1 Tiling Problem

We use the following notations and conventions:

- All points are in two dimension.
- A rectangle $R$ is a Cartesian product of two closed non-empty intervals $[a, b]$ and $[c, d]$. The area of $R$, denoted by $\text{Area}(R)$ is $(b - a) \times (d - c)$. Point $(x, y)$ is an interior point of $R$ if $a < x < b$ and $c < y < d$.
- Two rectangles are disjoint if they do not share any interior point.
- A polygonal region (or, simply a region) $F$ is an union of non-empty disjoint rectangles. The area of this region, denoted by $\text{Area}(F)$, is the sum of the areas of its rectangles.

We define the following tiling problem:

**Input:** A set $S = \{R_1, R_2, \ldots, R_n\}$ of $n$ rectangles and another rectangle $R$.

**Valid solution:** A subset $S' \subseteq S$ of rectangles such that $R \subseteq \bigcup_{R_i \in S'} R_i$.

**Objective:** minimize the area of the region covered outside $R$, i.e., $\text{Area}\left(\bigcup_{R_i \in S'} R_i \setminus R\right)$.

In the sequel, we will make use of the following easy observation.

**Observation 1.1.** Let $S' \subseteq S$ be a set of rectangles such that $\bigcup_{R_i \in S'} R_i \subseteq R$. Then, there is an optimal solution that contains all the rectangles in $S'$.

1.1 Inapproximability

Recall the following standard definition: an $(1+\delta)$-approximation algorithm for a minimization problem is a polynomial-time algorithm that finds a solution whose value is at most $(1+\delta)$ times the optimum.

**Theorem 1.2.** Assuming $P \neq \text{NP}$, our tiling problem does not admit an $(1+\delta)$-approximation algorithm for any $\delta < \frac{5}{171.6} \approx 0.0069$.

We prove the above theorem in the remainder of this section. The 3-MIS problem is defined as follows:

**Input:** a 3-regular graph $G = (V, E)$.

**Valid solution:** $V' \subset V$ such that every pair of vertices in $V'$ are independent, i.e., for all $u, v \in V'$ we have $\{u, v\} \notin E$.

**Objective:** maximize $|V'|$.

The following result appears in [1].

**Theorem 1.3.** [1] For every constant $\varepsilon > 0$ and $n' \geq 1$, it is NP-hard to decide if an instance of 3-MIS with $284n'$ nodes has the maximum size of independent set above $(140 - \varepsilon)n'$ or below $(139 + \varepsilon)n'$.
For our convenience, we will use the set-packing reformulation of 3-MIS. Let the edges of \( G \) be enumerated as \( e_1, e_2, \ldots, e_m \). Every vertex \( v \) generates a set \( S_v = \{e_i, e_j, e_k\} \) where \( e_i, e_j \) and \( e_k \) are the three edges incident on \( v \). Note that every set exactly three element, every edge \( \{u, v\} \) appears in exactly two sets \( S_u \) and \( S_v \) and two sets have no more than one element in common. An independent set \( V' \) of \( G \) is then equivalent to a collection of mutually disjoint set \( S_v \)'s for each \( v \in V' \). This leads us to the following equivalent reformulation of 3-MIS which we call 3-SET-PACKING.

**Input:** an universe \( U = \{e_1, e_2, \ldots, e_m\} \) of \( m \) elements and a collection of \( S = \{S_1, S_2, \ldots, S_n\} \) of \( n \) subsets of \( U \) such that:

- \( |S| = 3 \) for each \( S \in S \);
- every element \( u \in U \) appears in exactly two sets of \( S \);
- for any two sets \( S, S' \in S \) we have \( |S \cap S'| \leq 1 \).

**Valid solution:** a sub-collection \( S' \subset S \) such that every pair of sets in \( S' \) are mutually disjoint.

**Objective:** maximize the size of the packing \( |S'| \).

Theorem 1.3 can then be rephrased as follows.

**Theorem 1.4 (Rephrasing of Theorem 1.3).** [1] Let \( L \) be a language in \( \text{NP} \). Then, for every constant \( \varepsilon > 0 \), there exists a polynomial-time reduction that given an instance \( I \) of \( L \) produces an instance of \( I' = (U, S) \) of 3-SET-PACKING with \( (3 \times 284n')/2 = 426n' \) elements in \( U \) and \( 284n' \) sets in \( S \) such that:

- if \( I \in L \) then \( I' \) has a has at least \((140 - \varepsilon)n'\) disjoint sets;
- if \( I \notin L \) then \( I' \) has at most \((139 + \varepsilon)n'\) disjoint sets.

We reduce an instance of 3-SET-PACKING, as described in Theorem 1.4 above, to our tiling problem. We provide the construction step-by-step so that the intuition behind the reduction is clear. For notational convenience, let \( m = 426n' \), \( n = 284n' \) and denote the maximum number of disjoint sets in our instance of 3-SET-PACKING by \( \Psi \).

