Math C067 Random Variables

Richard Beigel

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What is a Random Variable?

- **A** *random variable* X **is a rule that assigns a numerical value to each outcome in the sample space of an experiment.**

  A random variable is not a variable.

  A random variable is a function (functions are also called "mappings").

  A random variable maps outcomes to real numbers.

  The **range** of a random variable (R_X) is the set of possible values that it can assign to outcomes.

- A *discrete random variable* can take on specific, isolated numerical values, like the outcome of a roll of a die, or the number of dollars in a randomly chosen bank account.

  (If you really want a precise definition of descrete, a random variable is discrete if there is a positive real number ϵ such that every pair of distinct values x, x' in R_X differ by at least ϵ, i.e., $(\exists \epsilon > 0)(\forall x, x' \in R_x)[x \neq x' \Rightarrow |x - x'| > 0]$. This won't be on the test.)

- A discrete random variable that can take on only finitely many values (like the outcome of a roll of a die) is called a *finite random variable.*

Examples.

Finite Random Variable Consider the experiment of flipping a coin three times. The sample space consists of all ordered triples of heads and tails, i.e.,

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Let X be the number of heads obtained, i.e., X is the function defined by

$$X(HHH) = 3, X(HHT) = 2, X(HTH) = 2, X(HTT) = 1$$

$$X(THH) = 2, X(THT) = 1, X(TTH) = 1, X(TTT) = 0$$

X is a finite random variable that can assume the 4 values 0, 1, 2, 3, i.e.,

$$R_X = \{0, 1, 2, 3\}.$$

Finite Random Variables Consider the experiment of rolling two dice.

The sample space consists of all ordered pairs of numbers between 1 and 6, i.e.,

$$S = \{(1,1),(1,2),\ldots,(6,6)\}.$$

Let X be the sum of the two numbers showing on the dice, i.e., X is the function

$$X(a,b) = a+b.$$

X is a finite random variable that can assume the 11 values 2, 3, 4, ..., 12, i.e.,

$$R_X = \{2,3,4,\ldots,12\}.$$

Let Y be the larger of the two numbers showing on the dice, i.e., Y is the function

$$Y(a,b) = \max(a,b).$$

Y is a finite random variable that can assume the 6 values 1, 2, 3, 4, 5, 6, i.e.,

$$R_Y = \{1,2,3,4,5,6\}.$$

Infinite Discrete Random Variable Consider the experiment of flipping a coin until it comes up heads.

The sample space consists of consists of all sequences that consist of some number of tails (possibly 0) followed by a single head, i.e.,

$$S = H, TH, TTH, TTTH, TTTTH, \ldots.$$

Let X be the number of times you flip the coin.

The possible values for X are 1, 2, 3, 4, ...so the range of X is the set of positive integers, i.e.,

$$R_X = \mathsf{Z}^+$$

Infinite Discrete Random Variable Consider the experiment of rolling a single die until you get a 2.

The sample space consists of consists of all sequences of numbers between 1 and 6 that end in a 2 and have no 2 before the end, i.e.,

$$S = 2, 12, 32, 42, 52, 62, 112, 132, 142, 152, 162, 312, \ldots$$

Let X be the number of times you roll the die.

The possible values for X are 1, 2, 3, 4, …so the range of X is the set of positive integers, i.e.,

$$R_X = \mathsf{Z}^+$$

Operations on Random Variables Let X and Y be random variables on the same sample space S, and let k be a number.

We can define new random variables like $X + k$, $X + Y$, $k \cdot X$, $X \cdot Y$, and X^2 by using the standard arithmetical operations on the functions X and Y:

- $X + k$ is the function defined by the formula

$$(X + k)(s) = X(s) + k$$

- $X + Y$ is the function defined by the formula

$$(X + Y)(s) = X(s) + Y(s)$$

- $k \cdot X$ is the function defined by the formula

$$(k \cdot X)(s) = k \cdot X(s)$$

- $X \cdot Y$ is the function defined by the formula

$$(X \cdot Y)(s) = X(s) \cdot Y(s)$$

- $X^2 = X \cdot X$, i.e., X^2 is the function defined by the formula

$$(X^2)(s) = X(s) \cdot X(s)$$

Notation

- $P(X = a)$ denotes the probability that X maps a random sample point to a, i.e.,

$$P(X = a) = P(\{s : X(s) = a\})$$

- $P(X \geq a)$ denotes the probability that X maps a random sample point to a number that is greater than or equal to a, i.e.,

$$P(X \geq a) = P(\{s : X(s) \geq a\})$$

- $P(X > a)$ denotes the probability that X maps a random sample point to a number that is greather than a, i.e.,

$$P(X > a) = P(\{s : X(s) > a\})$$

- $P(a \leq X \leq b)$ denotes the probability that X maps a random sample point to a number that is between a and b, i.e.,

$$P(a \leq X \leq b) = P(\{s : a \leq X(s) \leq b\})$$

- etc.

