On the Role of Mobility and Network Coding on Multi-message Gossip in Random Geometric Graphs

Gang Wang*, Zun Lin* and Jie Wu†

*School of Electrical and Information Engineering, Beihang University
Beijing, China, 100191
†Department of Computer and Information Sciences, Temple University
Philadelphia, PA, USA
Email: gwang@buaa.edu.cn, 69849545@qq.com, jiewu@temple.edu

***Abstract*—In this paper, the performance of network coding-based gossip algorithms—i.e. algebraic gossip algorithms—is analyzed on random geometric graphs under static and mobile environments. The lower bounds for the convergence time of algebraic gossip algorithms are derived based on the conductance, and these bounds are $O(n \log n \log \varepsilon^{-1} - \log n \log \varepsilon^{-1})$ with node mobility and $O\left(\frac{(n^{3/2}-n^{1/2}) \log \varepsilon^{-1}}{\log^{1/2} n}\right)$ without node mobility. Theoretical results show that algebraic gossip algorithms with node mobility converge $O\left(\frac{n^{1/2}}{\log^{3/2} n}\right)$ more quickly than without node mobility, and $O(\log n)$ more quickly than a gossip algorithm with node mobility but without network coding. Finally, we assess and compare the convergence time of various gossip algorithms, with both mobility and network coding. As corroborated by extensive numerical experimentation, integrated network coding with mobility can significantly improve the convergence time of information dissemination in dynamic environments.**

***Index Terms*—Network Coding, Algebraic Gossip, Mobility, Conductance, Random Geometric Graph.**

I. Introduction

With the increasing demand for sharing distributed information on large-scale communication networks, efficient information dissemination in large dynamic networks has emerged as an important research field. A random gossip algorithm is a simple and efficient algorithm for disseminating information over a wide range of distributed applications with the advantage of scalability, robustness, and effectiveness [1].

Because network coding has many advantages in terms of improving the network performance [2], Deb *et al.* developed an algebraic gossip algorithm by integrating gossip with network coding, rather than a naive store-and-forward mechanism, in order to rapidly disseminate all of the messages among all nodes [3]. The further generalized result was developed for an arbitrary graph in [4]. The performance of a uniform algebraic gossip algorithm was analyzed in [5] based on Projection Analysis method. Vasudevan *et al.* derived the uniform bound of the convergence time of algebraic gossip for an arbitrary graph, with an expected stopping time of $O(nlog^2n)$ rounds, where n denotes the number of nodes in the network [6]. A novel analytical approach to analyze the algebraic gossip algorithm based on Queuing Theory was proposed, which the convergence time in any graph for all-to-all communications was bounded by $O(n\Delta)$ rounds [7].

Node mobility offers many opportunities for improving certain aspects of a network's performance, such as its capacity. Sarwate *et al.* showed that node mobility could further reduce the convergence time for disseminating the information from randomized gossip [8]. [9] analyzed the influence of different mobility models on random gossip algorithms for single-message dissemination. A simple two-stage dissemination strategy that alternates between message flooding and random gossip was proposed for mobile ad-hoc networks in [10], and the convergence time of their algorithm was analyzed under velocity-constrained mobility models. Their proposed algorithm can approach the optimal bound within $O(\log n)$ rounds in terms of the convergence time, provided that the velocity is at least $v = \log n/k$, where k is the number of users intending to share a unique message with others.

The above results show that both network coding and node mobility can reduce convergence time. However, it is still unclear whether integrating network coding with node mobility will further improve the convergence time of gossip algorithms. Wang *et al.* conducted a theoretical analysis on the performance of multi-message algebraic gossip algorithms on random geometric graphs (RGG) [11].