Refer to Fig. 1. Geometrically, we have \( m \) vertical strips corresponding to \( m \) elements and \( n \) horizontal strips corresponding to the \( n \) sets of 3-SET-PACKING, as shown shaded in the figure. These strips define a grid in the natural way. The bounding box of this grid, shown by thick solid lines, is the rectangle \( R \) whose area we would need to cover with other rectangle. For a set, say \( S_\ell = \{e_i, e_j, e_k\} \), we have three unit squares \( \Delta_{i,\ell}, \Delta_{j,\ell} \) and \( \Delta_{k,\ell} \), at the intersection of the horizontal strip for \( S_\ell \), and the vertical strips for \( e_i, e_j \) and \( e_k \), respectively, as shown. The two unit squares in the vertical strip for an element \( e_i \) are horizontally offset so that their projections on the \( x \)-axis are separated. Finally, we have an additional set of \( m \) unit squares, one for each element \( e_i \) (denoted by \( \Delta_{i,\text{add}} \)), in a row vertically aligned with the unit square for the first occurrence of the element. Note that there are exactly three unit squares in each horizontal strip and exactly three unit squares in each vertical strip.

Now, we start defining our rectangles that can be used to cover \( R \) (refer to Fig. 2 for illustrations). The weight of a rectangle \( R' \) is denoted by \( w(R') \) and is equal to \( \text{Area}(R' \setminus R) \).
$S_\ell = \{e_i, e_j, e_k\}$

$S_\ell$ and $S'_\ell$ are the two sets that contain $e_i$. The unit squares corresponding to the two occurrences of $e_i$ are horizontally offset so that their projection on the $x$-axis are disjoint.

Figure 1: Geometric view of the grid and unit squares (not drawn to scale for clarity and does not show the all the rectangles). $S_\ell$ and $S'_\ell$ are the two sets that contain $e_i$. The unit squares corresponding to the two occurrences of $e_i$ are horizontally offset so that their projection on the $x$-axis are disjoint.
Figure 2: Rectangles in our cover for $e_i$ (not drawn to scale for clarity). For each of the two sets that contain $e_i$, we have a set rectangle (only one of them, $R_{i,j,k}$, is shown in the figure). For each $e_i$, there is also an extra upward and an extra downward rectangle. Forced rectangles (not shown in the figure) cover the area of $R$ outside the unit squares (the cross-hatched area).
Forced rectangles: We have a set of (at most $O(n^2)$) rectangles whose union covers exactly the area of $R$ excluding all the unit squares. Note that, by Observation 1.1, we may assume that all these rectangles are in an optimal solution. Thus, the only uncovered areas of $R$ are those areas corresponding to the unit squares.

Set rectangles: For every set, say $S_{\ell} = \{e_i, e_j, e_k\}$, we have a set-rectangle $R_{\{i,j,k\}}$ that covers the unit squares on the horizontal strip for $S_{\ell}$ such that $w(R_{\{i,j,k\}}) = \text{Area}(R_{\{i,j,k\}} \setminus R) = 3$.

Additional upward rectangles: For every element $e_i$, there is an additional upward rectangle $\text{UP}_i$ containing the additional unit square for $e_i$ and the unit square for the first occurrence of $e_i$ but not intersecting the unit square for the second occurrence of $e_i$ such that $w(\text{UP}_i) = \text{Area}(\text{UP}_i \setminus R) = 1.41$.

Additional downward rectangles: For every element $e_i$, there is an additional upward rectangle $\text{DOWN}_i$ containing the additional unit square for $e_i$ and the unit square for the second occurrence of $e_i$ but not intersecting the unit square for the first occurrence of $e_i$ such that $w(\text{DOWN}_i) = \text{Area}(\text{DOWN}_i \setminus R) = 1.41$.

Let $\text{OPT}$ denote the optimum value of the objective function of our instance of the tiling problem. We now prove the theorem by showing the following:

(completeness) If $\Psi \geq (140 - \varepsilon)n$ then $\text{OPT} \leq (1029.12 + 1.23 \varepsilon)n'$.

(soundness) If $\Psi \leq (139 + \varepsilon)n$ then $\text{OPT} \geq (1030.35 - 1.23 \varepsilon)n'$.

1.1.1 Proof of Completeness ($\Psi \geq (140 - \varepsilon)n$)  

For every set $S_{\ell} = \{e_i, e_j, e_k\}$ in the solution we select the set-rectangle $R_{\{i,j,k\}}$. The total weight of all these selected rectangles is exactly $3\Psi$. After this, we have the following:

• If an element $e_i$ belonged to one of the selected set-rectangles, then the vertical strip for $e_i$ contains two uncovered unit squares which we cover by $\text{UP}_i$ or $\text{DOWN}_i$, as appropriate, with a weight of 1.41. The total weight of all the selected rectangles in this manner is exactly $1.41 \times (3\Psi) = 4.23 \Psi$.