Probability Distributions. Suppose the random variable X assigns only a finite number of possible values to points in the sample space S, i.e., the range of X is a finite set

$$R_X = \{x_1, x_2, \ldots, x_n\}$$

where $x_1 < x_2 < \cdots < x_n$.

The *probability distribution of the random variable* X is the function f defined as

$$f(x_k) = P(X = x_k)$$

f is usually called the *distribution of* X.

Examples.

Probability Distribution of a Finite Random Variable Consider the experiment of flipping a coin three times. The sample space consists of all ordered triples of heads and tails, i.e.,

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Let X be the number of heads obtained, i.e., X is the function defined by

$$X(HHH) = 3, X(HHT) = 2, X(HTH) = 2, X(HTT) = 1$$

$$X(THH) = 2, X(THT) = 1, X(TTH) = 1, X(TTT) = 0$$

X is a finite random variable that can assume the 4 values 0, 1, 2, 3, i.e.,

$$R_X = \{0, 1, 2, 3\}.$$

The distribution of X is given in the table below:

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Probability Distribution of a Finite Random Variable Consider the experiment of rolling two dice.

The sample space consists of all ordered pairs of numbers between 1 and 6, i.e.,

$$S = \{(1,1), (1,2), \dots, (6,6)\}.$$

Let X be the sum of the two numbers showing on the dice, i.e., X is the function

$$X(a,b) = a + b.$$

X is a finite random variable that can assume the 11 values 2, 3, 4, ..., 12, i.e.,

$$R_X = \{2, 3, 4, \dots, 12\}.$$

$$P(X = k) = \begin{cases} \frac{k-1}{36} & \text{if } k \leq 7 \\ \frac{12-(k-1)}{36} & \text{if } k \geq 7 \end{cases}$$

The distribution of X is given in the table below:

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Let Y be the larger of the two numbers showing on the dice, i.e., Y is the function

$$Y(a,b) = \max(a,b).$$

Y is a finite random variable that can assume the 6 values 1, 2, 3, 4, 5, 6, i.e.,

$$R_Y = \{1, 2, 3, 4, 5, 6\}.$$

$$P(Y = k) = \frac{2k-1}{36}$$

The distribution of Y is given in the table below:

y	1	2	3	4	5	6
$f(y)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Probability Distribution of an Infinite Discrete Random Variable Consider the experiment of flipping a coin until it comes up heads.

The sample space consists of consists of all sequences that consist of some number of tails (possibly 0) followed by a single head, i.e.,

$$S = H, TH, TTH, TTTH, TTTTH, \ldots.$$

Let X be the number of times you flip the coin.

The possible values for X are 1, 2, 3, 4, ...

$$R_X = \mathsf{Z}^+$$

$$P(X = k) = \left(\frac{1}{2}\right)^k$$

The distribution of X is given in the table below:

x	1	2	3	4	5	...
$f(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	...

Probability Distribution of an Infinite Discrete Random Variable Consider the experiment of rolling a single die until you get a 2.

The sample space consists of consists of all sequences of numbers between 1 and 6 that end in a 2 and have no 2 before the end, i.e.,

$$S = 2, 12, 32, 42, 52, 62, 112, 132, 142, 152, 162, 312, \ldots$$

Let X be the number of times you roll the die.

The possible values for X are 1, 2, 3, 4, $\ldots$.

$$R_X = \mathsf{Z}^+$$

$$P(X = k) = \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6}$$

The distribution of X is given in the table below:

x	1	2	3	4	5	$\ldots$
$f(x)$	$\frac{1}{6}$	$\frac{5}{6} \cdot \frac{1}{6}$	$\left(\frac{5}{6}\right)^2 \cdot \frac{1}{6}$	$\left(\frac{5}{6}\right)^3 \cdot \frac{1}{6}$	$\left(\frac{5}{6}\right)^4 \cdot \frac{1}{6}$	$\ldots$

Expectation: Mean of a Finite Random Variable Suppose that X is a finite random variable with range $\{x_1, \ldots, x_n\}$ and distribution f. The *mathematical expectation* of X is the weighted average of all possible values of X. Each value is weighted by its probability, so that likely values affect the average more than unlikely values. The formula is

$$E(X) = x_1 \cdot f(x_1) + x_2 \cdot f(x_2) + \cdots + x_n \cdot f(x_n)$$

$E(X)$ is also called

- the *expected value* of X
- the *expectation* of X
- the *mean* of X
- μ (pronounced "mu," which is m in Greek)
- μ_X

If the distribution of X is given by a table

x	x_1	x_2	$\ldots$	x_n
$f(x)$	p_1	p_2	$\ldots$	p_n

then the formula for the expectation of X is

$$E(X) = x_1 \cdot p_1 + x_2 \cdot p_2 + \cdots + x_n \cdot p_n$$

Examples.