In this paper, we analyze the effect of network coding and mobility on gossip algorithms on random geometric graphs. We assume that the nodes move in a decentralized and distributed manner. During every timeslot, each node communicates with a neighbouring node that is randomly chosen by an exchange algorithm. Regardless of whether network coding and node mobility are considered, we analyze four gossip algorithms: the Mobile Algebraic Gossip (MAG) algorithm, the Static Algebraic Gossip (SAG) algorithm, the Mobile Gossip (MG) algorithm, and the Static Gossip (SG) algorithm. Based on the concept of k-conductance and mobile k-conductance in graph theory, we uniformly derive the bounds for the convergence time of these gossip algorithms in random geometric graphs, which is a basic and important topology for representing mobile networks. Our results show that when both network coding and node mobility are jointly utilized, the bounds for the convergence time can be improved by a magnitude of $O(\frac{n^{1/2}}{\log^{3/2} n})$ compared to when only network coding is used, and by a magnitude of $O(\log n)$ compared to when only node mobility is considered. The extensive numerical results confirm the theoretical results, which means that integrating network coding with node mobility in information dissemination improves the performance significantly.

The remainder of the paper is organized as follows. In Section II, we introduce the network model, algorithm, and protocols. Our main theoretical results and the essential proofs are provided in Section III, as well as the simulation results and a detailed analysis of other algorithms. Finally, conclusions are drawn in Section V.

II. MODEL AND PROTOCOL

A. Network Model

Consider a random geometric graph consisting of n nodes, denoted by the set of $V = \{1, 2, \ldots, n\}$ [12]. The n nodes are distributed independently and uniformly at random in the l-dimensional space R^l. Two nodes (u, v) are connected if and only if the distance between them is less than or equal to a threshold r, i.e. $d(u, v) \leq r$, where r is the threshold for this transmission radius. The probability of connection between the nodes clearly depends on their Euclidean distance.

Each message is represented by a r-dimensional vector with elements from the finite field F_q of size q. All the additions and the multiplications in the following description are assumed to be over F_q. We divide the time into timeslots. Initially, at timeslot $t = 0$, each node has only one message indexed by the elements in the set $M = \{m_1, m_2, \ldots, m_n\}$, and node i stores the message m_i, $i = 1, 2, \cdots, n$.

We assume that each node in the network intends to disseminate its message to all other nodes in the network. In gossip algorithms, a round is defined as the length of time during which nodes exchange their messages with each other. Assume that the overall time is divided into a number of rounds, and one round is divided into n consecutive timeslots. In a timeslot, only one node can be scheduled to spread its message to one of its neighbouring nodes, which is selected randomly and independently with a probability of $p = 1/d$, where d denotes the node's degree.

Three types of gossip algorithms (viz. Push, Pull, and Exchange) are defined as follows:

- Push: A message is transmitted from a transmitting node (the *caller*) A to a receiving node (the *callee*) B. The communication process is therefore initiated by the transmitting node A.
- Pull: A message is transmitted from Node B to Node A. The communication process is therefore initiated by the Node B.
- Exchange: A message is transmitted from Node A to Node B and, at the same time, another message is transmitted from Node B to Node A.

At timeslot t, the adjacency matrix $A(t)$ of an RGG is defined as follows: $A_{uv}(t) = 1$, for $u \neq v$, if nodes u and v are neighbours; otherwise $A_{uv}(t) = 0$. Let $N_u(t) = \{v \in \{1, 2, \ldots, n\} : A_{uv}(t) \neq 0\}$ to denote the set of neighbouring nodes of node u, and $d_u(t) = |N_u(t)|$ to denote the degree of node u, i.e. the number of its neighbouring nodes. The definition for the transitional probability matrix of the nodes is given by [13]:

$$P_{uv} = \begin{cases} \frac{1}{2} & \text{if } v = u \\ \frac{1}{2d} & \text{if } v \in N_u \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

where N_u is the set of nodes neighbouring u. When node v neighbours node u, Equation (1) ensures that the neighbours are selected randomly and uniformly.

B. Mobile Algebraic Gossip Algorithm

A mobile algebraic gossip algorithm was proposed in [11], and its information-dissemination process is described in Algorithm 1. Here, $E_i(t)$ represents the positional information for node i at timeslot t, and $P_{ij}(t)$ denotes the transitional probability between nodes i and j at timeslot t. Moreover, $A_i(t)$ is the random linear network-coded coefficient matrix used by node i in timeslot t, and $S(t)$ denotes the set of messages that node i has at timeslot t.