• If an element $e_i$ did not belong to one of the selected set-rectangles, then the vertical strip for $e_i$ contains three uncovered unit squares which we cover by using both $\text{UP}_i$ and $\text{DOWN}_i$. The total weight of all these rectangles is $(426n' - 3\Psi) \times 2 \times 1.41 = 2.82 \times (426n' - 3\Psi)$.

Thus, the total weight of all the rectangles in our solution is

$$3\Psi + 4.23\Psi + 2.82 \times (426n' - 3\Psi) = 1201.32n' - 1.23 \Psi \leq (1029.12 + 1.23 \varepsilon)n'$$

1.1.2 Proof of Soundness ($\Psi \leq (139 + \varepsilon)n$)  

We start with a given solution of our tiling problem. We may assume that the given solution does not contain an obvious polynomial-time local improvement, e.g., a rectangle $R$ whose removal still produces a valid solution, or all the rectangles $\text{UP}_i$, $\text{DOWN}_i$ and $R_{\{i,j,k\}}$ for any $i$. Also, we will repeatedly use of the following obvious observations:
• **at least one of the rectangles UP\_i or DOWN\_i is selected for each i;**

• if **none** of the set-rectangles corresponding to the two sets containing element e\_i are selected, then our solution **must** contain both the rectangles UP\_i and DOWN\_i.

We first “normalize” the solution in stages such that the value of the objective function for our tiling does **not** increase; each step assumes that the normalization of all previous steps has been carried out. To provide intuition, we label each step with its purpose.

(1) **Always select a set all whose elements are uncovered:** Suppose that a set rectangle R\_{i,j,k} corresponding to the set S_\ell = \{e_i, e_j, e_k\} is **not** selected. Let S_p = \{e_i, e_a, e_b\}, S_q = \{e_j, e_a', e_b'\} and S_r = \{e_k, e_a'', e_b''\} be the three other sets containing e_i, e_j and e_k respectively. Suppose that **none** of the set rectangles R\_{i,a,b}, R\_{i,a',b'} and R\_{i,a'',b''} were selected in our solution. Without loss of generality, assume that R\_{i,j,k} is vertically above the rectangles R\_{i,a,b}, R\_{i,a',b'} and R\_{i,a'',b''}. Thus, our solution must contain the rectangles UP\_i, DOWN\_i, UP\_j, DOWN\_j, UP\_k and DOWN\_k of total weight 6 \times 1.41 = 8.46. Suppose that we replace these rectangles by the set rectangle R\_{i,j,k}, DOWN\_i, DOWN\_j and DOWN\_k of total weight 3 + 3 \times 1.41 = 7.23. This produces a valid solution and in fact improves the total value of the solution.

(2) **No two selected sets should have an element in common:** Suppose that our solution contain two set rectangles R\_{i,j,k} and R\_{i',j',k'} such that \{e_i, e_j, e_k\} \cap \{e_{i'}, e_{j'}, e_{k'}\} \neq \emptyset. Without loss of generality, assume that e_i = e_{i'} (and thus e_j \neq e_{j'} and e_k \neq e_{k'}, since any two sets have exactly one element in common), R\_{i,j,k} is vertically above R\_{i',j',k'} and the other two rectangles corresponding to two sets that contain the elements e_{j'} and e_{k'} are vertically below R\_{i',j',k'}. We may assume that the rectangle DOWN\_i is in our solution, since otherwise the solution must contain UP\_i and replacing UP\_i by DOWN\_i produces a solution of the same value. We now have the following cases.

**Case 1:** our solution contains one of UP\_j’ or UP\_k’, say UP\_j’. (see Fig. 3). We remove R\_{i',j',k'} (of weight 3) and, if not already in our solution, add UP\_k’ (of weight 1.41) to our solution, thereby improving our solution.

**Case 2:** our solution contains neither UP\_j’ nor UP\_k’. We remove R\_{i',j',k'} (of weight 3) and add UP\_j’ and UP\_k’ (of total weight 2 \times 1.41 = 2.82) to our solution, thereby improving our solution.

(3) **Remove redundant rectangles:** If after Steps (1) and (2) above, we have a redundant rectangle, i.e., a rectangle whose unit squares are covered by other rectangles, we delete it.

After this normalization, using the same argument as in the proof of completeness, if we have x set rectangles in our solution then our new value of the solution y satisfies

\[ y = 3x + 4.23x + 2.82 \times (426n' - 3x) = 1201.32 \ n' - 1.23 \ x \]

which provide a solution of 3-SET-PACKING with x sets (by choosing the set \{e_i, e_j, e_k\} for every selected rectangle R\_{i,j,k}). Since \( x \leq \Psi \leq (139 + \varepsilon)n \), the desired claim follows.
Figure 3: Case 1 of the soundness proof.
References