Uniform Distribution If all possible values for a random variable X are equally likely then X is called a *uniform* random variable and f is called the *uniform distribution* on $\{x_1, \ldots, x_n\}$. The distribution of X is given by the table

x	x_1	x_2	$\ldots$	x_n
$f(x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\ldots$	$\frac{1}{n}$

The mean of a uniform random variable X is the ordinary mean of its range, R_X:

$$\begin{aligned} E(X) &= x_1 \cdot \frac{1}{n} + x_2 \cdot \frac{1}{n} + \cdots + x_1 \cdot \frac{1}{n} \\ &= \frac{x_1 + x_2 + \cdots x_n}{n} \end{aligned}$$

Mean of a finite random variable Consider the experiment of flipping a coin three times. The sample space consists of all ordered triples of heads and tails, i.e.,

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Let X be the number of heads obtained, i.e., X is the function defined by

$$X(HHH) = 3, X(HHT) = 2, X(HTH) = 2, X(HTT) = 1$$

$$X(THH) = 2, X(THT) = 1, X(TTH) = 1, X(TTT) = 0$$

X is a finite random variable that can assume the 4 values 0, 1, 2, 3, i.e.,

$$R_X = \{0, 1, 2, 3\}.$$

The distribution of X is given in the table below:

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The mean of X is given by the formula

$$\begin{aligned} E(X) &= 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} \\ &= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} \\ &= \frac{12}{8} \\ &= 1.5 \end{aligned}$$

Mean of a Finite Random Variable Consider the experiment of rolling two dice. The sample space consists of all ordered pairs of numbers between 1 and 6, i.e.,

$$S = \{(1,1),(1,2),\ldots,(6,6)\}.$$

Let X be the sum of the two numbers showing on the dice, i.e., X is the function

$$X(a,b) = a+b.$$

X is a finite random variable that can assume the 11 values 2, 3, 4, …, 12, i.e.,

$$R_X = \{2,3,4,\ldots,12\}.$$

$$P(X=k) = \begin{cases} \frac{k-1}{36} & \text{if } k \le 7 \\ \frac{12-(k-1)}{36} & \text{if } k \ge 7 \end{cases}$$

The distribution of X is given in the table below:

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The mean of X is given by the formula

$$\begin{aligned} E(X) &= 2\cdot\frac{1}{36}+3\cdot\frac{2}{36}+4\cdot\frac{3}{36}+5\cdot\frac{4}{36}+6\cdot\frac{5}{36}+7\cdot\frac{6}{36}+8\cdot\frac{5}{36}+9\cdot\frac{4}{36}+10\cdot\frac{3}{36}+11\cdot\frac{2}{36}+ \\ &= \frac{2}{36}+\frac{6}{36}+\frac{12}{36}+\frac{20}{36}+\frac{30}{36}+\frac{42}{36}+\frac{40}{36}+\frac{36}{36}+\frac{30}{36}+\frac{22}{36}+\frac{12}{36} \\ &= \frac{252}{36} \\ &= 7 \end{aligned}$$

Note: This could be seen without a calculation because the distribution is symmetric.

Let Y be the larger of the two numbers showing on the dice, i.e., Y is the function

$$Y(a,b) = \max(a,b).$$

Y is a finite random variable that can assume the 6 values 1, 2, 3, 4, 5, 6, i.e.,

$$R_Y = \{1,2,3,4,5,6\}.$$

$$P(Y=k) = \frac{2k-1}{36}$$

The distribution of Y is given in the table below:

y	1	2	3	4	5	6
$f(y)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

The mean of Y is given by the formula

$$\begin{aligned} E(Y) &= 1\cdot\frac{1}{36} + 2\cdot\frac{3}{36} + 3\cdot\frac{5}{36} + 4\cdot\frac{7}{36} + 5\cdot\frac{9}{36} + 6\cdot\frac{11}{36} \\ &= \frac{1}{36} + \frac{6}{36} + \frac{15}{36} + \frac{28}{36} + \frac{45}{36} + \frac{66}{36} \\ &= \frac{151}{36} \\ &\approx 4.19444444 \end{aligned}$$

Mean of an Infinite Random Variable Consider the experiment of flipping a coin until it comes up heads.

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Let X be the number of times you flip the coin.

The possible values for X are 1, 2, 3, 4, ...

$$R_X = \mathsf{Z}^+$$

$$P(X = k) = \left(\frac{1}{2}\right)^k$$

The distribution of X is given in the table below:

x	1	2	3	4	5	...
$f(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	...