Algorithm 1 Mobile Algebraic Gossip Algorithm

1: Suppose that during the t-th timeslot, node i is assigned to disseminate its message, its location $E_i(t)$, its encoded coefficient matrix $A_i(t)$, and its message set $S(t)$.
2: Node i moves to the next location $E_i(t+1)$ according to a mobility model at the $(t+1)$-th timeslot. It then randomly chooses one of its neighbours $j \in N(i)$ and computes the transitional probability $P_{ij}(t+1)$. Finally, it updates its transitional probability from $P_{ij}(t)$ to $P_{ij}(t+1)$.
3: Node i generates random coded coefficients in a finite field $G(p)$. Here, p is a prime number that is used to encode all of the node's own messages into a coded message, before sending it to node j. Meanwhile, node j finishes its coding operation and sends its coded message to node i.
4: Node i updates its encoded coefficient matrix $A_i(t+1)$ and its message set $S(t+1)$ in the same way as node j.
5: Steps 1–4 are repeated.

C. Performance Criteria

We are interested in discovering the expected time (i.e. the number of rounds) required for all of the nodes to receive (i.e. decode) all the messages, and also the time required to receive all the messages with high probability. Thus, we can define the performance criteria as follows:

***Definition* 1** [11]: The global convergence time is defined as

$$\boldsymbol{T}^{EX,RLC}_{global,mobile}(n,\varepsilon) = \sup\inf\{t : \mathbf{Pr}(dim\boldsymbol{A_i}(t) \neq n) < \varepsilon, \forall i \in \boldsymbol{V}\} \quad (2)$$

which represents the time required for all of the nodes in the network to receive all the messages with high probability $1-\varepsilon$, where $0 < \varepsilon < 1$.

***Definition* 2** [13]: the k-conductance of a graph G is defined as

$$\Phi(G) = \min_{S \subseteq V, |S| \leq k} \frac{\sum_{i \in S, j \in \bar{S}} a_{ij}}{|S|} \quad (3)$$

where S is a subset of V, $\bar{S}$ is the complementation of S, $|S|$ is the cardinality of the set S, and a_{ij} is an entry for the adjacency matrix of G.

In graph theory, the conductance of a graph $G = G(V, E)$ measures the connectivity of the graph. In [11], the definition for the conductance of a graph is generalized for mobile k-conductance, which is clearly formulated as k messages disseminating in a mobile ad-hoc network.

***Definition* 3** [11]: At timeslot t, the mobile conductance of a graph G is defined as

$$\Phi_k(P(t), E(t)) = \min_{S'(t) \subset V, |S'(t)| \leq k} \left\{ \frac{P_{ij}(n,r)}{|S'(t)|} E_{P(E(t+1)|E(t))}(N_{S'}(t+1)) \right\} \quad (4)$$

where $P(t) = [P_{ij}(t)]$ is the transition matrix, $P_{ij}(t)$ is the probability of transition from node i to node j, $E(t)$ is the positional vector for all nodes, and $P_{ij}(n,r)$ is the non-zero item for the stationary distribution of $P(t)$.

III. ANALYSING ALGEBRAIC GOSSIP ALGORITHMS IN MOBILE AD-HOC NETWORKS

Here, we consider algebraic gossip algorithms based on random linear network coding (RLNC) [14], which denote that upon receiving the messages from all other nodes, each node encodes the received message of other nodes with coefficients randomly selected from a finite field F_q before sending them. Let $S_u(t)$ be the set of all the RLNC-coded messages from node u at the timeslot t. If $x_u^i(t) \in S_u(t)$, where $i = 1, 2, \cdots, |S_u(t)|$, then $x_u^i(t) \in F_q^r$ is descripted as follows:

$$\left\{ x_u^i(t) | x_u^i(t) = \sum_{j=1}^{n} a_{ij}(t) m_j, a_{ij}(t) \in F_q \right\}. \quad (5)$$

Then the random encoded message from node u to node v is given by:

$$y_{uv}(t) = \sum_{i=1}^{|S_u(t)|} b_i(t) x_u^i(t) = \sum_{j=1}^{n} \left(\sum_{i=1}^{|S_u(t)|} a_{ij}(t) b_i(t) \right) \cdot m_j \quad (6)$$

where $b_i(t) \in F_q$ is the coded coefficient using RLNC before transmission, and $\sum_{i=1}^{|S_u(t)|} a_{ij}(t) b_i(t)$ denotes a new coded vector. Let the matrix $A_u(t) = [a_{ij}(t)]$, and let $i \in S_u(t)$ and $j = 1, 2, \cdots, n$. Then, $A_u(t)$ represents the matrix for the coded coefficient vectors of node u at timeslot t.

