An Optimal Routing Policy for Mesh-Connected Topologies

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Abstract – *We present a new routing policy, called maximum shortest paths (MP) routing policy, within the class of shortest-path routing policies for mesh-connected topologies which include popular 2-D and 3-D meshes, 2-D and 3-D tori, and n-dimensional hypercubes (n-cubes). In this policy, the routing message is always forwarded to a neighbor from which there exists a maximum number of shortest paths to the destination. An optimal routing defined in this paper is the one that maximizes the probability of reaching the destination from a given source without delays at intermediate nodes. We show that the MP routing policy is equivalent to the e-cube routing in n-cubes which is optimal, and it is also equivalent to the Badr and Podar's zig-zag (Z^2) routing policy in 2-D meshes which is also optimal. We prove that the Z^2 routing policy is not optimal in any $N \times N$ torus, where N is an even number larger than four. A routing algorithm is proposed to implement the MP routing policy in 2-D tori and it is proved to be at least suboptimal (optimal for some cases). Our approach is the first attempt to address optimal routing in the torus network which is still an open problem.*

Keywords: Mesh-connected topologies, multicomputers, shortest-path routing.

1 Introduction

Efficient routing of messages is critical to the performance of multicomputers. Basically, routing is the process of transmitting data from one node, called the *source* node, to another node called the *destination* node in a given multicomputer. The mesh-connected topology, which includes meshes, tori and hypercubes, is one of the most thoroughly investigated interconnection topologies for multicomputers.

In a shortest-path routing, only shortest paths (to the destination) are acceptable. A shortest-path routing policy is *optimal* [1] if it maximizes the probability of reaching the destination from a given source without delays at intermediate nodes. It is assumed that some of the outgoing links at a node may be unavailable due to competing traffic or physical link failure. In this paper, we present a new routing policy: *maximum shortest paths* (MP) routing policy within the class of shortest-path routing policies for mesh-connected topologies. In this policy, the routing message is always forwarded to a neighbor from which there exists a maximum number of shortest paths to the destination.

We show that the MP routing policy is equivalent to the e-cube routing in an n-cube which is optimal, and it is also equivalent to the Badr and Podar's zig-zag (Z^2) routing policy [1] in the 2-D meshes which is again optimal. We formally prove that the Z^2 routing algorithm is not optimal in any $N \times N$ torus, where N is an even number larger than four. By this we extend a result from [2] where only one counter example is given for a 6×6 torus. We also present a routing algorithm that implements the MP routing policy in 2-D tori and it is proved to be at least suboptimal. Our approach is the first attempt to achieve optimal routing in the torus network which is still an open problem.

2 Preliminaries

In a shortest-path routing, only shortest paths are acceptable. That is, each routing message is forwarded to a destination through a shortest path. If this message cannot be forwarded to the destination through a shortest path, it is simply discarded.

Let p be the probability that an acceptable outgoing link is healthy. Assume that this p is uniform across the whole network, it represents the probability that a message may successfully be transmitted to a neighbor along the selected link. $(1-p)$ represents the probability that a link fails. We denote $V = (v_1, v_2, ...v_m)$ the *eligible neighbor vector* of node v with respect to destination u in a given network, where an *eligible* *neighbor* of v is a neighbor closer to the destination u than node v in the network and m is the number of eligible neighbors. Note that m varies from node to node depending on the distance between the current node and the destination node. If a priority order is defined among v's eligible neighbors to which the routing message is forwarded, this order can be represented by $(v_1^{(k)}, v_2^{(k)}, ..., v_m^{(k)})$ which is a permutation of the eligible neighbor vector V, where k $(= m!)$ represents the kth permutation based on a permutation generator. Let $(p, p(1-p), p(1-p)^2, ..., p(1-p)^{m-1})$ be a *probability vector* associated with the eligible neighbor vector. In the ith element $p(1-p)^{i-1}$ of the probability vector, p represents the probability that the routing message is successfully forwarded to the ith neighbor in the given neighbor order and $(1-p)^{i-1}$ represents the probability that the first $i-1$ tries fail.

A shortest-path routing is optimal if it maximizes the probability of reaching the destination from a given source without delays at intermediate nodes. Let $S(v,u)$ be the maximum probability of delivery of a routing message from node v to node u. Then $S(v,u)$ satisfies the following recursive equations:

$$\begin{aligned} S(v,u) &= max_k\{\sum_{i=1}^{m} p(1-p)^{i-1} S(v_i^{(k)}, u)\} \\ S(u,u) &= 1 \end{aligned}$$

