 topologies $\frac{1}{1000}\sum_{k=1}^{1000}\zeta^*_{\text{scheme},k}$ and per topology improvement $\triangleq \frac{\zeta^*_{\text{scheme},k}-\zeta^*_{\text{baseline},k}}{\zeta^*_{\text{baseline},k}}$.

In our two-hop relay network simulations, we change three different setting parameters. These are: **(a)** the use of OpR where we consider OpR or no OpR, **(b)** node scheduling method where we use round robin scheduling or include the weight of scheduling as a new variable in the optimization problem. If the weights of scheduling are included in the optimization problem, we call such a scheme the optimal scheduling scheme, and **(c)** the objective function where we consider three objective functions. The objective functions that we use are: (1) Proportional fairness, such that the weight of session i is the available bandwidth for that session when no other sessions share the network; (2) strict fairness that requires the throughput of all of the sessions to be the same; (3) maximizing the sum of the throughput of the sessions.

Fig. 4 shows the average throughput achieved by our approximation scheme and the optimal one when no OpR and round robin scheduling is used. As shown in the figure, our scheme achieves 65%–95% of the optimal solution depending on the number of sessions. Also, our scheme performs similar to the *pairwise scheme* [12], which requires coordination among different nodes and has a complexity of $O(N^2)$. Figs. 6,7, and 8 show similar results when using OpR with the optimal scheduling.

Fig. 9 compares the achievable throughput of our approximation scheme compared to other XOR-based schemes. These schemes are COPE [7], CLONE [10], the capacity achieving scheme with XOR coding [16]. The results in the figure show that our scheme performs very close to the best XOR based scheme, while the

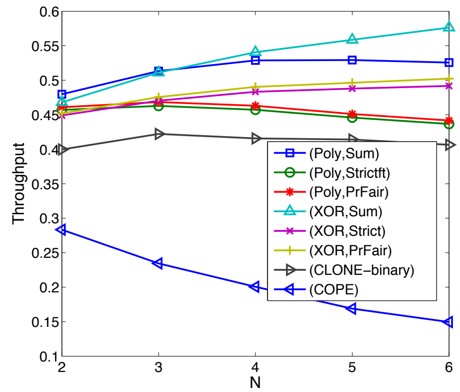


Fig. 9. Comparison of the achievable rate using different schemes.

best XOR based scheme has an exponential complexity. The figure also shows that our scheme improves the throughput $1.5-3.5$ folds compared to COPE, the state of the art XOR coding scheme and by 25% over CLONE, the state of the art XOR coding scheme optimized to work with lossy links. The figure shows that even when the objective is to achieve fairness, our scheme still improve the throughput over CLONE by $10-12\%$.

Fig. 10 shows the CDF function of the per topology improvement of our schemes compared to COPE when $N=6$. The results in the figure shows that there are topologies where the throughput improvement is four folds. Also for half of the topologies the improvement is over 2.7 folds. The results shows also that even when the objective is fairness, the throughput improvement is large. Figure 11 shows the CDF function for the per topology improvement over CLONE-binary for 1000 topologies when $N=6$. The results show that there are topologies where the throughput improvement witnessed by our scheme is over 60%. Also, when the objective is to achieve fairness, more than 90% of the topologies shows throughput improvement over CLONE-binary.

VI. Conclusion

In this paper, we have studied the capacity of lossy 2-hop relay networks. As the optimal solution has a high complexity, we provide a linear time approximation algorithm and characterize its performance using linear constraints. Our approximation algorithm uses random linear network coding and carefully mixes IANC with IRNC. We also show the effectiveness of our scheme through simulations.

For the achievable rate region in multihop wireless networks, we provide a linear programming formulation and show the improvements through simulations.