The mobile k-conductance for the algebraic gossip algorithm modelled on an RGG for mobile environments is analyzed in [11] , which is depicted as follows:

Theorem *1:* Let $S'(t) \subset V$ and $A(t) \subset \cup_{i=1}^{n} A_i(t)$. The mobile k-conductance for the algebraic gossip algorithm in the RGG is given by

$$\Phi_k(P(t), E(t)) = O\left(\frac{n-k}{n}\right), \quad (7)$$

and its global convergence time is therefore given by

$$T^{EX,RLC}_{global,mobile}(n,\varepsilon) = O(n \log n \log \varepsilon^{-1} - \log n \log \varepsilon^{-1}) \quad (8)$$

Proof: See [11]. ■

Based on Theorem 1, we can derive the k-conductances and the global convergence times for the mobile gossip algorithm, which is presented as follows:

Corollary 1: The k-conductance for the mobile gossip algorithm in the RGG is given by

$$\Phi_k(P(t), E(t)) = O\left(\frac{n-k}{k}\right), \quad (9)$$

and its global convergence time is therefore given by

$$T^{EX}_{global,mobile}(n,\varepsilon) = O\left(n \log n \log \varepsilon^{-1}\right). \quad (10)$$

Without considering node mobility, we have the following results.

Theorem *2:* The k-conductance of the static algebraic gossip algorithm in the RGG is given by

$$\Phi_k(P) = O\left(\frac{\sqrt{n \log n}}{k}\right) \quad (11)$$

and its global convergence time is therefore given by

$$T^{EX,RLC}_{global}(n,\varepsilon) = O\left(\frac{(n^{3/2} - n^{1/2}) \log \varepsilon^{-1}}{\log^{1/2} n}\right) \quad (12)$$

Proof: See Appendix A. ■

Based on Theorem 2, we can derive the k-conductances and the global convergence times for the static gossip algorithm, which is presented as follows:

Corollary 2: The k-conductance for the static gossip algorithm in the RGG is given by

$$\Phi_k(P) = O\left(\frac{\sqrt{n \log n}}{k}\right), \quad (13)$$

and its global convergence time is therefore given by

$$T^{EX}_{global}(n,\varepsilon) = O\left(\frac{n^{3/2} \log \varepsilon^{-1}}{\log^{1/2} n}\right). \quad (14)$$

All of the above results are summarized in Table I. From Table I, we can conclude the following:

(1) Network coding fully utilizes the additivity of information to reduce the time required for information dissemination. Thus, network coding can directly improve the convergence time of gossip algorithms, whether or not node mobility is considered.

(2) Node mobility can create a chance encounter between two nodes that are distant from each other, and this is equivalent to enlarging the node-transmission range. Thus, node mobility can reduce the convergence time for gossip algorithms in terms of movement. Moreover, node mobility has a much greater effect on the convergence time for information dissemination than network coding.

(3) Given the above theorem, we can conclude that mobile algebraic gossip algorithms that integrate node mobility with network coding can accelerate the convergence of information dissemination. Together, such an integration results in a faster convergence time than either node mobility or network coding alone. Specifically, gossip algorithms with both network coding and node mobility converge by the magnitude of $O(\frac{n^{1/2}}{\log^{3/2} n})$ faster than algorithms that do not consider node mobility, and $O(\log n)$ faster than a gossip algorithm with node mobility but without network coding.

All of the above results are verified with numerical simulations in the following section.