The *maximum shortest paths* (MP) routing policy proposed in this paper is a shortest-path routing policy. Our goal is to show that the MP routing policy is optimal for many mesh-connected networks. In the MP routing, the routing message is always forwarded to an eligible neighbor from which there exists a maximum number of shortest paths to the destination. Again we denote $(v_1, v_2, ..., v_m)$ the eligible neighbor vector of node v (with respect to a given destination node u), where each v_i is a neighbor of v. $P(v,u)$ represents the number of shortest paths from v to u. Then $P(v,u)$ satisfies the following equations:

$$\begin{aligned} P(v,u) &= \sum_{i=1}^{m} P(v_i, u) \\ P(u,u) &= 1 \end{aligned}$$

As an example, we consider routing in a 2-D mesh. Assume source $v = (i,j)$ and destination $u = (0,0)$, we use $P(v)$ to replace $P(v,u)$, the above equations become: $P(i,j) = P(i-1,j) + P(i,j-1)$ and $P(0,0) = 1$.

The zig-zag (Z^2) routing policy proposed by Badr and Podar [1] is another shortest path routing algorithm which is optimal in the mesh topology. Informally, the Z^2 policy states that the routing message should be sent towards the diagonal which denotes the set of nodes that have an equal number of rows and columns away from the destination node. It has been shown in [1] that Z^2 is optimal in 2-D meshes.

3 MP-based Routing

In [3], we show that the conventional e-cube routing is an example of MP routing in hypercubes and the MP routing policy is an optimal shortest-path routing in a 2-dimensional (2-D) mesh. It has been shown [1] that the Z^2 routing is optimal in a 2-D mesh; therefore, it can be used to implement the MP routing policy in a 2-D mesh. Here we foucs on optimal shortest-path routing on a 2-D torus which is a 2-D mesh with the wrap-around at the ends. Therefore, for some destination-source pairs there are more than two eligible neighbors. Specifically, for a $N \times N$ torus where N is even, there is one node that has four eligible neighbors and $2(N-2)$ source nodes ($N/2$th rows or columns away, but not both) for which three directions lie on the shortest path. Without loss of generality, we only consider source nodes that are $N/2$-column away. Note that nodes that are not at the $N/2$th column or row in a 2-D torus are equivalent to the one in a 2-D mesh. Therefore, in this case any optimal routing in a 2-D mesh is also optimal in a 2-D torus. Also, when either i or j in the source node (i,j) is larger than $N/2$, the shortest path uses wrap around links. To simplify our discussion, we assume that both i and j satisfy the constraint $0 \leq i,j \leq N/2$. We also assume that the destination node is always $(0,0)$ and the source node is denoted by (i,j).

In the following theorem, we show that Z^2 is not optimal for any $N \times N$ torus, where N is an even number larger than four. It is a generalization of Weller and Hajek's result [2], where only one counter example is given for a 6×6 torus.

Theorem 1: *The Z^2 routing algorithm is not optimal in any $N \times N$ torus, where N is an even number larger than four.*

Proof: It suffices to show one counter example for any given $N \times N$ torus, where N is an even number larger than four. Assume that node $(k, k-1)$ is at the $(N/2)$th column with respect to the source. That is, this node has three eligible neighbors: $(k-1, k-1)$, $(k, k-2)$, and $(k+1, k-1)$. Recall that a neighbor is said to be eligible if it is along one of the shortest paths from the current node to the destination node. We show that $S(k-1, k-1) < S(k, k-2)$, i.e., the diagonal node is not the neighbor that has the largest S value. Because $N \geq 3$, $k = \frac{N}{2} \geq 3$, node

$(k, k-2)$ has three eligible neighbors: $(k-1, k-2)$, $(k+1, k-2)$, and $(k, k-3)$. Based on the torus topology, we have $S(k-1,k-1) = \max\{pS(k-2,k-1)+ p(1-p)\ S(k-1,k-2), pS(k-1,k-2)\ +p(1-p)\ S(k-2,k-1)\}$ and $S(k,k-2) \geq pS(k-1,k-2)+ p(1-p)S(k+1,k-2)\ +p(1-p)^2S(k,k-3)$. Because nodes $(k-2,k-1),(k-1,k-2)$, and $(k+1,k-2)$ are identical with respect to the destination $(0,0)$, $S(k-2,k-1) = S(k-1,k-2) = S(k+1,k-2)$. Therefore, $S(k-1,k-1) = pS(k-2,k-1)+ p(1-p)\ S(k-1,k-2) < pS(k-1,k-2)\ +p(1-p)S(k+1,k-2)\ +p(1-p)^2S(k,k-3) = S(k,k-2)$ □

The MP routing on a 2-D torus works as follows: We need to find a location k (at the y dimension) along column i (the $N/2$th column), such that for any source whose j value is larger than k, the routing message should be forwarded along column i until reaching $t = (i,k)$, then the remaining part is equivalent to the routing in a 2-D mesh; therefore, the Z^2 routing can be used. For any source node whose value at the y dimension is equal to or smaller than k, the optimal routing is equivalent to the one in a 2-D mesh. This special point $t = (i,k)$ is called a *turning point.* Clearly, $k < i = N/2$. Our goal is to find the value of k in the turning point $t = (i,k)$.