The results presented in this paper opens many directions for future research. Following are some of these

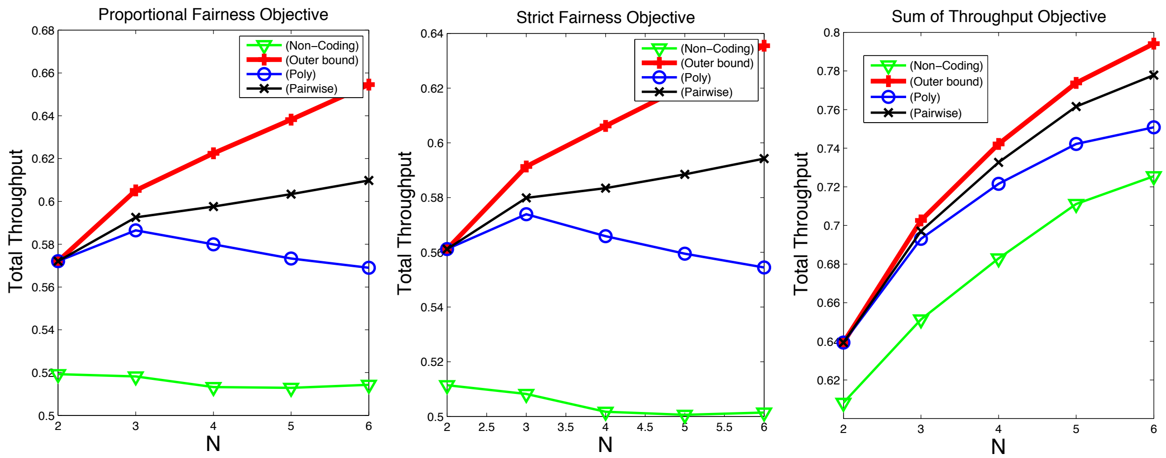


Fig. 8. Average throughput results for 1000 topologies with different objective functions when using OpR and optimal scheduling weights.

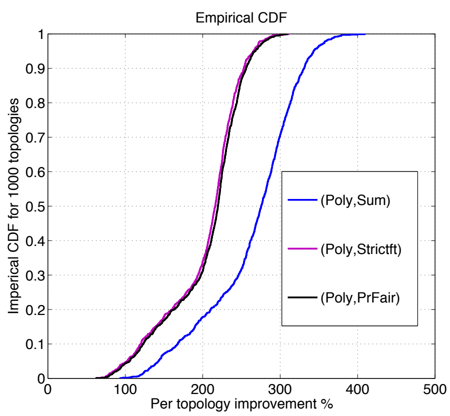


Fig. 10. The empirical CDF for the per topology improvement over COPE with $N = 6$.

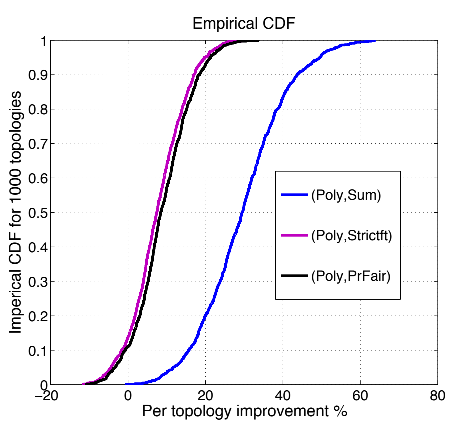


Fig. 11. The empirical CDF for the per topology improvement over CLONE-binary with $N = 6$.

directions: **(1)** Designing cross-layer and distributed algorithms for the multihop case. The loose coupling approach [3] can be used to achieve close to the capacity with minimal overhead. **(2)** Studying other fundamental questions about different types of coding like *pairwise network coding* in lossy wireless networks, which we studied before with lossless links [5], [18].

VII. Acknowledgment

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APPENDIX A
PROOF OF THEOREM 1

Proof: We note that $\sum_{i=1}^{N}|B_i| = n$. Therefore, there is enough time for the relay node to send the batches.

After the transmission of the N batches, d_i uses packets in batches $B_i, \ldots, B_N$ and the packets it has overheard to decode its own packets which belong to the set X_i. For d_i, the symbols that are coded together in batches $B_i, \ldots, B_N$ which are of interest to node d_i belong to one of these sets: 1) The set of symbols overheard previously by node d_i; 2) the set X_i which is of interest to node d_i; 3) any set of the form $X_j^{i^c}, \forall j > i$. Since we are using linear coding, d_i can subtract any linear combination of the packets it has overheard before. Therefore, the achievability of R_i in (1) is guaranteed, if d_i can decode all symbols in X_i and $X_j^{i^c}, \forall j > i$ from batches $B_i, \ldots, B_N$. Not all subsets of X_i exist in all B_i to B_N batches. Therfore, by using random coding and for d_i to decode all packets in the set X_i and in the sets $X_j^{i^c}, \forall j > i$, three conditions have to be guaranteed:

(1) Using (1), the number of packets in batches B_i to B_N that node d_i receives should be at least $n(p_{rd_i} - \sum_{j:j<i} \frac{p_{rd_i}}{p_{rd_j}}(R_j^{i^c} + \sum_{k:i>k>j}(R_j^{k^c k+1\ldots i})))$. This is because d_i is interested in decoding the packets in the sets X_i and $X_j^{i^c}$ from the batches B_i to B_N. The number of these packets is $nR_i + n\sum_{j:j>i} R_j^{i^c}$, and by (1), we have $nR_i + n\sum_{j:j>i} R_j^{i^c} \le n(p_{rd_i} - \sum_{j:j<i} \frac{p_{rd_i}}{p_{rd_j}}(R_j^{i^c} + \sum_{k:i>k>j} R_j^{k^c k+1\ldots i})), \forall i$.

(2) If packets in the set X_i^J are not part of the coded packets in all of the batches that d_i uses for decoding, then the number of packets that i receives from the batches where packets in X_i^J are used should be at least nR_i^J.

(3) The number of packets that node d_i receives from the batches that contain packets of the set $X_j^{i^c}, \forall j > i$ should be at least $nR_j^{i^c}$.

In the following, we show that the three conditions are satisfied:

(1) Node d_i receives packets from batches B_i to B_N through link (r, d_i). Therefore, the total number of coded packets that d_i uses for decoding from these batches are

$$\begin{aligned}
&(|B_i| + \ldots + |B_N|)p_{rd_i} \\
&= np_{rd_i}(1 - W_{i-1}) \\
&= np_{rd_i}(1 - \sum_{j:j<i}\frac{1}{p_{rd_j}}(R_j^{i^c} + \sum_{k:i>k>j}(R_j^{k^c k+1\ldots i}))) \\
&= n(p_{rd_i} - \sum_{j:j<i}\frac{p_{rd_i}}{p_{rd_j}}(R_j^{i^c} + \sum_{k:i>k>j}(R_j^{k^c k+1\ldots i}))).
\end{aligned}$$

(2) The symbols that belong to X_i and are not coded in all batches B_i to B_N belong to the sets $X_i^{l^c}$, $\forall l > i$. Symbols in the set $X_i^{l^c}$ are coded in batches B_i–B_{l-1}. Showing that from batch B_{l-1} $\forall l > i$, node i will receive $R_i^{i+1\ldots l-1 l^c}$ coded packets where symbols in $X_i^{i+1\ldots l-1 l^c}$ are used along with other symbols to generate the coded packets, which is equivalent to showing that $\forall l > i$. Node d_i will receive $R_i^{l^c}$ of the coded packets where symbols in $X_i^{l^c}$ are used along with other symbols to generate the coded packets. Let $W_l = n\sum_{i=1}^{l-1}\frac{1}{p_{rd_i}}W_{li}$, where $W_{li} = R_i^{(l+1)^c} + \sum_{k:i<k\le l} R_i^{k^c k+1\ldots l+1}$. We have $W_{li} - W_{(l-1)i} = R_i^{i+1\ldots l(l+1)^c} \ge 0, \forall l > i$. Therefore, the number of packets in batch B_{l-1} that node i receives will be

$$\begin{aligned}
&p_{rd_i}(W_{l-1} - W_{l-2}) \\
&= np_{rd_i}\big(\frac{1}{p_{rd_l}}W_{l-1l-2} + \sum_{j=1}^{l-3}\frac{1}{p_{rd_j}}(W_{l-1j} - W_{l-2j})\big) \\
&\ge R_i^{i+1\ldots i-1l^c}.\forall l > i.
\end{aligned}$$

(3) From Condition 2, the number of received packets by node d_j in the batches that contain coded packets in the set $X_j^{i^c}$ is $nR_j^{i^c}$. Since $i < j$, the channel between (r, d_j) is weaker than the channel between (r, d_i). Therefore, the number of received packets by node d_i in the batches that contain coded packets in the set $X_j^{i^c}$ is greater than or equal to $nR_j^{i^c}$. ∎