IV. SIMULATIONS

In this section, we present our performance analysis of the four gossip algorithms using numerical simulations. Initially, each node was placed at a random position within the simulation area, and the number of nodes ranged from 100 to 200. The Random Waypoint (RWP) Model [15] was used to formulate node mobility in the simulations. As the simulations progressed, each node moved within the simulation area, and paused momentarily at a certain location—during a so-called pause time—before randomly choosing a new location and moving there. Nodes moved at a velocity between V_{min} and V_{max} meters per second, where V_{min} and V_{max} denote the minimum velocity and maximum velocity of the node's movement, respectively. Each node continued in this way, alternating between pausing and moving to a new location throughout the duration of simulation. There is no pause time for the destination nodes.

The maximum transmission radius for each node in the RGG was set to $O\left(\sqrt{\frac{\log n}{n}}\right)$ to maintain node connectivity [16]. That is, each node could only receive signals from nodes within this transmission radius. We assumed that nodes do not have knowledge of the other nodes' states (e.g. the location, whether they caller or callee, etc.).

To better present the results of these experiments, we first simulated the distribution of the nodes' location with an RGG, as seen in Fig. 1. Obviously, it is considerably difficult for some neighbouring nodes to maintain connectivity, because that there are "holes" in the simulation area. This will adversely affect the convergence time of the information-dissemination process. Indeed, node mobility mitigates this problem to some extent, and this is one of our main conclusions.

In the following, we analyze the four gossip algorithms: the mobile algebraic gossip (MAG) algorithm, the static algebraic gossip (SAG) algorithm, the mobile gossip (MG) algorithm, and the static gossip (SG) algorithm. Here, the SG algorithm is a random gossip algorithm, which was analyzed in [1].

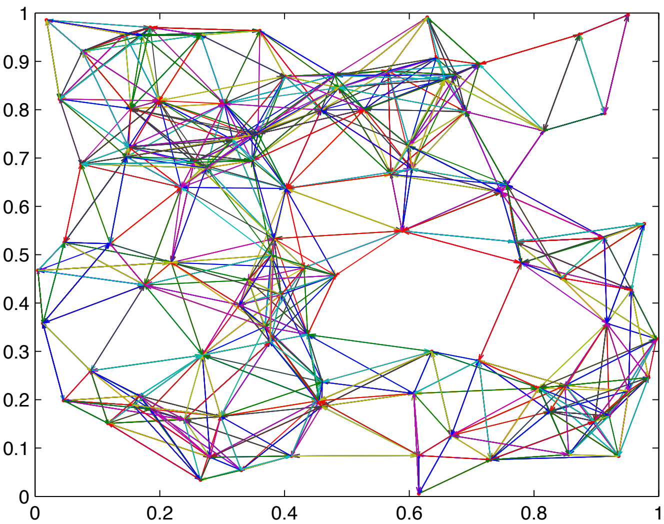

Fig. 1. Node Distribution under a Random Geometric Graph

For $\forall\varepsilon, 0 < \varepsilon < 1$, we calculated the time required for all nodes to receive all messages with probability of at least $1-\varepsilon$—i.e. the global convergence time. The results are shown in Fig. 2. In Fig. 2, the global convergence time increased as the parameter $1-\varepsilon$ increased. Moreover, as the parameter $1-\varepsilon$ increased, the gain from network coding decreased in the static and mobile networks. With high probability (hereafter "W.H.P."), $1-\varepsilon \leq 0.7$, suggesting that network coding offers a considerable advantage in terms of the convergence time. The global convergence time for the MAG algorithm was slower than that of the SAG algorithm, demonstrating that node mobility does not play a significant role in improving convergence at a low probability. Thus, we can conclude that node mobility is not always effective. Conversely, however, W.H.P. $1-\varepsilon \geq 0.85$. In this case, node mobility offers a considerable improvement in terms of the convergence time, regardless of whether network coding is used. Therefore, we selected an appropriate value ($1-\varepsilon = 0.9$) for comparing the convergence time between node mobility and network coding in the following simulations.

We analyzed the relation between the global convergence time and the number of nodes using various algorithms, as seen in Fig. 3. As more nodes were added, there was more of a difference between the convergence time of the MAG and SAG algorithms, much as there was between the MG and SG algorithms. This result confirms that node mobility is advantageous. Moreover, with many nodes, the global convergence time for MG was always faster than that of SG and SAG. This suggests that we can fully utilize node mobility, which increases chance encounters among nodes, without incurring a delay from network encoding and decoding.