Let $P(i,j)$ be the number of shortest paths from node (i,j) to node $(0,0)$. The following lemma reveals the number of shortest paths from (i,j) to $(0,0)$.

Lemma: *In a 2-D torus,* $P(i,j) = \binom{i+j}{j}$, $P(i,j) = 2\binom{i+j}{j}$, *and* $P(i,j) = 4\binom{i+j}{j}$, *where* (i,j) *is not at the* $N/2$*th column or row, at the* $N/2$*th column or row but not both, and at the* $N/2$*th column and row, respectively.*

Proof: We use induction to prove that when node (i,j) is not at the $N/2$th column or row and it has two eligible neighbors then $P(i,j) = \binom{i+j}{j}$. When node (i,j) has one eligible neighbor then either i or j is zero, there is only one path and the theorem clearly holds true in this case. Therefore, we only need to consider cases when both i and j are not zero. When $i+j=2$ the theorem clearly holds true and assume it holds for $i+j=k$. We now consider $P(i,j) = P(i-1,j)+P(i,j-1)$ for $i+j=k+1$. Based on the induction assumption, we have $P(i-1,j) = \binom{i+j-1}{j}$ and $P(i,j-1) = \binom{i+j-1}{j-1}$. Therefore,

$$P(i,j) = \binom{i+j-1}{j} + \binom{i+j-1}{j-1} = \binom{i+j}{j}.$$

When (i,j) is at the $(N/2)$th row or column (but not both), then there are two eligible neighbors (where either i or j is zero) or three eligible neighbors (where neigher i nor j is zero). For the two eligible neighbors cases there are two shortest paths. Also, $2\binom{i+j}{j}$ is equal to either $2\binom{j}{j}$ or $2\binom{i}{0}$. In both cases the result is two. Therefore, we only need to prove for the three eligible neighbors cases. When $i=j=1$ in a $2 \times N$ or a $N \times 2$ torus, clearly $P(i,j) = 2\binom{i+j}{j} = 2\binom{2}{1} = 4$. Assume this result holds for $i+j=k>2$. We consider the case where (i,j) $(i=\frac{N}{2})$ has three eligible neighbors $(i-1,j)$, $(i,j-1)$, and $(i+1,j)$. So nodes $(i-1,j)$ and $(i+1,j)$ are not at the $N/2$th column or row and node $(i,j-1)$ is still at the $N/2$th column. Based on the induction assumption and the result for nodes not at the $N/2$th column or row, we have $P(i-1,j) = P(i+1,j) = \binom{i+j-1}{j}$ and $P(i,j-1) = 2\binom{i+j-1}{j-1}$. Therefore,

$$\begin{aligned} P(i,j) &= P(i-1,j)+P(i+1,j)+P(i,j-1) \\ &= 2\binom{i+j-1}{j} + 2\binom{i+j-1}{j-1} \\ &= 2\binom{i+j}{j}. \end{aligned}$$

Similarly, the same result can be obtained for the case where (i,j) is at the $N/2$th row, i.e., $j=\frac{N}{2}$.

For node (i,j) that has four eligible neighbors: $(i-1,j)$, $(i+1,j)$, $(i,j-1)$, and $(i,j+1)$, each neighbor is at the $N/2$th column or row and neighbors and $P(i+1,j) = P(i-1,j)$ and $P(i,j-1) = P(i,j+1)$. Based on the result for nodes with three neighbors, we have

$$\begin{aligned} P(i,j) &= P(i-1,j)+P(i,j-1)+P(i+1,j)+P(i,j+1) \\ &= 2\binom{i+j-1}{j} + 2\binom{i+j-1}{j-1} + 2\binom{i+j-1}{j} \\ &\quad +2\binom{i+j-1}{j-1} = 4\binom{i+j}{j}. \end{aligned}$$

□

Theorem 2: *If the source is at the* $(N/2)$*th column and* N *is an even number larger than two, then the turning point* (t) *at the* $(N/2)$*th column is* $t = (N/2,k)$, *where* $N/2 = 2k+1$ *or* $N/2 = 2k$.

Proof: It suffices to prove that at the turning point $t = (i,k)$, where $i = N/2$, we have

$$k = max\{l \mid 2\binom{l+i-1}{l-1} \leq \binom{l+i-1}{l}\}$$

Note that $2\binom{k+i-1}{k-1} = P(i,k-1)$, where node $(i,k-1)$ is an eligible neighbor of the turning point (i,k), and $\binom{k+i-1}{k} = P(i-1,k)$ is another eligible neighbor of (i,k).