The mean of X is given by the formula

$$\begin{aligned} E(X) &= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + 5 \cdot \frac{1}{32} + \cdots \\ &= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots \\ &= 2 \end{aligned}$$

Note: Because its terms decrease **exponentionally** fast you can estimate this sum pretty well by adding up the first 10 terms on a calculator (the value you calculate will differ from the true value by approximately the last term that you use). This won't be on the test.

Probability Distribution of an Infinite Discrete Random Variable Consider the experiment of rolling a single die until you get a 2.

The sample space consists of consists of all sequences of numbers between 1 and 6 that end in a 2 and have no 2 before the end, i.e.,

$$S = 2, 12, 32, 42, 52, 62, 112, 132, 142, 152, 162, 312, \ldots$$

Let X be the number of times you roll the die.

The possible values for X are 1, 2, 3, 4, ...

$$R_X = \mathsf{Z}^+$$

$$P(X = k) = \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6}$$

The distribution of X is given in the table below:

x	1	2	3	4	5	...
$f(x)$	$\frac{1}{6}$	$\frac{5}{6} \cdot \frac{1}{6}$	$\left(\frac{5}{6}\right)^2 \cdot \frac{1}{6}$	$\left(\frac{5}{6}\right)^3 \cdot \frac{1}{6}$	$\left(\frac{5}{6}\right)^4 \cdot \frac{1}{6}$	...

The mean of X is given by the formula

$$\begin{aligned}
E(X) &= 1 \cdot \frac{1}{6} + 2 \cdot \frac{5}{6} \cdot \frac{1}{6} + 3 \cdot \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} + 4 \cdot \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} + 5 \cdot \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6} + \cdots \\
&= \frac{1}{6} + \frac{10}{36} + \frac{125}{216} + \frac{500}{1296} + \frac{3125}{7776} + \cdots \\
&= 6
\end{aligned}$$

Note: Because its terms decrease **exponentionally** fast you can estimate this sum pretty well by adding up the first 50 terms on a calculator (the value you calculate will differ from the true value by approximately 5 times the last term that you use). This won't be on the test.

Variance of a Finite Random Variable Recall that $\mu = E(X)$, the mean of X. Suppose that the distribution of X is given by the table

x	x_1	x_2	x_3	$\dots$	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\dots$	$f(x_n)$

The *variance* of X is given by the formula

$$\mathrm{Var}(X) = (x_1 - \mu)^2 \cdot f(x_1) + (x_2 - \mu)^2 \cdot f(x_2) + \cdots + (x_n - \mu)^2 \cdot f(x_n),$$

i.e., $\mathrm{Var}(X)$ is the weighted average of the values of $(X - \mu)^2$:

$$\mathrm{Var}(X) = E((X - \mu)^2)$$

An equivalent formula is proved in your book (it's not obvious):

$$\mathrm{Var}(X) = E(X^2) - \mu^2$$

The variance of X is also called σ^2 or σ_X^2.

The *standard deviation* of X is given by the formula

$$\sigma = \sigma_X = \sqrt{\mathrm{Var}(X)}$$

Example: Variance of a finite random variable. Consider the experiment of flipping a coin three times. The sample space consists of all ordered triples of heads and tails, i.e.,

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Let X be the number of heads obtained, i.e., X is the function defined by

$$X(HHH) = 3, X(HHT) = 2, X(HTH) = 2, X(HTT) = 1$$

$$X(THH) = 2, X(THT) = 1, X(TTH) = 1, X(TTT) = 0$$

X is a finite random variable that can assume the 4 values 0, 1, 2, 3, i.e.,

$$R_X = \{0, 1, 2, 3\}.$$

The distribution of X is given in the table below:

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The mean of X is given by the formula

$$\begin{aligned} E(X) &= 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} \\ &= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} \\ &= \frac{12}{8} \\ &= 1.5 \end{aligned}$$

Note: This could be seen without a calculation because the distribution is symmetric.

The variance of X is given by the formula

$$\begin{aligned} \mathrm{Var}(X) &= (0-1.5)^2 \cdot \frac{1}{8} + (1-1.5)^2 \cdot \frac{3}{8} + (2-1.5)^2 \cdot \frac{3}{8} + (3-1.5)^2 \cdot \frac{1}{8} \\ &= (-1.5)^2 \cdot \frac{1}{8} + (-0.5)^2 \cdot \frac{3}{8} + 0.5^2 \cdot \frac{3}{8} + 1.5^2 \cdot \frac{1}{8} \\ &= 1.5^2 \cdot \frac{1}{8} + 0.5^2 \cdot \frac{3}{8} + 0.5^2 \cdot \frac{3}{8} + 1.5^2 \cdot \frac{1}{8} \\ &= 0.75 \end{aligned}$$