We analyzed the relation between the global convergence time and the velocity of the nodes using various algorithms, as

TABLE I
PERFORMANCE COMPARISON OF GOSSIP ALGORITHMS MODELLED ON RANDOM GEOMETRIC GRAPHS

	Static Gossip	Static Algebraic Gossip	Mobile Gossip	Mobile Algebraic Gossip
k-Conductance	$O(\frac{\sqrt{n\log n}}{k})$	$O(\frac{\sqrt{n\log n}}{k})$	$O(\frac{n-k}{k})$	$O(\frac{n-k}{n})$
Conductance	$O(\frac{n^{5/2}}{\log^{1/2} n})$	$O(\frac{n^{5/2}}{\log^{1/2} n})$	$O(n^2\log n)$	$O(n^2\log n)$
Convergence Time	$O(\frac{n^{3/2}\log\varepsilon^{-1}}{\log^{1/2} n})$	$O(\frac{(n^{3/2}-n^{1/2})\log\varepsilon^{-1}}{\log^{1/2} n})$	$O(n\log n\log\varepsilon^{-1})$	$O((n-1)\log n\log\varepsilon^{-1})$

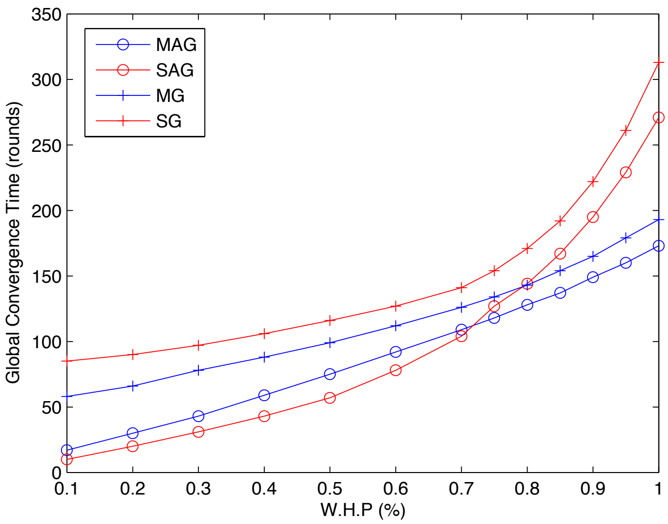

Fig. 2. Relation between the Global Convergence Time and W.H.P. $1-\varepsilon$ using Various Algorithms and 150 Nodes

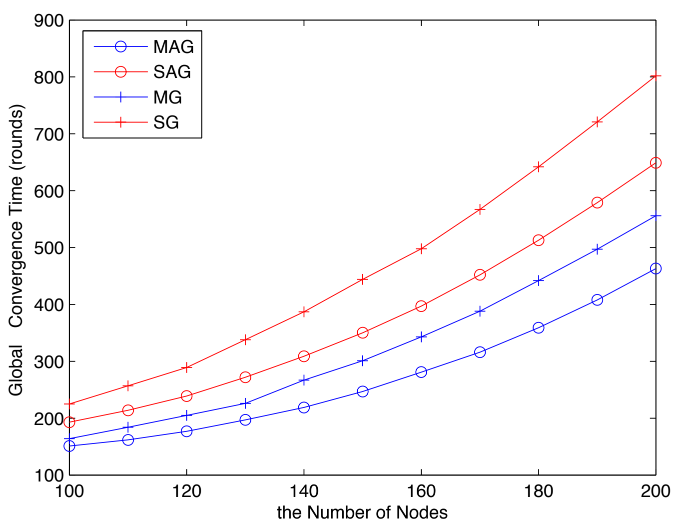

Fig. 3. Relation between the Global Convergence Time and the Number of Nodes with Various Algorithms

shown in Fig. 4. With a fixed number of nodes, network coding improved the global convergence time. Meanwhile, node mobility significantly reduced the global convergence time as the maximum velocity increased. This confirms that node mobility improves the performance of information dissemination.