Based on the properties of binomial numbers, it is equivalent to show that $2P(i,k) > P(i-1,k+1)$ and $2P(i,k-1) \leq P(i-1,k)$. That is,

$$2\binom{k+i}{k} > \binom{k+i}{k+1}$$

and

$$2\binom{k+i-1}{k-1} \leq \binom{k+i-1}{k}$$

For $i = 2k$, we have

$$\binom{k+i}{k+1} = \binom{3k}{k+1} = \frac{2k}{k+1}\binom{3k}{k} < 2\binom{3k}{k} = 2\binom{k+i}{k}$$

and

$$\binom{k+i-1}{k} = \binom{3k-1}{k} = 2\binom{3k-1}{k-1} = 2\binom{k+i-1}{k-1}$$

For $i = 2k+1$, we have

$$\binom{k+i}{k+1} = \binom{3k+1}{k+1} = \frac{2k+1}{k+1}\binom{3k+1}{k} < 2\binom{k+i}{k}$$

and

$$\binom{k+i-1}{k} = \binom{3k}{k} = \frac{2k+1}{k}\binom{3k}{k-1} > 2\binom{k+i-1}{k-1}$$

□

The MP routing on a 2-D torus works as follows: If the source is at the $N/2$th column (the same algorithm applies to the case where the source is at the $N/2$th row) then we consider the following two cases. Case 1: If j (the value at the y dimension) is larger than k, then the routing message should be forwarded along the y dimension until it reaches the kth row and then follow the Z^2 routing. Case 2: If j is smaller than k then follow the Z^2 routing directly. If the source is not at the $N/2$th column or row, then follow the Z^2 routing.

4 Discussions and Conclusions

We have shown that the optimal routing policy and the MP routing policy are the same in mesh and hypercube routing. We have also proved that the Z^2 algorithm is not optimal in a general 2-D torus network and have proposed a routing algorithm on a general 2-D torus network based on the MP policy. The next question is how close the MP policy is to the optimal routing policy in a 2-D torus network. Weller and Hajek [2] predict that the optimal policy for the torus seems unlikely to be of a simple closed form.

The following theorem reveals the relationship between $S(v)$, the maximum probability of delivery of a routing message at node v, and $P(v)$, the number of shortest paths from node v.

Theorem 3: *Let $S(v) = a_1p^k + a_2p^{k+1} + ... + a_np^m$, where a_1 is a positive integer. Then $P(v) = a_1$.*

Table 1: The P and S values for each node of a 6×6 torus

node v	(0,0)	(1,0)	(2,0)	(1,1)
$P(v)$	1	1	1	2
$S(v)$	1	p	p^2	$2p^2 - p^3$
node v	(3,0)	(2,1)	(3,1)	(2,2)
$P(v)$	2	3	8	6
$S(v)$	$2p^3 - p^4$	$3p^3 - 2p^4$	$8p^4 - 12p^5 + 6p^6 - p^7$	$6p^4 - 7p^5 + 2p^6$

Basically, the MP policy can be viewed as an approximation of the optimal policy where high order terms of p are eliminated. The question is whether these two policies are equivalent in a 2-D torus network. That is, it is not known that $S(v) > S(u)$ iff $P(v) > P(u)$, where $v = (i, j-1)$ and $u = (i-1, j)$.

We show that, for at least some cases, the MP routing is optimal in a 2-D torus. Table 1 shows the P's and S's for all the nodes in a 6×6 torus, assuming that $(0,0)$ is the destination node. Clearly, $S(v) > S(u)$ iff $P(v) > P(u)$ holds in this case. For example, $S(3,1) - S(2,2) = p^4(1-p)^2(2-p) > 0$ and $P(3,1) - P(2,2) = 8 - 6 > 0$. Also, $S(2,1) - S(3,0) = p^3(1-p)$ and $P(2,1) - P(3,0) = 3 - 2 > 0$. Therefore, the MP-based routing algorithm proposed in Section 3 is an optimal routing in a 6×6 torus, the turning point is at the node (3,1), where $N = 6$ and $k = 1$. It is still an open problem whether the MP routing policy is optimal in a general torus network. We conjecture that our MP-based routing algorithm is optimal in a general 2-D torus.

References

[1] S. Badr and P. Podar. An optimal shortest-path routing policy for network computers with regular mesh-connected topologies. *IEEE Transactions on Computers.* 38, (10), Oct. 1989, 1362-1371.

[2] T. Weller and B. Hajek. Comments on "An optimal shortest-path routing policy for network computers with regular mesh-connected topologies. *IEEE Transactions on Computers.* 43, (7), July 1994, 862-864.

[3] J. Wu. A new optimal shortest-path routing policy for multicomputers with mesh-connected topologies. Technical Report, TR-CSE-94-54, Florida Atlantic University, October 1994.