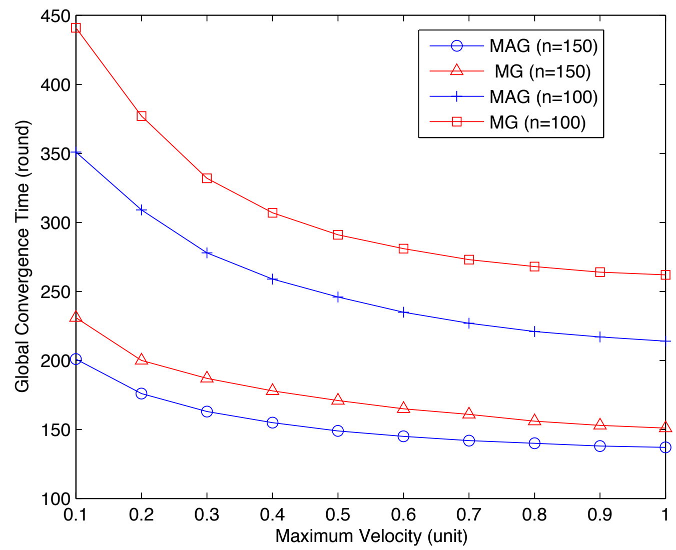

Fig. 4. Relation between the Global Convergence Time and the Maximum Velocity in Various Algorithms

V. CONCLUSION

In this paper, we studied the role of mobility and network coding on multi-message gossip in random geometric graphs, and derived the bounds for the convergence time of several different gossip algorithms. The results show that when both network coding and node mobility are used, the bounds for the convergence time can be improved by the magnitude of $O(\frac{n^{1/2}}{\log^{3/2} n})$ when only network coding is used, and by the margin of $O(\log n)$ when only node mobility is considered.

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APPENDIX A
PROOF FOR THEOREM 2

Let n nodes in a unit disk, and the movement of these nodes is an independent memoryless Markov process. Let V denote the set of all nodes, with the initial time $t = 0$. Moreover, we have rank$(\boldsymbol{A}_i(t)) = 1$, $\forall i \in V$, and let $t_k = \inf\{t : \boldsymbol{C}(t) \geq k\}$, where $\boldsymbol{C}(t)$ is the largest cardinality in $\boldsymbol{B}(t)$. Our goal is to make rank$(\boldsymbol{A}_i(t)) = n, \forall i \in V$. The global convergence time is divided into k time periods $[t_1, t_2), [t_2, t_3), \ldots, [t_k, t_{k+1})$. According to [13], when node i exchanges messages with node j, then we have the following inequality:

$$E(I_{ij}(t)) + E(I_{ji}(t)) \geq P'_{ij}(t) + \overline{P_{ij}}(t). \tag{15}$$

where $I_{ij}(t)$ be a characteristic function. The accumulative rank $\Delta(t) = \sum_{i=1}^{n}(rank(\boldsymbol{A}_i(t)) - 1)$ satisfy the following inequality for timeslot $t \in [t_k, t_{k+1})$:

$$\Delta(t+1) - \Delta(t) \geq \sum_{i \in V} \sum_{j \in V, j \neq i} (I_{ij} + I_{ji}) \tag{16}$$

By adopting expectations on both sides of the above inequality and using the result in Equation (15), we have

$$\begin{aligned} E(\Delta(t+1) - \Delta(t)) &\geq \sum_{i \in V} \sum_{j \in V, j \neq i} (E_(I_{ij}(t)) + E(I_{ji}(t))) \\ &\geq \sum_{i,j \in V, j \neq i, \boldsymbol{A}_i(t), \boldsymbol{A}_j(t) \in \boldsymbol{R}(t)} P'_{ij}(t) \\ &\quad + \sum_{i,j \in V, j \neq i, \boldsymbol{A}_i(t), \boldsymbol{A}_j(t) \in \widetilde{V}} \overline{P_{ij}}(t) \\ &= \sum_{i,j \in \boldsymbol{V}, j \neq i, \boldsymbol{A}_i(t), \boldsymbol{A}_j(t) \in \boldsymbol{R}(t)} P'_{ij}(t) \\ &\quad + \sum_{i,j \in \boldsymbol{V}, j \neq i, \boldsymbol{A}_i(t), \boldsymbol{A}_j(t) \in \boldsymbol{R}(t)} \frac{2P'_{ij}(t)}{d_i(t) - 2} \\ &\geq (1 + \frac{2}{k-2}) \cdot n \cdot \Phi_k(p) \end{aligned} \tag{17}$$

Let $g(t) = \Delta(t+1) - \Delta(t) - (1 + \frac{2}{k-2}) \cdot n \cdot \Phi_k(p)$. Similar with [4], we define $Z_k(t) = \sum_{i=t_k}^{t-1} g(i)\mathbf{1}_{\{i<t_{k+1}\}}$. It is easy to show that $Z_k(t_k) = 0$ and

$$\begin{aligned} &E(Z_k(t+1)|Z_k(t)) \\ &= E\left(\sum_{i=t_k}^{t} g(i)\mathbf{1}_{\{i<t_{k+1}\}} | \sum_{i=t_k}^{t-1} g(i)\mathbf{1}_{\{i<t_{k+1}\}}\right) \\ &= Z_k(t) + E\left(g(t)\mathbf{1}_{\{i<t_{k+1}\}} | \sum_{i=t_k}^{t-1} g(i)\mathbf{1}_{\{i<t_{k+1}\}}\right) \\ &\geq Z_k(t) \end{aligned} \tag{18}$$

which indicates that $Z_k(t)$ is a submartingale. Then, we have $E(Z_k(t+1)) \geq E(Z_k(t)) = 0$, and

$$\begin{aligned} &E\left(\sum_{i=t_k}^{t_{k+1}-1}(\Delta(i+1) - \Delta(i))\right) \\ &\geq E\left(\sum_{i=t_k}^{t_{k+1}-1}(1 + \frac{2}{k-2}) \cdot n \cdot \Phi_k(p)\right), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &E(\Delta(t_{k+1}) - \Delta(t_k)) \\ &\geq (1 + \frac{2}{k-2}) \cdot n \cdot \Phi_k(P(t), E(t)) \cdot E(t_{k+1} - t_k) \end{aligned} \tag{19}$$

according to *Lemma 5* in [4], Equation (17) is formulated as follows:

$$\begin{aligned} E(t_n) &\leq \frac{k-2}{k} \sum_{k=2}^{n-1} \frac{E(\Delta(t+1) - \Delta(t))}{n \cdot \Phi_k(p)} \\ &\quad + \frac{E(\Delta(t+1) - \Delta(t))}{n \cdot \Phi_k(p)}|_{k=1} \\ &\leq \sum_{k=1}^{n-1} \frac{2k}{n \cdot \Phi_k(p)} - \sum_{k=2}^{n-1} \frac{4}{n \cdot \Phi_k(p)} \\ &= \frac{2}{n} \cdot \widehat{\Phi}(p) - \frac{4}{n} \cdot \sum_{k=2}^{n-1} \frac{1}{\Phi_k(p)} \end{aligned} \tag{20}$$

According to Markov's inequality, Equation (15) implies that

$$P\left(t_n > \frac{\log \varepsilon^{-1}}{n}\left(\widehat{\Phi}(p) - \sum_{k=2}^{n-1} \frac{1}{\Phi_k(p)}\right)\right) < \varepsilon \tag{21}$$

The global convergence time for the static algebraic gossip-network model can therefore be expressed as follows:

$$\boldsymbol{T}_{global}^{EX,RLC}(n, \varepsilon) = O\left(\frac{\widehat{\Phi}(p) - \sum_{k=2}^{n-1} \frac{1}{\Phi_k(p)}}{n} \log \varepsilon^{-1}\right)$$

Then, the global convergence time for the static algebraic gossip model is given by

$$\boldsymbol{T}_{global}^{EX,RLC}(n, \varepsilon) = O\left(\frac{(n^{\frac{3}{2}} - n^{\frac{1}{2}}) \log \varepsilon^{-1}}{\log^{\frac{1}{2}} n}\right)$$